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by

Elena Popoviciu
Head of the Seminar

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Approximation operators - solutions and questions

OCTAVIAN AGRATINI

(CLUJ-NAPOCA)

ABSTRACT. The paper gathers some results concerning linear approximation operators and it raises three open problems.

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1 Introduction

A traditional feature of STMACO is to offer solutions regarding various existant problems from different mathematical areas and, in the same time, to present new open problems. In the present paper we follow this line. Practically, we never know the particular seed in our mind that will germinate. At first, in each section of the three, we indicate some recent results obtained in the theory of approximation of functions by linear positive operators. The focus of this note is to propose three research themes in connection with the above mentioned approaches. We point out that the results of both Section 2 and Section 3 are included in two distinct papers submitted abroad for publication. Section 4 incorporates a part of the communication delivered at the 5^{th} Conference of Probability and Statistics held in Bucharest, February, 2002.

2 Discrete operators without tails

The starting point is a sequence $(L_n)_{n\geq 1}$ of linear operators of discrete type, L_n being defined as follows

$$(2.1)\qquad (L_n f)(x) = \sum_{k=0}^{\infty} \phi_{n,k}(x) f\left(\frac{k}{a_n}\right), \quad x \geq 0,\ f \in \mathcal{F},$$

where $\mathcal{F}$ stands for the domain of L_n containing the set of all continuous functions on $\mathbb{R}_+ := [0,\infty)$ for which the series in (1.9) is convergent. Also we assume that the following requirements are fulfilled

$$(2.2)\qquad \begin{cases} k a_n^{-1} \leq k,\ k \in \mathbb{N},\ \text{with } \lim\limits_{n\to\infty} a_n = \infty, \\ \phi_{n,k} \in C^1(\mathbb{R}_+),\ \phi_{n,k} \geq 0 \text{ for every } k \in \mathbb{N}_0, \text{ and} \\ \sum\limits_{k=0}^{\infty} \phi_{n,k} = e_0,\ \sum\limits_{k=0}^{\infty} k a_n^{-1} \phi_{n,k} = e_1, \end{cases}$$

where e_j stands for the j-th monomial.

It is clear that the above classical construction requires an estimation of infinite sums which in a certain sense restricts the operators usefulness from the computational point of view. Thus, it is useful to consider partial sums which only have finite terms depending upon n and x. Roughly speaking, the operators will be truncated fading away their "tails".

In what follows, we also consider that a positive function $\psi \in \mathbb{R}^{\mathbb{N}\times\mathbb{R}_+}$, $\psi(n,\cdot) \in C(\mathbb{R}_+)$, exists with the property

$$(2.3)\qquad \psi(n,x)\phi'_{n,k}(x) = (k a_n^{-1} - x)\phi_{n,k}(x), \quad k \in \mathbb{N}_0,\ x \geq 0.$$

Moreover, we assume that ψ admits the following decomposition

$$(2.4)\qquad \psi(n,x) = \sum_{i=1}^{l} \frac{\psi_i(x)}{a_n^i},\ x \geq 0, \text{ where } \psi_i \in C(\mathbb{R}_+),\ i = \overline{1,l}.$$

Under all the above assumptions, one obtains that $(L_n)_{n\geq 1}$ converges to the identity operator, i.e. the sequence is an approximation process.

Keeping all assumptions we further define

$$(2.5)\qquad (L_{n,\delta}f)(x) := \sum_{k=0}^{[a_n(x+\delta(n))]} \phi_{n,k}(x) f\left(\frac{k}{a_n}\right), \quad x \geq 0,\ f \in \mathcal{F},$$

where $\delta = (\delta(n))_{n\geq 1}$ is a sequence of positive numbers.
The study of these operators can be developed in polynomial weighted spaces connected to the weights w_m, $w_m(x) = (1 + x^{2m})^{-1}$, $x \geq 0$. For every $m \in \mathbb{N}_0$, the spaces

$$E_m := \{f \in C(\mathbb{R}_+) : \ \|f\|_m := \sup_{x\geq 0} w_m(x)|f(x)| < \infty\}$$

endowed with the norm $\|\cdot\|_m$ and the natural order are Banach lattices.

Theorem 1. *Let $L_{n,\delta}$, $n \in \mathbb{N}$, be defined by (3.6).*
(i) If $\psi_i \in C^{2m-2}(\mathbb{R}_+)$, $i = \overline{1,l}$, and $\lim\limits_{n\to\infty} \sqrt{a_n}\delta(n) = \infty$ then $L_{n,\delta}f$ converges to f, uniformly on any compact $K \subset [0,\infty)$, for $f \in E_m \cap \mathcal{F}$.
(ii) If $\delta := M - x$, $M > 0$, the corresponding operators denoted by L_n^* map $C([0,M])$ into $C([0,M])$ and have the property

$$\lim_{n\to\infty} (L_n^* f)(x) = f(x) \text{ for all } f \in C([0,M]),$$

uniformly on every compact $K_M \subset [0,M)$.

Examples. We consider the particular case $a_n = n$, $n \in \mathbb{N}$.

1. Selecting $\phi_{n,k}(x) = \dfrac{(nx)^k}{k!}\exp(-nx)$, $x \geq 0$, the conditions (2.2), (2.3), (2.4) are fulfilled and we have $\psi(n,x) = x/n$. The operators L_n defined by (1.9) turn out into the Szász operators and in this case the operators from (3.6) have been investigated by Lehnhoff [5].

2. Selecting $\phi_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^{-n-k}$, $x \geq 0$, the conditions (2.2), (2.3), (2.4) are again fulfilled and we have $\psi(n,x) = x(x+1)/n$. Now, the operators L_n become the Baskakov operators and the operators from (3.6) have been investigated by J. Wang and S. Zhou [6].

In both examples $\mathcal{F}$ may coincide with E_2 and Theorem 1 encounters results obtained in the quoted papers.

As regards this approach we formulate the following

Problem 1. Develop a similar study for the integral extensions of L_n operators in Durrmeyer and Kantorovich sense.

3 A class of polynomials of Durrmeyer-type

M. Campiti and G. Metafune [3] replaced in the Bernstein polynomials,

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(k/n), \tag{3.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, and $x \in [0,1]$, the binomial coefficients by general ones satisfying similar recursive properties. The new sequence operators converges to an operator multiplied by an analytic function depending on the sequences of the sides of Pascal's triangle.
We fix two sequences of real positive numbers $a = (a_n)_{n\geq 1}$, $b = (b_n)_{n\geq 1}$ and for every $(n,k) \in \mathbb{N} \times \{0,1,\ldots,n\}$ we define the polynomials

$$q_{n,k}(x) = c_{n,k} x^k (1-x)^{n-k}, \quad x \in [0,1], \tag{3.2}$$

where the coefficients satisfy the following recursive formulas

$$c_{n+1,k} = c_{n,k} + c_{n,k-1}, \quad k = 1,\ldots,n, \quad c_{n,0} = a_n, \quad c_{n,n} = b_n. \tag{3.3}$$

For $f \in L_1([0,1])$, we consider polynomials having the form

$$(M_n f)(x) = (n+1) \sum_{k=0}^{n} q_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad x \in [0,1]. \tag{3.4}$$

If $a_j = b_j = 1$ for every $j = 1, 2, \ldots, n$, we have $c_{n,k} = \binom{n}{k}$ and M_n becomes the Bernstein modified operator in Durrmeyer sense.

Theorem 2. *Let the operator M_n be defined by (3.4). The following identities hold true*

$$(M_n e_0)(x) \tag{3.5}$$

$$= \sum_{m=1}^{n-1} (a_m x(1-x)^m + b_m x^m (1-x)) + a_n (1-x)^n + b_n x^n,$$

$$\int_0^1 (M_n f)(x) g(x) dx = \int_0^1 f(t)(M_n g)(t) dt$$

where f and g belong to $L_1([0,1])$. Particularly, M_n is a self-adjoint operator on the space $L_2([0,1])$.

From (3.5) we deduce $(M_n e_0)(0) = a_n$ and $(M_n e_0)(1) = b_n$. This means that the convergence of $(M_n)_{n \geq 1}$ implies the convergence of the sequence a and b. In what follows we assume that these sequences converges and set $\lim_{n\to\infty} a_n := l_a$, $\lim_{n\to\infty} b_n := l_b$. Because of the above assumption we can define the functions σ, τ, φ belonging to $\mathbb{R}^{[0,1]}$ as follows

$$\text{(3.6)} \qquad \begin{cases} \sigma(x) := \sum_{m=1}^{\infty} a_m x(1-x)^m, 0 < x \leq 1, \text{ and } \sigma(0) = l_a, \\ \tau(x) := \sum_{m=1}^{\infty} b_m x^m (1-x), 0 \leq x < 1, \text{ and } \tau(1) = l_b. \end{cases}$$

We also set

$$\text{(3.7)} \qquad \begin{cases} \mu(n) := \max_{m \leq n} \{a_m, b_m\}, \\ \lambda_n(x) := 2((n-3)x(1-x)+1)/(n+3), \\ \nu(n) := \sup_{j \geq n} \max\{|a_j - a_n|, |b_j - b_n|\}. \end{cases}$$

We present some results concerning the degree of convergence.

Theorem 3. *Let M_n be defined by (3.4) such that the sequences a and b converge. One has*

$$|(M_n f)(x) - f(x)(M_n e_0)(x)| \leq \mu(n)(1 + \lambda_n(x))\omega_f(1/\sqrt{n+2}),$$

$$|(M_n f)(x) - \varphi(x) f(x)|$$
$$\leq \mu(n)(1+\lambda_n(x))\omega_f(1/\sqrt{n+2}) + ((1-x)^n + x^n)\nu(n)|f(x)|.$$

If $f \in C([0,1])$ then $\|M_n f - \varphi f\| \leq 2\mu(n)\omega_f\left(\frac{1}{\sqrt{n+2}}\right) + \nu(n)\|f\|.$

If $f \in L_1([0,1])$ *then* $\|M_n f - \varphi f\|_1 \leq \frac{4}{3}\mu(n)\omega_f\left(\frac{1}{\sqrt{n+2}}\right) + \nu(n)\|f\|_1$.

Here $\varphi, \mu(n), \lambda_n(x), \nu(n)$ *are given by (3.6) and (3.7).*

We are able to completely describe the convergence of $(M_n)_{n\geq 1}$. This sequence converges on the space X ($X := C([0,1])$ or $X := L_1([0,1])$) if and only if the real sequences a and b converge. In this case we have

$$\lim_{n\to\infty} M_n f = \varphi f \text{ in the norm of the space } X, \text{ for every } f \in X.$$

Theorem 4. *Let* M_n *be defined by (3.4) such that the sequences* a *and* b *admit an upper bound less or equal to 1. Then* $M_n f$ *is a contraction in* $L_p([0,1])$ *for every* $f \in L_p([0,1])$, *where* $1 \leq p \leq \infty$.

Special cases. Let us consider a and b non-decreasing sequences. Choosing $d_n := a_n - a_{n-1}$, $d'_n = b_n - b_{n-1}$, $n \geq 1$, with the convention $a_0 = b_0 = 0$, one has $a_m = \sum_{k=1}^{m} d_k$, $b_m = \sum_{k=1}^{m} d'_k$ and from (3.6) we obtain

$$\varphi(x) = \sum_{m=1}^{\infty} d_m(1-x)^m + \sum_{m=1}^{\infty} d'_m x^m, \quad x \in [0,1].$$

For example, we can choose $d_m = 0$ and $d'_m = \frac{(\alpha)_m}{(\beta)_m m!}$, $m \geq 1$, where α and β are positive fixed numbers. Here $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$ for $k \geq 1$. This choice leads us to a hypergeometric function $\varphi(x) = {}_1F_1(\alpha,\beta;x) - 1$, $x \in [0,1]$. It is a convergent series for all values of x and by using Kummer's equation, we get

$$x\frac{d^2\varphi}{dx^2} + (\beta - x)\frac{d\varphi}{dx} - \alpha\varphi = \alpha.$$

We have free hands to give α and β various values thus obtaining for φ, functions with a great personality, as reflected in [1; *§13.6, page 509*].

Since we consider the sequence $(M_n f)_{n\geq 1}$ a fertile field of investigation, we propose

Problem 2. Find the iterates of the sequence and an asymptotic property as Voronovskaja-type formula. Study the convergence of derivatives of $M_n f$ for a differentiable function f.

4 Approximation processes of Feller type

In this section we consider the following:

$\{X_{n,j} : j = 1, 2, \ldots, n;\ n \in \mathbb{N}\}$ a triangular array of independent random variables (i.r.v.) such that for each fixed n, $X_{n,j}$, $j = \overline{1,n}$, are identically distributed (i.d.) with $E(X_{n,j}) = E_n(x)$ and finite variance $Var(X_{n,j}) = \sigma_n^2(x) > 0$, $j = \overline{1,n}$, where $x \in I \subset \mathbb{R}$ is a parameter;

$\{\lambda_{n,j} : j = 1, 2, \ldots, n;\ n \in \mathbb{N}\}$ a triangular array of positive numbers.

We construct the following sequence of linear operators

$$(4.1) \qquad (\Lambda_n h)(x) = E[h(Z_n)] = \int_{\mathbb{R}} h\Big(\Big(\sum_{j=1}^{n} \lambda_{n,j}\Big)u\Big)dF_{n,x}(u),\ h \in \mathcal{L},$$

where $Z_n = \sum_{j=1}^{n} \lambda_{n,j} X_{n,j}$, $F_{n,x}$ is the distribution function of Z_n and $\mathcal{L}$ stands for the domain of Λ_n containing all functions h for which $E[h(Z_n)] < \infty$.

Particular cases. Choosing $\lambda_{n,1} = \cdots = \lambda_{n,n} := \lambda_n$, $(\Lambda_n)_{n\geq 1}$ becomes a sequence studied by M.K. Khan [4]. If $X_{n,j}$, $j = \overline{1,n}$, are i.d. for all n, $E_n(x) = x$, $\sigma_n^2(x) = \sigma^2(x) > 0$ and $\varphi_n = n^{-1}$ then Λ_n reduces to the classical Feller operator.

More details about probabilistic methods and positive approximation processes can be found, e.g., in the monograph [2; §*5.2*].

By simple computation we obtain $\Lambda_n e_0 = e_0$, $\Lambda_n e_1 = \Big(\sum_{j=1}^{n} \lambda_{n,j}\Big) E_n$ and $\Lambda_n e_2 = \Big(\sum_{j=1}^{n} \lambda_{n,j}^2\Big)\sigma_n^2 + \Big(\sum_{j=1}^{n} \lambda_{n,j}\Big)^2 E_n^2$. Based on F. Altomare's and M. Campiti's monograph [2; §*5.1*] we indicate the rate of convergence.

Theorem 5. *Let Λ_n, $n \in \mathbb{N}$, be defined by (4.1). For every $h \in C(I)$ and $\alpha > 0$ holds true*

$$|(\Lambda_n h)(x) - h(x)| \leq (1 + n^{\alpha}\mu_2(\Lambda_{n,x}))\omega_1(h, n^{-\alpha/2}), \quad x \in I,$$

where $\omega_1(h,\cdot)$ *is the first modulus of smoothness associated to* h *and*

$$\mu_2(\Lambda_n,x) := \Big(\Big(\sum_{j=1}^{n}\lambda_{n,j}\Big)E_n(x)-x\Big)^2+\Big(\sum_{j=1}^{n}\lambda_{n,j}^2\Big)\sigma_n^2(x), \quad x\in I.$$

We mention that the classical results concerning Feller operators can be reobtained from the above. Now we formulate

Problem 3. By using probabilistic methods, establish an asymptotic estimate of the remainder.

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Babeş-Bolyai University
Fac. of Math. and Computer Science
Kogălniceanu 1, 3400 Cluj-Napoca
Romania
agratini@math.ubbcluj.ro