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On some wavelet type linear operators

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Abstract

In this paper is introduced a general class $(L_k)_{k\in\mathbb{Z}}$ of linear positive operators of wavelet type. The construction is based on two sequences of real numbers which verify some certain conditions. We also study some properties of the above operators. The main result consists in establishing a Jackson inequality by using the first modulus of smoothness.

1 Introduction

The subject of wavelet analysis is quite diverse and extensive, pioneering work being done by mathematicians, computer, mechanical and electrical engineers, physicists and also by experts in applied sciences such as geophysics and statistics. The last two decades have produced tremendous developments in the mathematical theory of wavelets and this fact can be justified reminding that more than 100 books have been issued. For example, L. Debnath's monograph [3] represents one of the most recent authoritative guide to wavelets.

Among numerous applications of this area we shall consider here the construction of wavelet type linear positive operators. In this respect, without any doubt, a major contribution is due to George Anastassiou and his

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collaborators. Anastassiou's quantitative approximation methods applied in wavelets' field are gathered in [1]. In this frame our research has been mainly motivated by paper [2], see also [1], *Chapter 6*.

Following the trend created by Franklin-Strömberg, the definition of *wavelets* is in connection with the bidimensional net $(2^k, j)$, $(k, j) \in \mathbb{Z} \times \mathbb{Z}$, where k denotes the *translation index* and j represents the *dilation index*.

From the point of view of approximation we can consider a general net having the form (a_k, b_j), $(k, j) \in \mathbb{Z}\times\mathbb{Z}$. Of course, regarding these sequences some additional conditions will be required. This new bi-dimensional net is more flexible than the previous one. Indeed, with the help of the above two general sequences, in order to approximate different kinds of functions, we can take advantage transforming the net in accordance with the problem data. We refer here to those signals f for which we are in position to obtain information in some certain points of the real line.

This is the motivation and, in the same time, the main idea of the present note. Our paper is designed as follows. In Section 2 we introduce some notations and we define the notion of *cvasi-scaling type function*. Some examples are provided. A general class $(L_k)_{k\in\mathbb{Z}}$ of linear positive operators of wavelet type is constructed in Section 3 and further on we give another look for L_k. In the last section we establish an inequality for estimating the degree of approximation. To do this, we involve the first modulus of smoothness of the approximated function. In order to ensure that (L_k) becomes an approximation process, a sufficient condition is given.

2 Preliminaries

Throughout the paper we consider $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. Let us define the following sets of real sequences:

$$\mathcal{S} := \{s = (s_n)_{n\in\mathbb{N}_0} : \ 0 \le s_0 \text{ and } s_k < s_{k+1} \text{ for every } k \in \mathbb{N}_0\},$$

$$\widetilde{\mathcal{S}}_1 := \{s = (s_m)_{m\in\mathbb{Z}} : \ (s_m)_{m\in\mathbb{N}_0} \in \mathcal{S} \text{ and } s_{-m} = s_m^{-1} \text{ for every } m \in \mathbb{N}_0\},$$

$$\widetilde{\mathcal{S}}_2 := \{s = (s_m)_{m\in\mathbb{Z}} \in \mathcal{S} \text{ and } s_{-m} := -s_m \text{ for every } m \in \mathbb{N}_0\}.$$

If $s \in \widetilde{S}_1 \cup \widetilde{S}_2$ then it is easy to see that $s_{-(m+1)} < s_{-m}$ for every $m \in \mathbb{N}_0$. Moreover we have: if $s \in \widetilde{S}_1$ then $s_0 = 1$, if $s \in \widetilde{S}_2$ then $s_0 = 0$.

In what follows we consider $a = (a_k)_{k \in \mathbb{Z}} \in \widetilde{S}_1$ and $b = (b_j)_{j \in \mathbb{Z}} \in \widetilde{S}_2$ as fix. Let $L_{1,loc}(\mathbb{R})$ be the vector space of the real-valued functions defined on $\mathbb{R}$ and integrable on any interval compact of the real line. Also, let $\varphi \in \mathbb{R}_+^{\mathbb{R}}$ be a bounded function verifying the conditions:

(C_1) φ belongs to $L_{1,loc}(\mathbb{R})$ and it has bounded support,

(C_2) a positive constant Γ exists with the property

$$\sum_{j=-\infty}^{\infty} \varphi(x + b_j) = \Gamma, \quad x \in \mathbb{R}. \tag{2.1}$$

The condition (C_1) implies that $\alpha > 0$ exists such that

$$supp\varphi \subset [-\alpha, \alpha]. \tag{2.2}$$

By using the above elements we define the functions

$$\varphi_{k,j}(x) := \sqrt{a_k}\varphi(a_k x + b_j), \quad x \in \mathbb{R}, \ (k, j) \in \mathbb{Z} \times \mathbb{Z}. \tag{2.3}$$

The same condition (C_1) guarantees that φ belongs to the Lebesgue space $L_2(\mathbb{R})$. As usual, the space is endowed with the inner product $(\cdot, \cdot)$ defined by

$$(h_1, h_2) = \int_{\mathbb{R}} h_1(t)h_2(t)dt, \quad h_1 \in L_2(\mathbb{R}), \ h_2 \in L_2(\mathbb{R}), \tag{2.4}$$

and it becomes a Banach space with the norm $\|\cdot\|_2$, $\|h\|_2 = \sqrt{(h, h)}$. For the sake of convenience, we make the following informal definition.

Definition 2.1. *A function $\varphi : \mathbb{R} \to [0, \infty)$ satisfying conditions (C_1) and (C_2) is called a cvasi-scaling type function.*

Example 2.2. Let us take the particular sequence $b = (j)_{j \in \mathbb{Z}} \in \widetilde{S}_2$.

1. The characteristic function $\chi_{[-1,1)}$ (often called *a rectangular pulse* or *a gate function*) verifies the conditions (C_1) and (C_2) with $\Gamma = 2$ and consequently $\varphi = \chi_{[-1,1)}$ is a cvasi-scaling type function.

2. Considering *a Chinese hat*

$$\varphi(x) = \begin{cases} x+1, & -1 \le x \le 0, \\ 1-x, & 0 < x \le 1, \\ 0, & x \in \mathbb{R} \setminus [-1,1], \end{cases}$$

the conditions (C_1) and (C_2) with $\Gamma = 1$ are fulfilled and φ is a cvasi-scaling type function.

We will denote by $\mathcal{D}_\alpha$, $\alpha \neq 0$, and by $\mathcal{T}_\beta$, $\beta \in \mathbb{R}$, the *dilation operator* respectively the *translation operator*. Recalling that $\mathcal{D}_\alpha f(x) = \sqrt{|\alpha|} f(\alpha x)$ and $\mathcal{T}_\beta f(x) = f(x+\beta)$ for every $f \in \mathbb{R}^{\mathbb{R}}$ and $x \in \mathbb{R}$, (2.3) can be written

$$\varphi_{k,j}(x) = \mathcal{D}_{a_k} \mathcal{T}_{b_j/a_k} \varphi(x), \quad x \in \mathbb{R}.$$

Based on these considerations and taking into account that *supp*φ is bounded, by a simple computation one obtains

Lemma 2.3. *Let φ be a cvasi-scaling type function. For every $f \in L_{1,loc}(\mathbb{R})$ the following identities*

$$(f, \varphi_{k,j}) = (\mathcal{D}_{a_{-k}} f, \varphi_{0,j}), \quad (k,j) \in \mathbb{Z} \times \mathbb{Z},$$

hold true.

3 The operators L_k

Let φ be a cvasi-scaling type function.

For every $f \in L_{1,loc}(\mathbb{R})$ and $k \in \mathbb{Z}$ we consider the operator

$$(L_k f)(x) := \sum_{j=-\infty}^{\infty} (f, \varphi_{k,j}) \varphi_{k,j}(x), \quad x \in \mathbb{R}, \tag{3.1}$$

where the functions $\varphi_{k,j}$ are given by (2.3), and $(\cdot,\cdot)$ is defined by (2.4).

Remark 3.1. *(i) Because of the function φ has bounded support, for any real x the summation in (3.1) involves only finite terms and consequently $(L_k f)(x)$ is well-defined on $\mathbb{R}$.*

(ii) The properties of the inner product imply that every L_k is a linear operator. Since $\varphi \geq 0$, relation (2.3) implies that L_k is also a positive operator.

(iii) In the particular case $a_k = 2^k$, $b_j = j$, $(k, j) \in \mathbb{Z} \times \mathbb{Z}$, operator L_k becomes operator A_k studied in [2], see also [1], Eq. (6.1). Now we have $\Gamma = \Delta = 1$.

In what follows we assume that the sequence $b = (b_j)_{j\in\mathbb{Z}} \in \widetilde{\mathcal{S}}_2$ verifies the condition

$$\text{a constant } \Delta \text{ exists such that } b_j - b_{j-1} = \Delta, \ j \in \mathbb{Z}. \tag{3.2}$$

The definition of the set $\widetilde{\mathcal{S}}_2$ implies that Δ is a positive constant and moreover, we have $b_j = j\Delta$, $j \in \mathbb{Z}$.

Lemma 3.2. *If the sequence $(b_j)_{j\in\mathbb{Z}} \in \widetilde{\mathcal{S}}_2$ satisfies (3.2) then one has*

$$\int_{\mathbb{R}} \varphi(x)dx = \int_{\mathbb{R}} \varphi(x + b_j)dx = \Gamma\Delta, \quad j \in \mathbb{Z}, \tag{3.3}$$

where φ is a cvasi-scaling type function.

Proof. Since $b_j = j\Delta$, $\Delta > 0$, we can write successively

$$\int_{\mathbb{R}} \varphi(x)dx = \sum_{j=-\infty}^{\infty} \int_{j\Delta}^{(j+1)\Delta} \varphi(x)dx = \sum_{j=-\infty}^{\infty} \int_0^{\Delta} \varphi(t + j\Delta)dt$$

$$= \int_0^{\Delta} \sum_{j=-\infty}^{\infty} \varphi(t + b_j)dt \overset{(2.1)}{=} \Gamma \int_0^{\Delta} dt = \Gamma\Delta.$$

The second identity is a direct consequence of the previous one. □

We shall present another look for L_k, $k \in \mathbb{Z}$, by using the dilation operator $\mathcal{D}_{a_{-k}}$ and the central operator L_0.

Theorem 3.3. *Let L_k, $k \in \mathbb{Z}$, be defined by (3.1). For every integer k and every function $f \in L_{1,loc}(\mathbb{R})$ one has*

$$L_k(f, x) = \sqrt{a_k} L_0(\mathcal{D}_{a_{-k}} f, a_k x), \quad x \in \mathbb{R}.$$

Proof. Since $a_0 = 1$, for $k = 0$ the statement is evident. For $k \in \mathbb{Z}^*$, by using Lemma 2.3 and formula (2.3), we get

$$(L_k f)(x) = \sum_{j=-\infty}^{\infty} (\mathcal{D}_{a_{-k}} f, \varphi_{0,j}) \sqrt{a_k} \varphi_{0,j}(a_k x), \quad x \in \mathbb{R},$$

and the proof of our theorem is complete. □

4 The main result

Now we estimate $|L_k f - \overline{c} f|$, where $\overline{c}$ is a certain constant and f belongs to the space $C(\mathbb{R}) \subset L_{1,loc}(\mathbb{R})$.

Theorem 4.1. *Let L_k, $k \in \mathbb{Z}$, be defined by (3.1) such that (3.2) is fulfilled. For every function $f \in C(\mathbb{R})$ the following inequality*

$$|(L_k f)(x) - \overline{c} f(x)| \le \overline{c} \omega_f(2\alpha a_{-k}), \quad k \in \mathbb{Z}, \ x \in \mathbb{R},$$

holds true, where constant $\overline{c}$ is defined by $\overline{c} = \overline{c}(\Gamma, \Delta) := \Gamma^2 \Delta$, α is given at (2.2) and ω_f represents the modulus of continuity associated to f.

Proof. At the first step we observe that relation (2.1) implies

$$\sum_{j=-\infty}^{\infty} \varphi(a_k x + b_j) = \Gamma, \quad x \in \mathbb{R}, \ k \in \mathbb{Z}. \tag{4.1}$$

Let $x \in \mathbb{R}$ and $k \in \mathbb{Z}$ be fix. By using both (3.1), (4.1) and (2.3) we can write

$$|(L_k f)(x) - \Gamma^2 \Delta f(x)| = \Big| \sum_{j=-\infty}^{\infty} (f, \varphi_{k,j}) \varphi_{k,j}(x) - \Gamma \Delta f(x) \sum_{j=-\infty}^{\infty} \varphi(a_k x + b_j) \Big|$$

$$\leq \sum_{j=-\infty}^{\infty} |\sqrt{a_k}(f,\varphi_{k,j}) - \Gamma\Delta f(x)|\varphi(a_k x + b_j)$$

$$= \sum_{j\in I_{x,k}} |\sqrt{a_k}(f,\varphi_{k,j}) - \Gamma\Delta f(x)|\varphi(a_k x + b_j), \tag{4.2}$$

where $I_{x,k} := \{j \in \mathbb{Z}|\ a_k x + b_j \in [-\alpha,\alpha]\}$ and α appears at (2.2).

Further on, with the help of (3.3) and (2.3) we obtain

$$|\sqrt{a_k}(f,\varphi_{k,j}) - \Gamma\Delta f(x)| = \left|a_k \int_{\mathbb{R}} f(u)\varphi(a_k u + b_j)du - \Gamma\Delta f(x)\right|$$

$$= \left|\int_{\mathbb{R}} f(a_k^{-1}t)\varphi(t+b_j)dt - f(x)\int_{\mathbb{R}} \varphi(t+b_j)dt\right|$$

$$\leq \int_{\mathbb{R}} |f(a_k^{-1}t) - f(x)|\varphi(t+b_j)dt$$

$$= \int_{-\alpha-b_j}^{\alpha-b_j} |f(a_k^{-1}t) - f(x)|\varphi(t+b_j)dt. \tag{4.3}$$

Since $f \in C(\mathbb{R})$, we have $|f(u) - f(v)| \leq \omega_f(|u-v|)$, $(u,v) \in \mathbb{R}\times\mathbb{R}$, and choosing $(u,v) := (a_k^{-1}t, x) \in [-(\alpha+b_j)a_k^{-1}, (\alpha-b_j)a_k^{-1}]^2$ we get

$$|f(a_k^{-1}t) - f(x)| \leq \omega_f(2\alpha a_k^{-1}).$$

Returning to (4.3) we have

$$|\sqrt{a_k}(f,\varphi_{k,j}) - \Gamma\Delta f(x)| \leq \omega_f(2\alpha a_k^{-1}) \int_{-\alpha-b_j}^{\alpha-b_j} \varphi(t+b_j)dt$$

$$= \omega_f(2\alpha a_k^{-1}) \int_{\mathbb{R}} \varphi(t+b_j)dt = \Delta\Gamma\omega_f(2\alpha a_k^{-1}),$$

and consequently, relation (4.2) combined with (4.1) implies

$$|(L_k f)(x) - \Gamma^2\Delta f(x)| \leq \Delta\Gamma\omega_f(2\alpha a_k^{-1}) \sum_{j\in I_{x,k}} \varphi(a_k x + b_j)$$

$$= \Delta\Gamma\omega_f(2\alpha a_k^{-1}) \sum_{j=-\infty}^{\infty} \varphi(a_k x + b_j) = \Delta\Gamma^2\omega_f(2\alpha a_{-k}).$$

□

From Theorem 4.1 we give the following

Corollary 4.2. *Let the operators L_k, $k \in \mathbb{Z}$, be defined by (3.1) so that (3.2) holds true. If $\lim_{k\to\infty} a_k = \infty$ then the sequence $(\bar{c}^{-1}L_k)_{k\geq 0}$ has the approximation property, in other words*

$$\lim_{k\to\infty} L_k f = \bar{c}f,$$

uniformly on any interval compact of the real axis, for every $f \in C(\mathbb{R})$.

Note. We must mention that at the celebration of Professor D. D. Stancu, we presented the paper entitled "Stancu polynomials revisited", written in February, 2002. Following the advice of Professor D. D. Stancu, that paper was submitted for publication in *Revue d'Analyse Numerique et de Theorie de l'Approximation*, Cluj-Napoca, and will appear in tome 31(2002), no.1.

The present note elaborated in August 2002 has not yet been presented at any meeting.

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