Mathematical Analysis and Approximation Theory

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THE MYSTERIOUS WAVELETS WORLD

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ABSTRACT. This survey paper contains the basic ideas of windowed Fourier transform, wavelet transforms, wavelet bases and multiresolution analysis, providing important information that introduces the reader at the forefront of current research.

1. PRELUDE

"Wavelets are without doubt an exciting and intuitive concept. The concept brings with it a new way of thinking, which is absolutely essential and was entirely missing in previously existing algorithms."

Yves Meyer

In the last two decades more than 100 books and monographs on the subject of wavelets have been issued. As a detail (see [9, *Preface*]): in the Wavelet Literature Survey edited by the Institute of Numerical Analysis and Applied Mathematics of Vienna University are quoted 976 papers on this topic, all of them having been written between 1985-1993. From 1993, the number of articles dedicated to wavelet transforms and their applications has increased yearly. A natural motivation consists in the fact that this subject is diverse and extensive, pioneering work being done by mathematicians, computer, mechanical and electrical engineers, physicists and also by experts in applied sciences such as geophysics and statistics. This way, different approaches appear in the presentation of the wavelet theory.

Roughly speaking, our aim is to go in the opposite direction, which means that by using the vast information we shall try to briefly present the fundamental aspects of the wavelet theory. In other words, we shall try to deliver to a mathematician who is novice in the wavelet world, sufficient information in order to get the picture of this universe generated by a lucky marriage between the results of the signal processing community and the results in multiresolution analysis. In order that we be able to present a large range of information in a concise way, we have avoided, as a rule, the proofs of the theorems. On the other hand, we have provided lots of examples so as to clarify the notions involved.

This friendly guide to wavelets is based on the short course *An Introduction to Wavelets* consisting in eight hours and given by the author at the Department of Mathematics of the University of Bari in the frame of the Socrates Programme, during the period of April 22^{nd} – May 5^{th}, 2002.

We hope it can motivate some readers to get involved with this research area and to convince them of the fact that wavelets are a tool rich mathematical content and great potential for varied applications.

2. Preliminaries

On some spaces of functions

Let a be a fixed real positive number. For every integer n we consider

$$e_n:\mathbb{R}\to\mathbb{C},\quad e_n(t)=e^{2n\pi it/a}=\cos\frac{2n\pi t}{a}+i\sin\frac{2n\pi t}{a}. \tag{2.1}$$

Clearly: $e_n(t+a)=e_n(t)$, $e_0(t)=1$, $\overline{e}_n=e_{-n}$ and $|e_n(t)|=1$ for every $(n,t)\in\mathbb{Z}\times\mathbb{R}$.
For every $N\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ we denote by $\mathcal{T}_N$ the complex linear space of trigonometric polynomials of degree no greater than N. If $p:\mathbb{R}\to\mathbb{C}$ is an element of $\mathcal{T}_N$ then $p(t)=\sum_n c_ne_n(t)$ where $c_n\in\mathbb{C}$ and $n=\overline{-N,N}$.
This space is endowed with the inner product defined for every p and q belonging to $\mathcal{T}_N$ as follows $(p,q):=\int_0^a p(t)\overline{q}(t)dt$. Consequently, it is also a normed space with the norm defined by $\|p\|=\sqrt{(p,p)}$, $p\in\mathcal{T}_N$.

If we set

$$\begin{cases} a_n:=c_n+c_{-n},\\ b_n:=i(c_n-c_{-n}),\end{cases}\quad\text{then}\quad\begin{cases} c_n=(a_n-ib_n)/2,\\ c_{-n}=(a_n+ib_n)/2,\end{cases} \tag{2.2}$$

where $n=\overline{0,N}$, and every $p\in\mathcal{T}_N$ admits the representations

$$p(t)=\sum_{n=-N}^{N}c_ne_n(t)=\frac{a_0}{2}+\sum_{n=1}^{N}a_n\cos\frac{2n\pi t}{a}+b_n\sin\frac{2n\pi t}{a},\quad t\in\mathbb{R}. \tag{2.3}$$

Remark 2.1. If e_n is defined by (2.1) then $(e_n)_{n\in\mathbb{Z}}$ is an orthogonal system. One has $(e_n,e_m)=a\delta_{n,m}$, $\delta_{n,m}$ - Kronecker's symbol.
Theorem 2.2. *For every $p\in\mathcal{T}_N$ defined by (2.3) the following relations hold true:*
(i) $c_n=a^{-1}(p,e_n)$, $n=\overline{-N,N}$;

(ii) Fourier formulas

$$a_n = \frac{2}{a}\int_0^a p(t)\cos\frac{2n\pi t}{a}dt, \quad b_n = \frac{2}{a}\int_0^a p(t)\sin\frac{2n\pi t}{a}dt, \quad n = \overline{1,N};$$

(iii) Parseval's identity

$$\sum_{n=-N}^{N} |c_n|^2 = \frac{1}{a}\int_0^a |p(t)|^2 dt. \tag{2.4}$$

Further on, we consider the space

$$L_P^2(0,a) := \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ is } a - \text{periodic and } \int_0^a |f(t)|^2 dt < \infty \right\},$$

endowed with the inner product defined by $(f,g) := \int_0^a f(t)\overline{g}(t)dt$ and with the norm $\|\cdot\|_2$, $\|f\|_2 := (f,f)^{1/2}$.
We recall: $\mathcal{T}_N \le L_P^2(0,a)$ and $\|f\|_2 = 0 \Leftrightarrow f = 0$ a.e. on $(0,a)$.
Theorem 2.3. *Let $f \in L_P^2(0,a)$. There exists a unique polynomial $f_N \in \mathcal{T}_N$ with the property $\|f - f_N\|_2 = \min_{p\in\mathcal{T}_N} \|f - p\|_2$ and it is given by*

$$f_N(t) = \sum_{n=-N}^{N} c_n e_n(t), \text{ where } c_n = \frac{1}{a}\int_0^a f(t)e_{-n}(t)dt. \tag{2.5}$$

One has $\|f - f_N\|_2^2 = \|f\|_2^2 - a\sum_{n=-N}^{N} |c_n|^2$ (*Bessel's equality*) and consequently, $\sum_{n=-N}^{N} |c_n|^2 \le \frac{\|f\|_2^2}{a}$, $(\forall)\ N \in \mathbb{N}_0$ (*Bessel's inequality*). The polynomial f_N is called the *orthogonal projection of f onto $\mathcal{T}_N$* or *element of the best approximation to f in $\mathcal{T}_N$*.
Theorem 2.4. *Let $f \in L_P^2(0,a)$ and $f_N \in \mathcal{T}_N$ defined by (2.5). The sequence $(f_N)_{N\ge 0}$ is strongly convergent to f, this meaning*

$$\lim_{N\to\infty} \|f - f_N\|_2 = \lim_{N\to\infty} \int_0^a |f(t) - f_N(t)|^2 dt = 0. \tag{2.6}$$

Remark 2.5. (i) The identity (2.6) guarantees that $f = \sum_{n=-\infty}^{\infty} c_n e_n$ almost everywhere on $\mathbb{R}$. This, in general, does not imply pointwise convergence!

(ii) If $f_N \in \mathcal{T}_N$ is the orthogonal projection of f onto $\mathcal{T}_N$ then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = \frac{1}{a}\int_0^a |f(t)|^2 dt,$$

where c_n and a_n, b_n are defined by (2.5) respectively (2.2).
We denote by l^2 the space of all square-summable bi-infinite sequences of complex numbers $x = (x_n)_{n\in\mathbb{Z}}$ - this meaning $\sum_{k\in\mathbb{Z}} |x_k|^2 < \infty$ - endowed with the inner product defined by $(x, y) = \sum_{k\in\mathbb{Z}} x_k \overline{y}_k$.
It is known that both l^2 and $L^2_P(0, a)$ are Hilbert spaces.
Theorem 2.6. *The operator* $\phi : L^2_P(0, a) \to l^2$, $\phi(f) = (\sqrt{a}c_n(f))_{n\in\mathbb{Z}}$ *where* $c_n(f) = a^{-1}(f, e_n)$, *verifies the following identity*

$$(\phi(f), \phi(g)) = (f, g),$$

for every f *and* g *belonging to* $L^2_P(0, a)$ *(*ϕ *is an isometric operator).*
We also consider the space

$$L^1_P(0, a) := \left\{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is } a - \text{periodic and } \int_0^a |f(t)|dt < \infty\right\},$$

endowed with the norm $\|\cdot\|_1$, $\|f\|_1 = \int_0^a |f(t)|dt$.
One has $L^2_P(0, a) \subset L^1_P(0, a)$ and $\|f\|_1 \le \sqrt{a}\|f\|_2$, $f \in L^2_P(0, a)$.

Fourier transform in $L^1(\mathbb{R})$

Definition 2.7. Let f belong to $L^1(\mathbb{R})$. The Fourier transform $\mathcal{F}f$ of f (also denoted by $\widehat{f}$) is defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} f(x)dx. \tag{2.7}$$

The conjugate Fourier transform $\overline{\mathcal{F}}f$ of f is defined by

$$\overline{\mathcal{F}}f(\xi) = \int_{\mathbb{R}} e^{2\pi i\xi x} f(x)dx. \tag{2.8}$$

Remark 2.8. (i) Often, the Fourier transform of f is given by

$$(Ff)(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx.$$

Clearly, $(Ff)(2\pi\xi) = (\mathcal{F}f)(\xi)$. We chose the relation (2.7) because of the operator $\mathcal{F}$ is an isometric operator on the space $L^2(\mathbb{R})$ as we will see later. However, this change do not alter the theory of Fourier transforms at all.

(ii) The Fourier transform $\widehat{f}$ of f not necessary belong to $L^1(\mathbb{R})$. For example we easily deduce

$$\text{(2.9)} \quad \text{if } f = \chi_{[a,b]} \text{ then } \widehat{\chi}_{[a,b]}(\xi) = \begin{cases} b-a, & \xi = 0, \\ \dfrac{\sin \pi\xi(b-a)}{\pi\xi} e^{-i\pi\xi(a+b)}, & \xi \neq 0, \end{cases}$$

and $\widehat{\chi}_{[a,b]} \notin L^1(\mathbb{R})$.
We notice that in science and engineering, the characteristic function $\chi_{[a,b]}$ is often called a *rectangular pulse* or *gate function.*
Further on, we gather the most significant properties of $\mathcal{F}f$ for $f \in L^1(\mathbb{R})$.
Theorem 2.9. *(1) If $f \in L^1(\mathbb{R})$ then $\mathcal{F}f$ is continuous and bounded on $\mathbb{R}$. $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ is a linear continuous operator and $\|\widehat{f}\|_\infty \leq \|f\|_1$. The identity $\lim_{|\xi|\to\infty} |\widehat{f}(\xi)| = 0$ holds true.*
(2) If f and g belong to $L^1(\mathbb{R})$ then $f\widehat{g}$, $\widehat{f}g$ belong to $L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} f(t)\widehat{g}(t)dt = \int_{\mathbb{R}} \widehat{f}(x)g(x)dx.$$

(3) If $m_k f \in L^1(\mathbb{R})$, $k = \overline{0,n}$, then $\widehat{f}^{(k)}(\xi) = \widehat{(-2\pi i x)^k f(x)}(\xi)$, $k = \overline{1,n}$. Here m_k indicates the monomial $m_k(x) = x^k$, $x \in \mathbb{R}$.
(4) If $f \in C^n(\mathbb{R}) \cap L^1(\mathbb{R})$ and $f^{(k)} \in L^1(\mathbb{R})$, $k = \overline{1,n}$, then $\widehat{f^{(k)}}(\xi) = (2\pi i\xi)^k \widehat{f}(\xi)$.
(5) If $f \in L^1(\mathbb{R})$ and $\operatorname{supp}(f)$ is bounded then $\widehat{f} \in C^\infty(\mathbb{R})$.
(6) If $f \in L^1(\mathbb{R})$ then $\overline{\mathcal{F}f} = \overline{\mathcal{F}}\,\overline{f}$ and $(\mathcal{F}f)_\sigma = \overline{\mathcal{F}}f = \mathcal{F}f_\sigma$. Here h_σ is defined by $h_\sigma(x) = h(-x)$, $x \in \mathbb{R}$.
(7) If $f \in L^1(\mathbb{R})$ is an odd (even) function then $\widehat{f}$ is odd (even) function.
(8) If $f \in L^1(\mathbb{R})$ then $\widehat{T_a f}(\xi) = e^{-2\pi i a\xi}\widehat{f}(\xi)$, $T_a\widehat{f}(\xi) = \widehat{e^{2\pi i x a} f(x)}(\xi)$. Here T_a indicates the shift operator ($T_a h(x) = h(x-a)$, $x \in \mathbb{R}$).
(9) Inverse Fourier transform.
If f and $\widehat{f}$ belong to $L^1(\mathbb{R})$ then $\overline{\mathcal{F}}\widehat{f}(t) = f(t)$ for every point t where f is continuous.
(10) If $f \in C^2(\mathbb{R})$ and $\{f, f', f''\} \subset L^1(\mathbb{R})$ then $\widehat{f} \in L^1(\mathbb{R})$.
(11) If $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$ then $\mathcal{F}\widehat{f}(x) = f_\sigma(x) = f(-x)$.
(12) If f, g belong to $L^1(\mathbb{R})$ then $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$, $\xi \in \mathbb{R}$.
(13) If $f, \widehat{f}, g, \widehat{g}$ belong to $L^1(\mathbb{R})$ then $\widehat{fg}(\xi) = \widehat{f} * \widehat{g}(\xi)$, $\xi \in \mathbb{R}$.
Remark 2.10. (i) Property (9) says: the function f can be recovered from $\widehat{f}$ by using the operator $\overline{\mathcal{F}}\mathcal{F}$ at every point x where f is continuous.

(ii) For a given $f \in L^p(\mathbb{R})$ we can consider

$$\Phi_f(x) := \sum_{k=-\infty}^{\infty} f(x + ak), \quad x \in \mathbb{R}, \tag{2.10}$$

where $a > 0$ is fixed. The first question: really, Φ_f is a function? For $p = 1$ the answer is positive and this can be read as follows:
If $f \in L^1(\mathbb{R})$ then the series defined by (2.10) converges to the a-periodic function Φ_f. One has $\Phi_f \in L^1_P(0, a)$ and $\|\Phi_f\|_{L^1_P(0,a)} \le \|f\|_1$.

Fourier transform in $L^2(\mathbb{R})$

Definition 2.11. The function $f : \mathbb{R} \to \mathbb{C}$ has (f.f.d.) property (function with fast diminution) if for every $p \in \mathbb{N}_0$, $\lim_{|x|\to\infty} |x^p f(x)| = 0$ holds true.

Theorem 2.12. *(1) If $f \in L^1_{loc}(\mathbb{R})$ has (f.f.d.) property then $m_p f \in L^1(\mathbb{R})$, for every $p \in \mathbb{N}_0$.*
(2) If $f \in L^1(\mathbb{R})$ has (f.f.d.) property then $\widehat{f} \in C^\infty(\mathbb{R})$.
(3) If $f \in C^\infty(\mathbb{R})$ and $f^{(k)} \in L^1(\mathbb{R})$ for every $k \in \mathbb{N}_0$ then $\widehat{f}$ has (f.f.p.) property.
We define

$$\mathcal{D}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \mid f \in C^\infty(\mathbb{R}) \text{ and } f^{(k)} \text{ has (f.f.d.) property, } (\forall) k \in \mathbb{N}_0\}.$$

Clearly, $\mathcal{D}(\mathbb{R}) \subset L^1(\mathbb{R})$ and for $f \in \mathcal{D}(\mathbb{R})$ one has $f' \in \mathcal{D}(\mathbb{R})$, $\widehat{f} \in \mathcal{D}(\mathbb{R})$, $qf \in \mathcal{D}(\mathbb{R})$, where q is an arbitrary polynomial.

Definition 2.13. The sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in \mathcal{D}(\mathbb{R})$, converges to 0 in the space $\mathcal{D}(\mathbb{R})$ if

$$(\forall)\ p \in \mathbb{N}_0,\ (\forall)\ q \in \mathbb{N},\ \lim_{n\to\infty} \sup_{x\in\mathbb{R}} |x^p f_n^{(q)}(x)| = 0.$$

We can prove: if $f_n \to 0$ in $\mathcal{D}(\mathbb{R})$ then one has

$$f_n \to 0 \text{ in } L^1(\mathbb{R}), \quad f_n' \to 0 \text{ in } \mathcal{D}(\mathbb{R}), \quad \widehat{f_n} \to 0 \text{ in } \mathcal{D}(\mathbb{R}).$$

Theorem 2.14. *The Fourier transform $\mathcal{F}$ is a linear continuous bijective operator mapping $\mathcal{D}(\mathbb{R})$ into $\mathcal{D}(\mathbb{R})$. Its inverse transform is $\overline{\mathcal{F}}$, consequently the relations*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x)dx, \quad f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \widehat{f}(\xi)d\xi$$

are equivalent for every f belonging to $\mathcal{D}(\mathbb{R})$.

Theorem 2.15. *$\mathcal{D}(\mathbb{R})$ is a linear subspace of $L^2(\mathbb{R})$ dense in $L^2(\mathbb{R})$.*

Theorem 2.16. (Plancherel-Parseval' identities) *If f and g belong to $\mathcal{D}(\mathbb{R})$ then the following relations hold*

$$\int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}}(\xi)d\xi = \int_{\mathbb{R}} f(x)\overline{g}(x)dx, \quad \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |f(x)|^2 dx.$$

Theorem 2.17. *The Fourier transform $\mathcal{F}$ (respectively $\overline{\mathcal{F}}$) is one continuous extendable to $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. Keeping the same notations for the extensions, the following properties hold:*
(1) $\mathcal{F}\overline{\mathcal{F}}f = \overline{\mathcal{F}}\mathcal{F}f = f$, $\mathcal{F}\mathcal{F}f = f_\sigma$,
(2) $\int_{\mathbb{R}} f(x)\overline{g}(x)dx = \int_{\mathbb{R}} \mathcal{F}f(\xi)\overline{\mathcal{F}g}(\xi)d\xi$, $\|f\|_2 = \|\mathcal{F}f\|_2$,
(3) $\int_{\mathbb{R}} \mathcal{F}f(t)g(t)dt = \int_{\mathbb{R}} f(u)\mathcal{F}g(u)du$,
for every function f and g belonging to $L^2(\mathbb{R})$.

3. The first brick in the wall: WFT

At the beginning we briefly discuss about signals presenting some aspects regarding their classification.
From the mathematical point of view a signal is a function of time. We can express a signal $f(t)$ in terms of its amplitude and phase as follows: $f(t) = a(t)\exp(i\theta(t))$.
A. If the variable belongs to an interval I then the signal $x = x(t)$, t in I, is called *analogical signal*. Usually, this type of signal is a continuous function of time t, with the exception of perhaps a countable number of jump continuities. If the variable is discrete then the function $x = (x_n)_{n\in\mathbb{Z}}$ is called *discrete* (or *digital*) *signal*. Usually, it is obtained by discretization of an analogical signal.
Examining Theorem 2.6 we deduce

- *Every analogical signal $f \in L^2_P(0,a)$ can be identified with the digital signal $(\sqrt{a}c_n)_{n\in\mathbb{Z}} \in l^2$, where $c_n = a^{-1}(f, e_n)$.*

B. Also we can speak about *deterministic* and *random* (or *stochastic*) signals. A signal is called deterministic if it can be determined explicitly, in terms of a mathematical relationship. A deterministic signal is referred to as *periodic* or *transient* if the signal repeats continuously at regular intervals of time respectively decays to zero after a finite time interval.
Practically, in nature there are random or stochastic signals in the sense that they cannot be determined precisely at any given instant of time. Probabilistic and statistical information is required for description of random signals. It is necessary to consider a particular random process that

can produce a set of time-histories, known as an *ensemble*. This can represent an experiment producing random datum which is repeated n times to give an ensemble of n separate records. The *average value* at time t over the ensemble x is defined by $\langle x(t)\rangle = \lim\limits_{n\to\infty} n^{-1}\sum_{k=1}^{n} x_k(t)$, where x takes any one of a set of values x_k, $k=\overline{1,n}$.
The average value of the product of two samples taken at two separate times t_1 and t_2 is called the *autocorrelation function* R, for each separate record, defined by $R(\tau) = \lim\limits_{n\to\infty} n^{-1}\sum_{k=1}^{n} x_k(t_1)x_k(t_2)$, $\tau = t_1 - t_2$.
C. On the other hand a signal can be *stationary* or *non-stationary*. In the first case the properties of the signal are invariant over time. The ideal tool for studying this type is the Fourier transform. In other words, stationary signals decompose canonically into linear combinations of waves (sines and cosines). A signal is non-stationary if the values of $\langle x(t)\rangle$ and $R(\tau)$ vary with time, even if the change in time is very slow.
Let $f \in L^1(\mathbb{R})$ be a signal. Physically, the Fourier integral (2.7) measures oscillations of f at the *frequency* $\omega = 2\pi\xi$. The frequency is measured by ξ in terms of Hertz. Also, $\widehat{f}(\omega/2\pi)$ is called the *frequency spectrum* of a signal or waveform $f(t)$. It seems equally justified to refer to $f(t)$ as the waveform in the *time domain* and $\widehat{f}(\omega/2\pi)$ as the waveform in the *frequency domain*. The continuous Fourier transform is not satisfactory for many applications. It is appropriate only for stationary signals. For any signal $f(t)$ the transform $\widehat{f}(\omega)$ gives information on the frequency content over the entire signal for the frequency ω. To extract frequency information at even a single ω, it requires an infinite amount of time, $(-\infty < t < \infty)$, using both past and future information of the signal. Another trouble of standard Fourier method is that it is quite inadequate for dealing with signals whose frequency content changes over time. The formula for $\widehat{f}(\omega)$ does not even reflect frequencies that envolve with time. So, the Fourier transform analysis cannot provide any information regarding either a time evolution of spectral characteristics or a possible localization with respect to the time variable. Transient signals require the idea of frequency analysis that is local in time.
In what follows we present an attempt to correct these deficiencies. This approach is due to Dennis Gabor (1946), a physicist and engineer who won the 1971 Nobel Prize in physics. He introduced the *windowed Fourier transform* (WFT) to measure localized frequency components of sound waves. Generally speaking, this major idea was to use a time-localization *window function*, say $g_a(t-b)$, for extracting local information from the Fourier transform of a signal, where the parameter a measures the width of the window and the parameter b is used to translate the window in order to

cover the whole time domain. This way we use a window function in order to localize the Fourier transform, then shift the window to another position, and so on.

In this direction an elementary idea is the following: if the signal $f(t)$ is given, we distort it over a bounded interval, say $[t_1, t_2]$. Practically, we multiply f by the function $\chi_{[t_1,t_2]}$ and then we apply $\mathcal{F}$. The initial spectrum $\widehat{f}(\lambda)$ is substituted by $\widehat{\chi_{[t_1,t_2]} f}(\lambda) = \widehat{\chi}_{[t_1,t_2]} * \widehat{f}(\lambda)$, see Theorem 2.9, property (13); $\widehat{\chi}_{[t_1,t_2]}$ is given at (2.9). Choosing the interval $[-A, A]$ one has

$$s_A(\lambda) := \widehat{\chi}_{[-A,A]}(\lambda) = \begin{cases} 2A, & \lambda = 0, \\ \dfrac{\sin 2\pi A\lambda}{\pi\lambda}, & \lambda \neq 0, \end{cases} \tag{3.1}$$

where s_A is called the *Shannon sampling function.*

Definition 3.1. A non trivial function $w \in L^2(\mathbb{R})$ is called a window function if $m_1 w \in L^2(\mathbb{R})$. The center t^* and radius Δ_w of a window function w are defined to be

$$t^* := \frac{1}{\|w\|_2^2} \int_{\mathbb{R}} t|w(t)|^2 dt,$$

and

$$\Delta_w := \frac{1}{\|w\|_2} \Big\{ \int_{\mathbb{R}} (t - t^*)^2 |w(t)|^2 dt \Big\}^{1/2},$$

respectively. The width of the window function w is defined by $2\Delta_w$.

Examples 3.2. Let $A > 0$.

(i) The rectangular time-window $\chi_{[-A,A]}$ and frequency-window s_A given by (3.1).

(ii) The triangle time-frequency window

$$w(t) = \begin{cases} \dfrac{t}{A} + 1, & t \in [-A, 0], \\ -\dfrac{t}{A} + 1, & t \in (0, A], \\ 0, & t \in \mathbb{R} \setminus [-A, A], \end{cases} \qquad \widehat{w}(\lambda) = \begin{cases} A, & \lambda = 0, \\ \dfrac{1}{A}\left(\dfrac{\sin \pi A\lambda}{\pi\lambda}\right)^2, & \lambda \neq 0 \end{cases}$$

(iii) The Gaussian time-frequency window with parameter $\alpha > 0$

$$w_\alpha(t) = Ae^{-\alpha t^2}, \quad \widehat{w}_\alpha(\lambda) = A\sqrt{\frac{\pi}{\alpha}} e^{-\pi^2 \alpha^{-1} \lambda^2}.$$

Since $\int_{\mathbb{R}} e^{-\beta t^2} dt = \sqrt{\pi/\beta}$, $\beta > 0$, for w_α window we obtain

$$\|w_\alpha\|_2^2 = A^2 \sqrt{\pi/(2\alpha)}, \quad t^* = 0, \quad \Delta_{w_\alpha} = 1/(2\sqrt{\alpha}).$$

Under the normalized condition $\int_{\mathbb{R}} w_\alpha(t)dt = 1$ (in other words, $\widehat{w}_\alpha(0) = 1$) we get $A = \sqrt{\alpha/\pi}$.

Definition 3.3. Let $w \in L^2(\mathbb{R})$ be a window function.
The operator $W : L^2(\mathbb{R}) \to \mathbb{C}^{\mathbb{R}\times\mathbb{R}}$, $f \mapsto W_f$, where

$$W_f(\lambda, b) = \int_{\mathbb{R}} f(t)\overline{w}(t-b)e^{-2\pi i\lambda t}dt, \tag{3.2}$$

is called *the windowed Fourier transform* (WFT) or *the continuous Gabor transform.*

Clearly, the operator W is linear. If for every pair $(\lambda, b) \in \mathbb{R}\times\mathbb{R}$ we consider the functions $w_{\lambda,b}$,

$$w_{\lambda,b}(t) = w(t-b)e^{2\pi i\lambda t}, \quad t \in \mathbb{R}, \tag{3.3}$$

then (3.2) can be written $W_f(\lambda, b) = (f, w_{\lambda,b})$.

Theorem 3.4. (*Gabor formulas*). *Let $w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a window such that $|\widehat{w}|$ is an even function and $\|w\|_2 = 1$. Let $w_{\lambda,b}$, $(\lambda, b) \in \mathbb{R}\times\mathbb{R}$, be defined by (3.3). For every $f \in L^2(\mathbb{R})$ we consider the coefficients $W_f(\lambda, b)$ defined by (3.2). The following relations hold true.*

(1) $\iint\limits_{\mathbb{R}\times\mathbb{R}} |W_f(\lambda, b)|^2 d\lambda db = \|f\|_2^2$ *(conservation of energy).*

(2) $f(t) = \iint\limits_{\mathbb{R}\times\mathbb{R}} W_f(\lambda, b)w_{\lambda,b}(t)d\lambda db$ *(reconstruction formula).*

This identity will be read as follows

$$\text{if } g_A(t) := \iint\limits_{\substack{|\lambda|\le A\\ b\in\mathbb{R}}} W_f(\lambda, b)w_{\lambda,b}(t)d\lambda db, \text{ then } \lim_{A\to\infty} \|g_A - f\|_2 = 0.$$

In time, various other functions have been used as window functions instead of the Gaussian function that was originally introduced by Gabor.

4. Swimming on wavelets

In order to present the definition of *wavelets* we can follow three different trends created respectively by Franklin-Strömberg, Grossmann-Morlet, Littlewood-Paley-Stein.

In [7; *Chapter 2*] is done a well documented presentation of wavelets from a historical perspective. Also, the historical tree of wavelet theory appears in [1; *page 2*]. Further on, we follow the Franklin-Strömberg direction. In this case the analysis of a signal f is obtained by restricting the Littlewood-Paley analysis to the set L in $(0, \infty)\times\mathbb{R}$ consisting of the points $(2^{-j}, k2^{-j})$, $j, k \in \mathbb{Z}$.

According to (2.6) and (2.5) any $f \in L^2_P(0, 2\pi)$ has a Fourier series representation

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e_n(t), \quad c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e_{-n}(x) dx. \tag{4.1}$$

There are two distinct features in the above Fourier series.
(i) The signal f is decomposed into a sum of infinitely mutually orthogonal components, that is $(c_n e_n, c_m e_m) = 0$ for every $n \neq m$. Moreover, Remark 2.1 guarantees that $(e_n/\sqrt{2\pi})_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(0, 2\pi)$.
(ii) The orthonormal basis $(e_n/\sqrt{2\pi})_{n\in\mathbb{Z}}$ is generated by *dilation* of a single function $w(t) = e^{it}$. Indeed, $w(2\pi n t) = e_n(t)$, for all integers n.
Briefly, we say:

- *every 2π-periodic square-integrable function is generated by integral dilations of the basic function w, which is a sinusoidal wave.*

For any large integer $|n|$, the wave e_n has high frequency and for $|n|$ with small value, the wave e_n has low frequency. Consequently, an arbitrary signal $f \in L^2(0, 2\pi)$ is composed of waves with various frequencies.
Now, we are going to examine the space $L^2(\mathbb{R})$. Practically we are looking for waves that generate $L^2(\mathbb{R})$. We notice that $w \notin L^2(\mathbb{R})$ and any element of this space must "decay" to zero at $\pm\infty$. Consequently, we look for small waves, or *wavelets*, to generate $L^2(\mathbb{R})$.
As a single function w generates the entire space $L^2(0, 2\pi)$, we try to obtain a single function, say ψ, to generate the entire space $L^2(\mathbb{R})$. At this moment a problem appears: if the wave ψ has very fast decay, how can it cover the whole real line? The solution is: the wave ψ must shift along $\mathbb{R}$. We consider all the integral shifts of ψ, namely $\psi(x-k)$, $k \in \mathbb{Z}$. We need waves with various frequencies partitioned in *frequency bands*. For computational reasons, we will use integral powers of 2 for frequency partitioning.

- *We consider the wavelets $x \mapsto \psi(2^j x - k)$, $(j,k) \in \mathbb{Z} \times \mathbb{Z}$, obtained from a single wave ψ by binary dilation (2^j, $j \in \mathbb{Z}$) and by dyadic translation ($k/2^j$, $k \in \mathbb{Z}$).*

The function ψ is called *mother wavelet*. This mother "gives birth" to an infinity of small waves by two operations - dilations and translations. Every new-born child will be called by using (j,k)-index:

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \quad x \in \mathbb{R}, \tag{4.2}$$

j denoting the *dilation index* and k representing the *translation index*. Dilation by larger j compresses the function on the x-axis. Altering k has the effect of sliding the function along the x-axis.

Lemma 4.1. *(i) If $f \in L^2(\mathbb{R})$ then for every $(j,k) \in \mathbb{Z} \times \mathbb{Z}$ one has*

$$\|f(2^j \cdot -k)\|_2 = 2^{-j/2}\|f\|_2.$$

(ii) If $\psi \in L^2(\mathbb{R})$ has the property $\|\psi\|_2 = 1$ then the functions $\psi_{j,k}$ defined by (4.2) have the same property.

Definition 4.2 (in the sense of Franklin and Strömberg). A function ψ belonging to $L^2(\mathbb{R})$ is called orthogonal wavelet if the functions $\psi_{j,k}$, (j,k) in $\mathbb{Z} \times \mathbb{Z}$, defined at (4.2) form an orthonormal basis of $L^2(\mathbb{R})$.

Remark 4.3. Let ψ be an orthogonal wavelet. For all integers j, k, l, m, one has

$$(\psi_{j,k}, \psi_{l,m}) = \int_{\mathbb{R}} \psi_{j,k}(x)\overline{\psi_{l,m}}(x)dx = \delta_{j,l}\delta_{k,m}.$$

Every signal $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} c_{j,k}\psi_{j,k}(x), \text{ where } c_{j,k} = (f, \psi_{j,k}). \tag{4.3}$$

This series is called *wavelet series* and $c_{j,k}$ represent the *wavelets coefficients.*

We recall that the convergence of the series in (4.3) is in $L^2(\mathbb{R})$, meaning

$$\lim_{M_1,M_2,N_1,N_2\to\infty} \left\| f - \sum_{j=-M_2}^{N_2} \sum_{k=-M_1}^{N_1} c_{j,k}\psi_{j,k} \right\|_2 = 0.$$

All these above statements are implied by the notion of orthonormal basis in the Hilbert space $L^2(\mathbb{R})$.

An example: the Haar system

The Haar function H is a piecewise constant transform given by

$$H(x) := \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x) = \begin{cases} 1, & 0 \le x < 1/2, \\ -1, & 1/2 \le x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Clearly, $\int_{\mathbb{R}} H(x)dx = 0$ and $\|H\|_2 = 1$.

This function is the embryo of the solution of the following problem formulated by Haar (1910): does there exist an orthonormal system $h_0, h_1, h_2, \ldots$ of functions defined on $[0,1]$ such that for every $f \in C([0,1])$, the series $\sum_{j\ge0}(f, h_j)h_j$ converges to f uniformly on $[0,1]$?

For $n \ge 1$ Haar wrote $n = 2^j + k$, $j \ge 0$, $0 \le k < 2^j$, and defined $h_n(x) = 2^{j/2}H(2^j x - k)$. Clearly, $int(\text{supp}h_n) = I_n := (k2^{-j}, (k+1)2^{-j})$, which is included in $[0,1)$ for $0 \le k < 2^j$. To complete the set, define $h_0(x) = 1$ on $[0,1)$. Then $(h_n)_{n\ge0}$ is an orthonormal basis for $L^2([0,1])$.

The uniform approximation of f by $\sum_{k=0}^{n}(f,h_k)h_k$ is nothing more than the classical approximation of a continuous function by step functions whose values are the mean values of $f(x)$ on the appropriate dyadic intervals. This approximation can be criticized that the "atoms" h_n used to construct the continuous function f are not themselves continuous functions.

Theorem 4.4. *The Haar function defined by (4.4) is an othogonal wavelet in sense of Definition 4.2.*

Proof. Practically it is necessary to show two things: the set $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ defined by (4.2) with $\psi \equiv H$ is orthonormal and any function $f \in L^2(\mathbb{R})$ can be approximated arbitrarily well by a finite linear combination of the $\psi_{j,k}$'s.
Taking into account that supp$\psi_{j,k} = [k2^{-j},(k+1)2^{-j}]$ the first statement is not difficult to be shown. A minute proof can be found, for example, in [8; *Theorem 1.1*].
The proof technique of the second statement leads us to a discussion of the principles of wavelet analysis and in what follows we shall present it. Since $\lim_{j_1\to\infty}\int_{-2^{j_1}}^{2^{j_1}} f^2(x)dx = \int_{\mathbb{R}} f^2(x)dx$, we can approximate f arbitrarily well in the L^2 sense by choosing a large integer j_1. Thus $f|_{[-2^{j_1},2^{j_1})}$ represents the first approximation of f and this restriction is further approximated by a piecewise constant function over all small intervals of the form $[l2^{-j_0},(l+1)2^{-j_0}) := I_{j_0,l}$; the integer j_0 is chosen to be large enough to make the approximation as good as desired. Our problem comes to approximating a function with bounded support with the help of piecewise constant functions having bounded supports.
Let f^{j_0} be a function such that it is piecewise constant on intervals of length 2^{-j_0} as described above and it has supp$f^{j_0} = [-2^{j_1},2^{j_1}]$. We denote by $f_l^{j_0}$ the constant value of the function f^{j_0} on the interval $I_{j_0,l}$. At this moment we decompose f^{j_0} as the sum of two functions

$$f^{j_0} = f^{j_0-1} + g^{j_0-1}, \tag{4.5}$$

where f^{j_0-1} is an approximation to f^{j_0} that is piecewise constant over intervals of length $2^{-(j_0-1)}$, twice as large as before. According to our convention, we put $f_l^{j_0-1} := f^{j_0-1}(x)$, $x \in I_{j_0-1,l}$, and the value $f_l^{j_0-1}$ is obtained by averaging the two corresponding constant values of the function f^{j_0}, more precisely $f_l^{j_0-1} = (f_{2l}^{j_0} + f_{2l+1}^{j_0})/2$.
By using (4.5) we can define a detail function g^{j_0-1} which is piecewise constant over the same intervals as those for f^{j_0}. It follows that

$$g_{2l}^{j_0-1} = f_{2l}^{j_0} - f_l^{j_0-1} = f_{2l}^{j_0} - \frac{1}{2}(f_{2l}^{j_0} + f_{2l+1}^{j_0}),$$

$$g_{2l+1}^{j_0-1} = f_{2l+1}^{j_0} - f_l^{j_0-1} = \frac{1}{2}(f_{2l+1}^{j_0} - f_{2l}^{j_0}) = -g_{2l}^{j_0-1}.$$

Using the Haar function (4.4) we can get the expression for the detail function g^{j_0-1} in terms of dilated and translated Haar wavelets ($H \equiv \psi$):

$$g_{j_0-1}(x) = \sum_{l=-2^{j_1+j_0-1}+1}^{2^{j_1+j_0-1}} g_{2l}^{j_0-1}\psi(2^{j_0-1}x - l).$$

Consequently, we deduce $f^{j_0} = f^{j_0-1} + \sum_l c_{j_0-1,l}\psi_{j_0-1,l}$, where

$$c_{j,k} = (f^{j_0}, \psi_{j,k}) = 2^{j/2}\int_{k2^j}^{(k+1)2^j} f^{j_0}(x)\psi(2^j x - k)dx. \tag{4.6}$$

The approximation function f^{j_0-1} can be decomposed again, giving

$$f^{j_0} = f^{j_0-1} + g^{j_0-1} = (f^{j_0-2} + g^{j_0-2}) + g^{j_0-1}.$$

The new function f^{j_0-2} has the properties: $\mathrm{supp} f^{j_0-2} = \mathrm{supp} f^{j_0}$ and it is piecewise constant over the intervals $I_{j_0-2,l}$. Using the wavelet representation of g^{j_0-2}, one has

$$f^{j_0} = f^{j_0-2} + \sum_l c_{j_0-2,l}\psi_{j_0-2,l} + \sum_l c_{j_0-1,l}\psi_{j_0-1,l},$$

where the coefficients $c_{j,k}$ are given at (4.6).

Repeating the procedure, we get $f^{j_0} = f^{-j_1} + \sum_{j=-j_1}^{j_0-1}\sum_l c_{j,k}\psi_{j,k}$. The coarser approximation f^{-j_1} has two constant pieces: $f^{-j_1}|_{[0,2^{j_1})} = f_0^{-j_1}$ and $f^{-j_1}|_{[-2^{j_1},0)} = f_{-1}^{-j_1}$. Now the entire support of f^{j_0} has been represented. Further on we double the support of the approximation to f^{j_0}. from -2^{j_1+1} to 2^{j_1+1}. The function f^{-j_1} can be broken down: $f^{-j_1} = f^{-(j_1+1)} + g^{-(j_1+1)}$, where

$$f^{-(j_1+1)}|_{[0,2^{j_1+1})} = 2^{-1}f_0^{-j_1}, \quad f^{-(j_1+1)}|_{[-2^{j_1+1},0)} = 2^{-1}f_{-1}^{-j_1} \quad \text{and}$$

$$g^{-(j_1+1)} = 2^{-1}f_0^{-j_1}\psi(2^{-(j_1+1)}x) - 2^{-1}f_{-1}^{-j_1}\psi(2^{-(j_1+1)}x + 1).$$

After k steps one obtains $f^{j_0} = f^{-(j_1+k)} + \sum_{j=-(j_1+k)}^{j_0-1}\sum_k c_{j,k}\psi_{j,k}$, where

$$\mathrm{supp} f^{-(j_1+k)} = [-2^{j_1+k}, 2^{j_1+k}], \quad f^{-(j_1+k)}|_{[0,2^{j_1+k})} = 2^{-k}f_0^{-j_1}$$

and $f^{-(j_1+k)}|_{[-2^{j_1+k},0)} = 2^{-k}f_{-1}^{-j_1}$.

Using only the sequence of detail functions to approximate f^{j_0}, we can obtain the error of approximation

$$\left\| f^{j_0} - \sum_{j=-(j_1+k)}^{j_0-1}\sum_k c_{j,k}\psi_{j,k}\right\|_2 = \|f^{-(j_1+k)}\|_2 = \sqrt{2}^{j_1-k}\sqrt{|f_0^{-j_1}|^2 + |f_{-1}^{-j_1}|^2}.$$

By choosing k large enough, we can make the error as small as we wish. □

5. Multiresolution analysis

We need *multi*-resolution because the resolution (i.e. the details of the function that we can see) will be governed by the frequencies, i.e. by our dilations though the integer parameter n. For each resolution we have a space of basis functions obtained by translation of a basic function obtained with a fixed parameter k. Consequently we work with several spaces at a different resolution, this meaning *multiresolution*.
Briefly, this concept is related to

- *the study of signals of different levels of resolution as a limit of successive approximations, each of them being a finer version of* f.

MRA and TSR

The proof of Theorem 4.4 is based on the decomposition of a piecewise constant approximation function into a *coarser approximation* and a *detail function*. Using the Haar function and the corresponding piecewise constant approximations, for each level j, one can construct f^j, an approximation of the original signal f.

- *The approximation can be written as the sum of the next coarser approximation* f^{j-1} *and a detail function, say* g^{j-1}. (5.1)

Further on each detail function g^j can be written as a linear combination of the corresponding $\psi_{j,k}$-functions ($\psi_{j,k}(x) = 2^{j/2}H(2^j x - k)$).
At this point, for each $j \in \mathbb{Z}$ we define a function space V_j,

$$V_j := \{f \in L^2(\mathbb{R}) |\ f \text{ is piecewise constant on } [k2^{-j}, (k+1)2^{-j}),\ k \in \mathbb{Z}\}.$$

The sequence $(V_j)_{j\in\mathbb{Z}}$ represents a ladder of subspaces of increasing resolution, as j increases. Each subspace V_j consists of functions that are piecewise constant over intervals of exactly twice the length of those for V_{j-1}. The above function sequence enjoys the following properties:

$$(P_1) \quad \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots; \tag{5.2}$$

$$(P_2) \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\},\ \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}); \tag{5.3}$$

$$(P_3) \quad f \in V_j \text{ if and only if } f(2\cdot) \in V_{j+1},\ j \in \mathbb{Z}; \tag{5.4}$$

$$(P_4) \quad f \in V_0 \text{ implies } f(\cdot - k) \in V_0, \text{ for all } k \in \mathbb{Z}. \tag{5.5}$$

In other words: (P_1) indicates that we deal with an ascensional sequence; (P_2) contains both the separable property and the density property; (P_3) demonstrates that each V_j is a scaled version of the original space V_0; (P_4) means the invariance of V_0 of the integer shifts. By using the shift operator T_a we can rewrite (P_4): $f \in V_0$ implies $T_k f \in V_0$, for all $k \in \mathbb{Z}$. If the scale $j = 0$ is associated with V_0, then the scale 2^{-j} is associated with V_j.

Setting $P^j f$ for the *projection* of a function f onto the space V_j (or the *best approximation* to f in V_j), relation (5.1) implies

$$P^j f = P^{j-1} f + g^{j-1}.$$

The detail function g^{j-1} is the "residual" between two approximations and it can be written in terms of dilated and translated wavelets as follows

$$P^j f = P^{j-1} f + \sum_{k\in\mathbb{Z}} (f, \psi_{j-1,k})\psi_{j-1,k}. \tag{5.6}$$

The decomposition can be extended recursively

$$P^j f = P^{j_0} f + \sum_{l=j_0}^{j-1} g^l = P^{j_0} f + \sum_{l=j_0}^{j-1} \sum_{k\in\mathbb{Z}} (f, \psi_{l,k})\psi_{l,k}.$$

Now, regarding to the sequence $(V_j)_{j\in\mathbb{Z}}$ we ask the following property to be fulfilled.

(P_5) There exists a function $\phi \in V_0$ such that the set $\{\phi_{0,k} : \ k \in \mathbb{Z}\}$, $\phi_{0,k} = \phi(\cdot - k)$, constitutes an orthonormal basis for V_0, that is

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx = \sum_{k\in\mathbb{Z}} |(f, \phi_{0,k})|^2 \text{ for all } f \in V_0. \tag{5.7}$$

For example, in the Haar case one choice for ϕ is $\phi(x) = \chi_{[0,1)}(x)$, $x \in \mathbb{R}$.

Remark 5.1. The function ϕ which satisfies (5.7) is called the *scaling function* (or the *father wavelet*) since its dilates and translates constitute orthonormal bases for all V_j subspaces, which are scaled version of V_0.

If the central space V_0 is generated by a single function $\phi \in L^2(\mathbb{R})$ in the sense that $V_0 = \overline{sp\{\phi_{0,k} : k \in \mathbb{Z}\}}$, then all the subspaces V_j are also generated by the same ϕ, namely

$$V_j = \overline{sp\{\phi_{j,k} : k \in \mathbb{Z}\}},\ j \in \mathbb{Z},\ \text{where } \phi_{j,k}(x) := 2^{j/2}\phi(2^j x - k).$$

Definition 5.2. A *multiresolution analysis* (MRA) *generated by the scaling function* ϕ consists of a sequence $(V_n)_{n\in\mathbb{Z}}$ of embedded closed subspaces of $L^2(\mathbb{R})$ that satisfy the following conditions: (5.2), (5.3), (5.4), (5.5), (5.7).

We have seen that a basis for V_0 is given by translates of the father function ϕ. The reciprocal of the translation distance is called the *resolution* of this basis. One could say that

- *the resolution gives the number of basis functions per unit length.*

Let us set by definition the resolution of V_0 to be 1. The projection $P^0 f$ gives an approximation of f at resolution 1. The projection $P^j f$ of f onto V_j gives the approximation of f at resolution 2^j.

Remark 5.3. (i) Sometimes, requirement (P_5) is relaxed by assuming that $\{\phi_{0,k} = \phi(\cdot - k), k \in \mathbb{Z}\}$ is a *Riesz basis* for V_0, that is, for every $f \in V_0$, there exists a unique sequence $(c_k)_{k\in\mathbb{Z}} \in l^2$ such that

$$\lim_{N\to\infty} \left\| f - \sum_{k=-N}^{N} c_k \phi_{0,k} \right\|_2 = 0$$

and there exist two real positive constants A and B (independent of f) such that

$$A \sum_{k\in\mathbb{Z}} |c_k|^2 \leq \|f\|_2^2 \leq B \sum_{k\in\mathbb{Z}} |c_k|^2.$$

In this case we have a MRA with a Riesz basis.
Clearly, (5.7) implies that $(\phi(\cdot - k))_{k\in\mathbb{Z}}$ is a Riesz basis for V_0 with $A = B = 1$.
(ii) Typical examples of scaling functions ϕ are the m^{th} order cardinal B-splines N_m, $m \in \mathbb{N}$, defined recursively by convolution: $N_1 := \chi_{[0,1)}$ and

$$N_m(x) := \int_{\mathbb{R}} N_{m-1}(x-t)N_1(t)dt = \int_0^1 N_{m-1}(x-t)dt, \ m \geq 2.$$

One has supp$N_m = [0, m]$ and $N_m(x) > 0$, for $0 < x < m$. Setting

$$\mathcal{B}_m := \{N_m(\cdot - k) : k \in \mathbb{Z}\} \text{ and } V_0^m := \overline{sp(\mathcal{B}_m)},$$

with an additional effort we can prove that $\mathcal{B}_m$ is a Riesz basis in V_0^m.
Theorem 5.4. *Let $(V_j)_{j\in\mathbb{Z}}$ be a MRA in the sense of Definition 5.2.*
If $f \in L^2(\mathbb{R})$ and $P^j f$ is the projection of f onto V_j then

$$\lim_{j\to\infty} \|P^j f - f\|_2 = 0 \ \text{and} \lim_{j\to-\infty} \|P^j f\|_2 = 0.$$

Theorem 5.5. *If ϕ generates a MRA on $L^2(\mathbb{R})$ then one has*

$$\phi(x) = \sum_{n=-\infty}^{\infty} p_n \phi(2x - n), \quad x \in \mathbb{R}, \tag{5.8}$$

where $(p_n)_{n\in\mathbb{Z}}$ belongs to the space l^2.
Equation (5.8) is called the *dilation equation.* It involves both x and $2x$ and is often referred to as the *two-scale relation* (TSR). A third name used for (5.8) is the *refinement equation* because it displays $\phi(x)$ in the defined space V_1. This space has the finer scale 2^{-1} and it contains $\phi(x)$ which has scale 1. To avoid trivialities we look for a solution of (5.8) with $\int_{\mathbb{R}} \phi(x)dx \neq 0$. Moreover, suppose we normalize ϕ so that

$$\int_{\mathbb{R}} \phi(x)dx = 1. \tag{5.9}$$

On the father wavelet

We investigate (TSR) in order to reveal properties of the father wavelet.

Theorem 5.6. *Let $(V_j)_{j\in\mathbb{Z}}$ be a multiresolution analysis generated by the scaling function ϕ. Let $(p_n)_{n\in\mathbb{Z}}$ be given by (5.8). The following properties hold true*

(1) $\sum_{n=-\infty}^{\infty} p_n = 2.$

(2) $\mathcal{F}\phi(\xi) = \prod_{j=1}^{\infty} H(2^{-j}\xi)$ *and* $\mathcal{F}\phi(2\xi) = H(\xi)\mathcal{F}\phi(\xi)$, *where*

$$H(\xi) = \frac{1}{2}\sum_{n=-\infty}^{\infty} p_n e^{-2n\pi i\xi}, \quad \xi \in \mathbb{R}. \tag{5.10}$$

(3) $\sum_{k\in\mathbb{Z}} |\mathcal{F}\phi(x+k)|^2 = 1,\ x \in \mathbb{R}.$

Supposing that (5.9) is fulfilled, we also have

(4) $|H(\xi)|^2 + \left|H\left(\xi + 2^{-1}\right)\right|^2 = 1,\ \xi \in \mathbb{R}$, *where H is given by (5.10).*

(5) $\sum_{n\in\mathbb{Z}} p_{2n} = \sum_{n\in\mathbb{Z}} p_{2n+1} = 1.$

(6) $\sum_{k\in\mathbb{Z}} \phi(x-k) = \sum_{k\in\mathbb{Z}} \phi(k) = 1.$

Examples 5.7. We give two solutions of the equation (5.8).

1. Set $p_0 = p_1 = 1$. The solution is the *box function* $\phi = \chi_{[0,1)}$.

2. Set $p_1 = 1$, $p_0 = p_2 = 2^{-1}$. The solution is the *hat function*

$$\phi(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Since we need a more systematic approach of (TSR) we present three general construction methods for the father ϕ.

A. BY ITERATION

At first step we fix ϕ_0 (for example $\phi_0 = \chi_{[0,1)}$) and then we consider the recurrence relation

$$\phi_j(x) = \sum_{n\in\mathbb{Z}} p_n \phi_{j-1}(2x-n).$$

For j tending to infinity we obtain the scaling function ϕ.

Examples 5.8.

1. For $p_0 = p_1 = 1$ and $\phi_0 = \chi_{[0,1)}$, the box remains invariant, $\phi_j = \phi_0$, $j \ge 1$.

2. For $p_1 = 1$, $p_0 = p_2 = 2^{-1}$ and $\phi_0 = \chi_{[0,1)}$, the hat function appears as $j \to \infty$.
3. For $p_0 = p_3 = 2^{-2}$, $p_1 = p_2 = 3 \cdot 2^{-2}$ and $\phi_0 = \chi_{[0,1)}$, the solution is a quadratic spline

$$\phi(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ -2x^2 + 6x - 3, & 1 < x \le 2 \\ (3-x)^2, & 2 < x \le 3 \\ 0, & x \in \mathbb{R} \setminus [0,3]. \end{cases}$$

4. For $p_0 = (1+\sqrt{3})/4$, $p_1 = (3+\sqrt{3})/4$, $p_2 = (3-\sqrt{3})/4$, $p_3 = (1-\sqrt{3})/4$ and $\phi_0 = \chi_{[0,1)}$ the corresponding father wavelet ϕ is called D_4 (D for Daubechies and 4 because only four coefficients p_k are non-zero).

B. BY FOURIER ANALYSIS

Firstly we determine H function defined by (5.10). Secondly, by using Theorem 5.6, see (2), we obtain $\mathcal{F}\phi$. Taking into account Theorem 2.17, property (1), we get $\mathcal{F}\mathcal{F}\phi = \phi_\sigma$ and consequently ϕ.

Example 5.9. We return at Examples 5.8, the case $p_0 = p_1 = 1$. We obtain $H(\xi) = (1 + e^{-2\pi i \xi})/2$ and consequently, see property (2) at Theorem 5.6,

$$\begin{aligned} \mathcal{F}\phi(\xi) &= \prod_{j=1}^{\infty} H(2^{-j}\xi) = \lim_{N\to\infty} H(2^{-1}\xi)H(2^{-2}\xi)\ldots H(2^{-N}\xi) \\ &= \frac{1-\exp(-2\pi i \xi)}{2\pi i \xi} = \int_0^1 e^{-2\pi i \xi x}\,dx, \end{aligned}$$

where $\xi \neq 0$. This Fourier transform $\mathcal{F}\phi$ appears at (2.9), consequently ϕ is the box function.

C. BY RECURSION

Suppose $\phi(x)$ is known at integer values $x = k$. Then the dilation equation defines $\phi(x)$ at half integers $x = k/2$. Repeating this process yields $\phi(x)$ at all dyadic points $x = k/2^j$, $(k, j) \in \mathbb{Z} \times \mathbb{Z}$.

This is an algorithm often used in practice and it is suited for the case when (TSR) is described by a finite sum. In what follows we assume that there exist the integers $N' < N''$ with the property

$$p_{N'} \neq 0, \quad p_{N''} \neq 0, \quad p_k = 0 \text{ for } k < N' \text{ and for } k > N''. \tag{5.11}$$

Theorem 5.10. *Let ϕ be a solution of the dilation equation (5.8). If the relations (5.11) hold true then* $\mathrm{supp}\phi \subset [N', N'']$.

Note that in order to obtain ϕ we use *method A* (by iteration), choosing ϕ_0 such that $\mathrm{supp}\phi_0$ is compact.

Under the assumptions (5.11), (TSR) can be rewritten

$$\phi(x) = \sum_{k=0}^{N} p_k \phi(2x-k), \quad p_0 p_N \neq 0. \tag{5.12}$$

Since we are looking for $\phi \in C(\mathbb{R})$, Theorem 5.10 implies $\phi(0) = \phi(N) = 0$. At first we obtain $\phi(k)$, $k = \overline{1, N-1}$. We choose $x := k$, $k = \overline{1, N-1}$ in (5.12) and we solve a linear system of the form $v = Pv$, where the matrix P is given by $P := (p_{2j-k})_{1\le j,k \le N-1}$ and $v = (\phi(1), \phi(2), \ldots, \phi(N-1))^t$. Here j represents the row-index and k is the column-index. Because of ϕ generates a partition of the unit (see (6), Theorem 5.6) we obtain the values $\phi(k)$, $k \in \mathbb{Z}$, as follows: we find the eigenvector of the eigenvalue 1 and we impose $\sum_{k=1}^{N-1} \phi(k) = 1$. Next, we define ϕ as a piecewise linear function taking the values $\phi(k)$ on $\mathbb{Z}$. More precisely, we consider

$$\phi_0(x) = \phi(x)(k+1-x) + \phi(k+1)(x-k), \quad x \in [k, k+1].$$

Finally we obtain ϕ_j by using the relations $\phi_{j+1}(x) = \sum_{k=N'}^{N''} p_k \phi_j(2x-k)$, $j \ge 0$. The functions ϕ_j are piecewise linear functions having the knots $k/2^j \in [0, N]$, $k \in \mathbb{Z}$.

Example 5.11. We choose $p_0 = \mu$, $p_1 = -\mu^{-1}$ where $\mu = (1+\sqrt{5})/2$ is the golden ratio. In concordance with Theorem 5.6, property (5), we must take $p_3 = \mu$ and $p_2 = 1 - \mu$. The matrix P is given by

$$P = \frac{1}{2}\begin{pmatrix} 1-\sqrt{5} & 1+\sqrt{5} \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix}.$$

The solution of the system $v = Pv$ is $v = a(1\ 1)^t$ and normalized condition implies $a = 1/2$. One has $\phi(1) = \phi(2) = 1/2$ and for all $k \in \mathbb{Z} \setminus \{1, 2\}$, $\phi(k) = 0$ hold. By recursion we obtain the values $\phi(k/2^j)$, $(j, k) \in \mathbb{Z} \times \mathbb{Z}$.

On the mother wavelet

Let $(V_j)_{j\in\mathbb{Z}}$ be a MRA. Since $V_j \subset V_{j+1}$ we define the orthogonal complement of V_j in V_{j+1} for every $j \in \mathbb{Z}$ so that we have

$$V_{j+1} = V_j \oplus W_j, \quad j \in \mathbb{Z}, \tag{5.13}$$

and $V_j \perp W_k$ for $k \neq j$. Here $\oplus$ indicates *orthogonal sum*. Thus

$$\bigoplus_{j=-\infty}^{m} W_j = V_{m+1} \text{ and } \bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}). \tag{5.14}$$

The last decomposition is usually called an *orthogonal decomposition* of $L^2(\mathbb{R})$. This means that the decomposition of any signal $f \in L^2(\mathbb{R})$ as the (infinite) sum of functions $g_j \in W_j$, $f = \cdots + g_{-1} + g_0 + g_1 + \ldots$, is not only unique, but these components of f are also mutually orthogonal, as described by $(g_l, g_j) = 0$, $l \neq j$.
Moreover, the spaces W_j inherit the scale property (P_3) of V_j, $j \in \mathbb{Z}$, in other words $v \in W_0 \Leftrightarrow v(2^j \cdot) \in W_{j+1}$, $j \in \mathbb{Z}$.
Now, we are looking for a function $\psi \in W_0$ such that $\{\psi_{0,k} : k \in \mathbb{Z}\}$ to become an orthogonal basis in W_0. This function is called *mother-wavelet*. Similarly as in (4.2) we consider

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad x \in \mathbb{R}, \quad (j,k) \in \mathbb{Z} \times \mathbb{Z}. \tag{5.15}$$

- *As the father-wavelet generates orthonormal bases in V_j, the mother-wavelet generates orthonormal bases in W_j, $j \in \mathbb{Z}$.*

Theorem 5.12. *Let $(V_j)_{j\in\mathbb{Z}}$ a MRA of $L^2(\mathbb{R})$ generated by the scaling function ϕ. Let $(W_j)_{j\in\mathbb{Z}}$ be defined by (5.13). If $\psi \in W_0$ such that $(\psi_{0,k})_{k\in\mathbb{Z}}$ is an orthonormal basis in W_0 then a function $G \in L^2_P(0,1)$ exists enjoying of the following properties*

$$(\mathcal{F}\psi)(2\lambda) = G(\lambda)(\mathcal{F}\phi)(\lambda), \tag{5.16}$$
$$|G(\lambda)|^2 + |G(\lambda + 1/2)|^2 = 1 \tag{5.17}$$
$$H(\lambda)\overline{G}(\lambda) + H(\lambda + 1/2)\overline{G}(\lambda + 1/2) = 0, \tag{5.18}$$

where H is given by (5.10).
Since $\psi \in W_0 \subset V_1$ and $\{\phi_{1,k} : k \in \mathbb{Z}\}$ is a basis in V_1, a sequence $(q_k)_{k\in\mathbb{Z}}$ belonging to l^2 exists such that

$$\psi(t) = \sum_{k=-\infty}^{\infty} q_k \phi(2t - k), \quad t \in \mathbb{R}. \tag{5.19}$$

Associated to the mother-wavelet ψ, we consider $G \in L^2_P(0,1)$ as follows

$$G(\lambda) = \frac{1}{2}\sum_{k=-\infty}^{\infty} q_k e^{-2k\pi i\lambda}, \quad \lambda \in \mathbb{R}. \tag{5.20}$$

How to find the functions G and ψ

Firstly we are looking for functions G which verify (5.17) and (5.18). Writing $G(\lambda) = \exp(-2\pi i\lambda)\overline{U}(\lambda)$, where $U \in L^2_P(0,1)$, relation (5.18) leads us to the identity

$$H(\lambda)U(\lambda) = H(\lambda + 1/2)\,U(\lambda + 1/2), \quad \lambda \in \mathbb{R}.$$

This means that HU has 2^{-1}-period and G satisfies

$$\overline{H}(\lambda)G(\lambda) = e^{-2\pi i\lambda}\overline{\theta}(\lambda), \tag{5.21}$$

where θ is a 2^{-1}-periodic function. Substituting (5.21) in (5.17) and taking into account $|\overline{H}|^2 = |H|^2$, $|H(\lambda)|^2 + |H(\lambda+1/2)|^2 = 1$, one obtains

$$|\theta(\lambda)| = |H(\lambda)H(\lambda+1/2)|. \tag{5.22}$$

For example, θ_α, $\alpha \in \mathbb{R}$, defined by $\theta_\alpha(\lambda) = H(\lambda)H(\lambda+1/2)e^{-2\pi i\alpha}$, $\lambda \in \mathbb{R}$, satisfies (5.22). By using θ_α, the function G is given by

$$G(\lambda) = e^{-2\pi i(\lambda-\alpha)}\overline{H}(\lambda+1/2). \tag{5.23}$$

By direct computational we can prove that all these functions satisfy both (5.17) and (5.18). At this moment G and father ϕ are known. From (5.16) we get $\mathcal{F}\psi$ and, further on, mother ψ is born.

Theorem 5.13. *Let ϕ be a normalized solution of (TSR). If ψ is defined by its Fourier transform $\mathcal{F}\psi(\lambda) = G(\lambda/2)(\mathcal{F}\phi)(\lambda/2)$, $\lambda \in \mathbb{R}$, where $G(\lambda/2) = 2^{-1}\exp(-2\pi i(\lambda/2-\alpha))\sum_{n\in\mathbb{Z}}\overline{p}_n \exp(n\pi i(\lambda+1))$, then $(T_k\psi)_{k\in\mathbb{Z}}$ is an orthonormal basis in W_0 space and $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R})$.*

Theorem 5.14. *Let H, G be defined by (5.10) respectively (5.20). If (5.23) holds true with $\alpha = 2^{-1}$ then one has $(\forall)\ k \in \mathbb{Z}$, $q_k = (-1)^k\overline{p}_{1-k}$ and the mother-wavelet ψ is given by*

$$\psi(t) = \sum_{k\in\mathbb{Z}}(-1)^k\overline{p}_{1-k}\phi(2t-k).$$

Examples 5.15. Knowing the scale function ϕ, by using Theorem 5.14 we can determine ψ. We keep in mind Examples 5.7.

1. For the box function $\phi = \chi_{[0,1)}$ the mother ψ is the Haar function, (4.4).

2. For the hat function ϕ, the mother ψ is the following

$$\psi(x) = \begin{cases} -1/2 - x, & -1/2 \le x < 0, \\ 3x - 1/2, & 0 \le x < 1/2, \\ -3x + 5/2, & 1/2 \le x < 1, \\ x - 3/2, & 1 \le x \le 3/2, \\ 0, & x \in \mathbb{R}\setminus[-1/2, 3/2]. \end{cases}$$

6. Wavelet transforms

Wavelet decompositions and reconstructions

We want to come to algorithms for wavelet decomposition and wavelet reconstruction.

Let $(V_j)_{j\in\mathbb{Z}}$ be a MRA generated by the scaling function $\phi \in L^2(\mathbb{R})$ and let f be a signal in $L^2(\mathbb{R})$. For a given $\varepsilon > 0$, the property (5.3) guarantees that an integer n and a function $f_n \in V_n$ exist such that $\|f - f_n\|_2 < \varepsilon$, in other words, f can be approximated as closely as desired by an f_n, for some $n \in \mathbb{Z}$. Since $V_n = V_{n-1} \oplus W_{n-1}$, f_n has a unique decomposition

$$f_n = f_{n-1} + g_{n-1}, \text{ where } f_{n-1} \in V_{n-1} \text{ and } g_{n-1} \in W_{n-1}.$$

By repeating this process, we have

$$f_n = g_{n-1} + g_{n-2} + \cdots + g_{n-m} + f_{n-m}, \quad f_j \in V_j, \quad g_j \in W_j. \tag{6.1}$$

This type of decomposition is called *wavelet decomposition.* By using the bases of spaces V_j and W_j we can write

$$\begin{cases} f_j(x) = \sum_{k\in\mathbb{Z}} p_{j,k}\phi_{j,k}(x), \quad p_{j,k} = (f, \phi_{j,k}) \\ g_j(x) = \sum_{k\in\mathbb{Z}} q_{j,k}\psi_{j,k}(x), \quad q_{j,k} = (f, \psi_{j,k}). \end{cases} \tag{6.2}$$

We set $\widetilde{p}_j := (p_{j,k})_{k\in\mathbb{Z}} \in l^2$ and $\widetilde{q}_j := (q_{j,k})_{k\in\mathbb{Z}} \in l^2$. The decomposition is to find $\widetilde{p}_{n-1}$ and $\widetilde{q}_{n-1}$ from $\widetilde{p}_n$ and the reconstruction is to recover $\widetilde{p}_n$ from $\widetilde{p}_{n-1}$ and $\widetilde{q}_{n-1}$.

Remark 6.1. Every signal $f \in L^2(\mathbb{R})$ can be unique decomposed under the form $\sum_{j\in\mathbb{Z}} g_j$. where g_j, $j \in \mathbb{Z}$, are defined by (6.2) and are called the *voices* of f. For every integer n, $f_n = \sum_{j\le n-1} g_j$ represents the orthogonal projection of f onto V_n.

Theorem 6.2. *If ϕ is the father-wavelet and ψ is the corresponding mother-wavelet then the following identities hold true*

$$\begin{aligned} \phi_{n,k}(x) &= \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} p_{l-2k}\phi_{n+1,l}(x), \ x \in \mathbb{R}, \\ \psi_{n,k}(x) &= \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} q_{l-2k}\phi_{n+1,l}(x), \ x \in \mathbb{R}, \\ (\phi_{n,k}, \phi_{n+1,l}) &= \frac{1}{\sqrt{2}} p_{l-2k}, \ (\psi_{n,k}, \phi_{n+1,l}) = \frac{1}{\sqrt{2}} q_{l-2k}, \end{aligned}$$

where the l^2-sequences $(p_n)_n$, $(q_n)_n$ are defined by (5.8) respectively (5.19).
This theorem and relation (6.2) imply

$$p_{n,k} = \int_{\mathbb{R}} f(x)\overline{\phi}_{n,k}(x)dx = \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} \overline{p}_{l-2k} p_{n+1,l}. \tag{6.3}$$

Let us define an operator (named *filter*) $\mathcal{H}: l^2 \to l^2$, $a = (a_k) \mapsto \mathcal{H}a$,

$$(\mathcal{H}a)_k = \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} \overline{p}_{l-2k} a_l.$$

From (6.3) we have $\widetilde{p}_n = \mathcal{H}\widetilde{p}_{n+1}$. Analogously, we define the filter $\mathcal{G}: l^2 - l^2$, $a \mapsto \mathcal{G}a$ by

$$(\mathcal{G}a)_k = \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} \overline{q}_{l-2k} a_l,$$

and by using the same arguments we obtain $\widetilde{q}_n = \mathcal{G}\widetilde{p}_{n+1}$.
We recall: if $\mathcal{A}$ is an filter on l^2, then the *adjoint* $\mathcal{A}^*: l^2 \to l^2$ is defined by $(\mathcal{A}a, b) = (a, \mathcal{A}^*b)$.
The matrix representation of the adjoint operator is the Hermitian conjugate of the matrix representation of the operator. Hence the adjoints of $\mathcal{H}$ and $\mathcal{G}$ are given by

$$(\mathcal{H}^*a)_k = \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} p_{k-2l} a_l, \quad (\mathcal{G}^*a)_k = \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} q_{k-2l} a_l.$$

We are now in a position to express $\widetilde{p}_{n+1}$ in terms of $\widetilde{p}_n$ and $\widetilde{q}_n$. Taking into account that $f - f_{n+1} \perp V_{n+1}$, by using (6.2) and Theorem 6.2 we get

$$\begin{aligned} p_{n+1,k} &= (f, \phi_{n+1,k}) = (f_{n+1}, \phi_{n+1,k}) = (f_n + g_n, \phi_{n+1,k}) \\ &= \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} p_{k-2l} p_{n,l} + \frac{1}{\sqrt{2}} \sum_{l\in\mathbb{Z}} q_{k-2l} q_{n,l}, \end{aligned}$$

and consequently $\widetilde{p}_{n+1} = \mathcal{H}^*\widetilde{p}_n + \mathcal{G}^*\widetilde{q}_n$.
With the help of matrix representation, the decomposition-reconstruction can be written as follows

$$\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix} \widetilde{p}_{n+1} = \begin{pmatrix} \widetilde{p}_n \\ \widetilde{q}_n \end{pmatrix}, \quad \widetilde{p}_{n+1} = [\mathcal{H}^* \ \mathcal{G}^*] \begin{pmatrix} \widetilde{p}_n \\ \widetilde{q}_n \end{pmatrix}.$$

Integral wavelet transform

We associate to a function $\psi \in \mathbb{C}^{\mathbb{R}}$ the family $(\psi_{a,b})$, $a > 0$, $b \in \mathbb{R}$, where

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right). \tag{6.4}$$

Theorem 6.3. *Let ψ belong to $L^2(\mathbb{R})$ and $\psi_{a,b}$ be defined by (6.4). For every $a > 0$ and $b \in \mathbb{R}$ the following properties hold true*
(i) $\psi_{a,b} \in L^2(\mathbb{R})$ and $\|\psi_{a,b}\|_2 = \|\psi\|_2$;
(ii) if ψ is a window function in the sense of Definition 3.1 with its center t^* and its radius Δ_ψ then $\psi_{a,b}$ is a window function with the center $at^* + b$ and the radius $a\Delta_\psi$;

(iii) if $\widehat{\psi}$ is a window function with its center λ^* and its radius $\Delta_{\widehat{\psi}}$ then $\widehat{\psi}_{a,b}$ is a window function with the center $a^{-1}\lambda^*$ and the radius $a^{-1}\Delta_{\widehat{\psi}}$.

Definition 6.4. If $\psi \in L^2(\mathbb{R})$ satisfies the *admissibility condition*

$$C_\psi := \int_{-\infty}^{0} \frac{|\widehat{\psi}(\lambda)|^2}{|\lambda|} d\lambda = \int_{0}^{\infty} \frac{|\widehat{\psi}(\lambda)|^2}{|\lambda|} d\lambda < \infty, \tag{6.5}$$

then ψ is called a *basic wavelet.* Relative to every basic wavelet ψ, the *integral wavelet transform (IWT) on* $L^2(\mathbb{R})$ is the following operator $W_\psi : L^2(\mathbb{R}) \to \mathbb{C}^{\mathbb{R}_+^* \times \mathbb{R}}$, $f \mapsto W_\psi f$, where

$$(W_\psi f)(a,b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt, \quad a > 0, \quad b \in \mathbb{R}. \tag{6.6}$$

$(W_\psi f)(a,b)$ are called the coefficients of f relative to ψ.
The IWT was introduced by Grossmann and Morlet.

Remark 6.5. (i) In (6.5), the equality of two integrals is not a very restrictive condition. For example, if ψ is a real valued function then $\widehat{\psi}(-\lambda) = \overline{\widehat{\psi}(\lambda)}$ and the equality holds true.

(ii) The admissibility condition can be written as follows

$$C_\psi = \int_0^\infty \frac{|\psi(\lambda)|^2}{\lambda} d\lambda = \int_0^\infty \frac{|\psi(-\lambda)|^2}{\lambda} d\lambda.$$

(iii) The operator W_ψ is linear. By using (6.4) and Theorem 2.17 - (2) - the coefficients of f relative to ψ can be written

$$(W_\psi f)(a,b) = (f, \psi_{a,b}) = (\widehat{f}, \widehat{\psi}_{a,b}), \tag{6.7}$$

and according to Schwarz inequality and Theorem 6.3(i) one has

$$(\forall)\ f \in L^2(\mathbb{R}),\ (\forall)\ (a,b) \in (0,\infty) \times \mathbb{R}, \quad |(W_\psi f)(a,b)| \le \|f\|_2 \|\psi\|_2.$$

(iv) If $\psi \in L^2(\mathbb{R})$ is a basic wavelet such that both ψ and $\widehat{\psi}$ are windows functions, then ψ is called a *window basic wavelet.* In this case ψ has the following properties

$$\{\psi, \widehat{\psi}\} \subset L^1(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{\infty} \psi(x) dx = 0.$$

(v) Examining Theorem 6.3 and relations (6.6), (6.7) we deduce

1. The IWT gives local information of a signal f with a *time-window* $[b + at^* - a\Delta_\psi, b + at^* + a\Delta_\psi]$. This window narrows, for small values of a and widens for allowing a to be large.

2. The IWT also gives local unformation of $\widehat{f}$ with a *frequency-window* $[(\lambda^* - \Delta_{\widehat{\psi}})/a, (\lambda^* + \Delta_{\widehat{\psi}})/a]$. The ratio between center frequency and bandwidth is $\lambda^*/(2\Delta_{\widehat{\psi}})$, independent of the scaling a.

3. Considering λ^*/a to be the frequency variable λ, we may create the $t - \lambda$ plane as the time-frequency plane. With the help of two previous windows we form a rectangular *time-frequency window*

$$[b + at^* - a\Delta_\psi, b + at^* + a\Delta_\psi] \times [(\lambda^* - \Delta_{\widehat{\psi}})/a, (\lambda^* + \Delta_{\widehat{\psi}})/a].$$

This window has finite area given by $4\Delta_\psi\Delta_{\widehat{\psi}}$ (independent of a and b). It narrows for detecting high-frequency phenomena (small $a > 0$) and it widens for investigating low-frequency behavior (large $a > 0$).

Theorem 6.6. *If ψ is a mother wavelet in the sense of Definition 4.2 then the coefficients $c_{j,k}$ of the series wavelet (4.3) can be expressed by using IWT as follows*

$$c_{j,k} = (W_\psi f)(2^{-j}, k2^{-j}), \quad (j,k) \in \mathbb{Z} \times \mathbb{Z}.$$

Theorem 6.7. *Let $\psi \in L^2(\mathbb{R})$ be a window basic wavelet. For any two signals f and g belonging to $L^2(\mathbb{R})$ one has*

$$\int_0^\infty \Big(\int_{\mathbb{R}} (W_\psi f)(a,b)\overline{(W_\psi g)(a,b)}db \Big) \frac{da}{a^2} = C_\psi \langle f, g \rangle,$$

where C_ψ is given by (6.5).

Corollary 6.8. *If $\psi \in L^2(\mathbb{R})$ is a window basic wavelet then every signal $f \in L^2(\mathbb{R})$ verifies*

$$\int_0^\infty \Big(\int_{\mathbb{R}} |(W_\psi f)(a,b)|^2 db \Big) \frac{da}{a^2} = C_\psi \|f\|_2^2.$$

Theorem 6.9. (The reconstruction formula) *Let $\psi \in L^2(\mathbb{R})$ be a window basic wavelet. If $f \in L^2(\mathbb{R})$ then*

$$f(t) = \frac{1}{C_\psi} \int_0^\infty \Big(\int_{\mathbb{R}} (W_\psi f)(a,b)\psi_{a,b}(t) db \Big) \frac{da}{a^2}.$$

This formula must be understood as follows: if

$$f_\varepsilon(t) = \frac{1}{C_\psi} \int_\varepsilon^\infty \Big(\int_{\mathbb{R}} (W_\psi f)(a,b)\psi_{a,b}(t) db \Big) \frac{da}{a^2}, \quad \varepsilon > 0,$$

then $\lim\limits_{\varepsilon \to 0+} \|f - f_\varepsilon\|_2 = 0$.

Remark 6.10. In practical research, some mathematical requirements wavelets are not fulfilled. They surrender and the rigorous frames are broken. For example, Jean Morlet used a window basic wavelet defined by $\psi(t) = e^{-t^2/2} \cos 5t, \quad t \in \mathbb{R}$.
Since $\widehat{\psi}(0) = \sqrt{2\pi} e^{-25/2} > 0$, (6.5) is not verified ($C_\psi = \infty$), but the value $\widehat{\psi}(0)$ is approx. 10^{-5} and practically is considered zero.

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