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ABSTRACT. In this paper we are dealing with a general class of linear and positive operators of discrete type. We investigate the convergence of the operators and we give estimates of the rate of convergence by using the classical modulus of continuity, Ditzian-Totik weighted moduli, as well as the weighted K-functional of second order. In some cases we prove that these operators leave invariant the class of increasing functions respectively the convex and the Hölder continuous functions. Also a Voronovskaja type formula is established and some concrete examples are presented.[1]

1. INTRODUCTION

The positive approximation processes have been the object of many investigations. In the last years, it comes out a further development of their study in connection with some evolution problems via semigroup theory. We point out that F. Altomare and his Bari school deepened the study of elliptic-parabolic equations by means of positive operators. New types of linear operators have been introduced in order to

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enlarge the class of evolution equations whose solutions can be approximated by constructive approximation processes. A detailed analysis of these aspects can be found in the monograph [3].

Motivated by this research direction we deal with a sequence of positive linear approximation operators of discrete type. The paper is split into four sections. The next section is devoted to construct this general class of operators and to present some concrete examples. In Section 3 we investigate the convergence of the operators giving general estimates in terms both of the modulus of continuity and of Ditzian-Totik weighted moduli. Furthermore we study some particular cases in which these operators leave invariant the classes of monotone functions, convex functions and Lipschitz functions. In the last section we focus our attention to establish a Voronoskaja type formula. This result provides a link with the generation problem for certain differential operators.

2. Construction of the sequence $(L_{n,\lambda})_{n\geq 1}$

Let D be a general interval of the real line. The Landau symbols will be denoted by $o(\cdot)$ and $\mathcal{O}(\cdot)$, as usual. Also e_j stands for the j-th monomial, $e_j(t) = t^j$, $t \in D$, j being a non-negative integer. Throughout the paper we will denote by $C(D)$ the vector space of all real-valued continuous functions on D. Also $C_B(D)$ represents the subspace of $C(D)$ of all real-valued bounded continuous functions on D endowed with the natural order and the sup-norm $\|\cdot\|$ defined by $\|f\| = \sup_{x\in D} |f(x)|$, $f \in C_B(D)$, with respect to which it becomes a Banach lattice. We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each integer $n \geq 1$ we consider a set of indexes $\Delta_n \subset \mathbb{N}_0$ and a net on D namely $(x_{n,k})_{k\in\Delta_n}$ with the following property: for every $k \in \Delta_n$, there $\gamma_k > 0$ exists such that $x_{n,k} = o(n^{-\gamma_k})$. Let $(\phi_{n,k})_{k\in\Delta_n}$ be a sequence of continuous functions on D verifying the following conditions:

$$\phi_{n,k} \geq 0, \quad \sum_{k\in\Delta_n} \phi_{n,k} = 1, \tag{1}$$

$$\sum_{k\in\Delta_n} \phi_{n,k}(x) x_{n,k} = x, \quad x \in D, \tag{2}$$

$$\sum_{k\in\Delta_n} \phi_{n,k}(x) x_{n,k}^2 = x^2 + \frac{\psi_n(x)}{u(n)}, \quad x \in D, \tag{3}$$

where $\psi_n \in C(D)$ and $u(n) = \mathcal{O}(n^\alpha)$, $n \to \infty$, for some constant $\alpha > 0$. Furthermore we require

$$\lim_{n\to\infty} \frac{\psi_n(x)}{u(n)} = 0 \text{ uniformly on any compact } K \subset D. \tag{4}$$

The above data satisfying conditions (1)-(4) can be indicated briefly as

$$\left\langle D, \Delta_n, x_{n,k}, \phi_{n,k}(x); \psi_n, u \right\rangle, \quad (n,k) \in \mathbb{N} \times \Delta_n, \quad x \in D. \tag{5}$$

Actually, this system leads us to a particular case of a sequence of linear and positive operators of discrete type which, in time, has been investigated in many papers. For any $f \in C(D)$ it defines the operators

$$(l_n f)(x) := \sum_{k\in\Delta_n} \phi_{n,k}(x) f(x_{n,k}), \quad x \in D, \quad n \in \mathbb{N}. \tag{6}$$

Under our assumptions we have $l_n e_0(x) = 1$, $(l_n e_1)(x) = x$ and $(l_n e_2)(x) = x^2 + \dfrac{\psi_n(x)}{u(n)}$. Consequently, by using the well-known theorem of Bohman-Korovkin it follows that $\lim\limits_{n\to\infty} (l_n f)(x) = f(x)$ uniformly on each compact $K \subset D$, for all $f \in C(D)$.

We just recall the best known and intensively studied operators given by particular cases of the systems (5).

1° (Bernstein operators) $b_n : \left\langle [0,1], \{0,1,\dots,n\}, \dfrac{k}{n}, \dbinom{n}{k} x^k(1-x)^{n-k}; e_1 - e_2, e_1 \right\rangle$, i.e.

$$b_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k}.$$

2° (Favard-Szasz-Mirakjan operators) $s_n : \left\langle [0,\infty), \mathbb{N}_0, \dfrac{k}{n}, e^{-nx}\dfrac{(nx)^k}{k!}; e_1, e_1 \right\rangle$, i.e.

$$s_n(f)(x) := e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}.$$

3° (Baskakov operators) $v_n : \left\langle [0,\infty), \mathbb{N}_0, \dfrac{k}{n}, \dbinom{n+k-1}{k} x^k(1+x)^{-n-k}; e_1 + e_2, e_1 \right\rangle$, i.e.

$$v_n(f)(x) := \sum_{k=0}^{\infty} \binom{n+k-1}{k} f\left(\frac{k}{n}\right) x^k(1+x)^{-n-k}.$$

4° (Stancu operators)

$$d_n^{\langle\alpha\rangle} : \left\langle [0,1], \{0,1,\dots,n\}, \frac{k}{n}, \binom{n}{k} w_{n,k}(x,\alpha); (n\alpha+1)(e_1 - e_2), (1+\alpha)e_1 \right\rangle$$

where

$$w_{n,k}(x,\alpha)=\frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=0}^{n-k-1}(1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)},$$

α being a real parameter which may depend on the natural number n. It turns out that

$$d_n^{\langle\alpha\rangle}(f)(x):=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)w_{n,k}(x,\alpha).$$

If $0\le\alpha=\alpha(n)\to 0$, $n\to\infty$, then the sequence converges to the identity operator. As a particular case we have $d_n^{\langle 0\rangle}=b_n$. Also, if $\alpha=-1/n$ then this operator becomes the Lagrange interpolation operator corresponding to the equally spaced nodes k/n.

In what follows we use two sequences of operators of the type (6), namely

$$l_n^{(1)}:\left\langle[0,1],I_n,x_{n,k},\lambda_{n,k}(x);\psi_{1,n},u_1\right\rangle,\quad (n,k)\in\mathbb{N}\times I_n,\quad x\in[0,1],$$

$$l_n^{(2)}:\left\langle[0,\infty),J_n,x_{n,k},\mu_{n,k}(x);\psi_{2,n},u_2\right\rangle,\quad (n,k)\in\mathbb{N}\times J_n,\quad x\in[0,\infty),$$

such that $0\in I_n\cap J_n$ and $x_{n,0}=0$. Regarding these sequences we impose the following admissibility condition to be satisfied: for any $n\in\mathbb{N}$, a function $\widetilde{\psi}_n\in C([0,\infty))$ exists such that

$$x_{n,p}\psi_{2,p}(x)=u_2(p)\frac{\widetilde{\psi}_n(x)}{u_1(n)},\quad p\in I_n,\quad x\ge 0. \tag{7}$$

If this condition is fulfilled then the pair $(l_n^{(1)},l_n^{(2)})$ will be called compatible approximation processes.

Let S be a set of indexes. For every $\beta\in S$ we will fix a function $w_\beta\in C([0,\infty))$ such that $\lim_{x\to\infty}w_\beta(x)e_2(x)$ is finite and $w_\beta(x)>0$ for $x>0$. By using this weighted function w_β we introduce the space

$$C_{w_\beta}:=\{f\in C([0,\infty)):\ \sup_{x\ge 0}w_\beta(x)|f(x)|<\infty\}.$$

Endowed with the natural order and the norm $\|\cdot\|_{w_\beta}$ defined by

$$\|f\|_{w_\beta}:=\|w_\beta f\|,\quad f\in C_{w_\beta},$$

the space becomes a Banach lattice. Clearly, the test functions e_j, $j\in\{0,1,2\}$, belong to C_{w_β} and taking into account the properties of w_β one has

$$\|f\|_{w_\beta}\le\|w_\beta\|\|f\|,\ \text{for every } f\in C_B([0,\infty)).$$

Using this net of weights $w = (w_\beta)_{\beta \in S}$ we set

$$C_w := \bigcup_{\beta \in S} C_{w_\beta}$$

which becomes a vector subspace of $C([0, \infty))$.

Finally we consider a function $\lambda \in C([0, \infty))$ such that $0 \le \lambda(x) \le 1$ for every $x \ge 0$. Now we are able to present the announced sequence of operators

$$(8) \qquad (L_{n,\lambda} f)(x) = \sum_{p \in I_n} \sum_{k \in J_p} \lambda_{n,p}(\lambda(x)) \mu_{p,k}(x) f(x_{n,p} x_{p,k} + (1 - x_{n,p})x), \quad x \ge 0,$$

where $f \in C_w$.

Considering the function $f_{n,p,x} : [0, \infty) \to \mathbb{R}$ defined by

$$(9) \qquad f_{n,p,x}(t) := f(x_{n,p} t + (1 - x_{n,p})x) \quad (t \ge 0)$$

for fixed $x \ge 0$, $n \in \mathbb{N}$ and $p \in I_n$, we can describe of the operators $L_{n,\lambda}$, $n \in \mathbb{N}$, in terms of the $l_n^{(2)}$ operators as follows

$$(10) \qquad (L_{n,\lambda} f)(x) = \sum_{p \in I_n} \lambda_{n,p}(\lambda(x)) (l_p^{(2)} f_{n,p,x})(x), \quad x \ge 0.$$

If $0 \in I_n$ then $l_0^{(2)}$ denotes the identity operator on C_w.

We are going to present two examples.

Example A. Let $S = (0, \infty)$. For any $\beta \in S$ we consider the weighted function w_β, $w_\beta(x) = \exp(-\beta x)$. We choose $l_n^{(1)} = b_n$ and $l_n^{(2)} = s_n$ the Bernstein and the Favard-Szasz-Mirakjan operator of order n, respectively. Plainly we obtain $u_1(n) = n$, $n \in \mathbb{N}$, $u_2(p) = p$, $p \in \mathbb{N}_0$, $\psi_{2,p}(x) = x$, $x \ge 0$, and $\widetilde{\psi}_n(x) = x$, $x \ge 0$. Thus, condition (7) is fulfilled. It turns out that

$$(L_{n,\lambda} f)(x) = (M_{n,\lambda} f)(x) =$$
$$= \sum_{p=0}^{n} \sum_{k=0}^{\infty} \binom{n}{p} \lambda^p(x)(1 - \lambda(x))^{n-p} e^{-px} \frac{(px)^k}{k!} f\left(\frac{k}{n} + \left(1 - \frac{p}{n}\right) x\right),$$

where $f \in E_\infty := \bigcup_{\beta > 0} \{g \in C([0, \infty)) : \sup_{x \ge 0} \exp(-\beta x)|f(x)| < \infty\}$. If $\lambda = e_0$, then M_{n,e_0} becomes the n^{th} Favard-Szasz-Mirakjan operator s_n. The $M_{n,\lambda}$ operator was introduced by F. Altomare and I. Carbone and studied in several papers [2], [4], [6].

Example B. Let $S = \{2, 3, 4, \dots\}$. For any $m \in S$ we consider the function w_m, $w_m(x) = (1 + x^m)^{-1}$. Now we choose $l_n^{(1)} = b_n$ and $l_n^{(2)} = v_n$ the Bernstein and the

Baskakov operator of n^{th} order. We have $u_1(n) = n$, $n \in \mathbb{N}$, $u_2(p) = p$, $p \in \mathbb{N}_0$, $\psi_{2,p}(x) = x+x^2$, $x \geq 0$, and consequently $\widetilde{\psi}_n(x) = x+x^2$, $x \geq 0$. Again the compatible condition (7) is verified and we obtain

$$(L_{n,\lambda}f)(x) := (B_{n,\lambda}f)(x) =$$
$$= \sum_{p=0}^{n} \binom{n}{p} \frac{\lambda^p(x)(1-\lambda(x))^{n-p}}{(1+x)^p} \sum_{k=0}^{\infty} \binom{p+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n} + \left(1-\frac{p}{n}\right)x\right),$$

where $f \in \bigcup_{m\geq 2} E_m$, $E_m := \left\{ g \in C([0,\infty)) : \sup_{x\geq 0} \frac{|f(x)|}{1+x^m} < \infty \right\}$. Clearly B_{n,e_0} is the n^{th} Baskakov operator. The $B_{n,\lambda}$ operator was introduced and studied by F. Altomare and E.M. Mangino [5].

3. Properties of the sequence $(L_{n,\lambda})_{n\geq 1}$

At first we will emphasize the convergence of our sequence and we will also give estimates of the rate of convergence.

Theorem 1. *Let the operator $L_{n,\lambda}$ be defined by (8). The following identities hold true:*

(i) $L_{n,\lambda}e_j = e_j$, $j \in \{0,1\}$,

(ii) $L_{n,\lambda}e_2 = e_2 + \frac{\lambda\widetilde{\psi}_n}{u_1}$,

where $\widetilde{\psi}_n$ is introduced by (7).

Proof. (i) Since the sequences $(l_n^{(j)})_{n\geq 1}$, $j \in \{1,2\}$, are of the type described by (5), they verify both condition (1) and (2). Thus

$$(L_{n,\lambda}e_0)(x) = \sum_{p\in I_n}\sum_{k\in J_p} \lambda_{n,p}(\lambda(x))\mu_{p,k}(x) = \sum_{p\in I_n} \lambda_{n,p}(\lambda(x)) = 1$$

and

$$(L_{n,\lambda}e_1)(x) = \sum_{p\in I_n} \lambda_{n,p}(\lambda(x))x_{n,p} \sum_{k\in J_p} \mu_{p,k}(x)x_{p,k} + \sum_{p\in I_n} x\lambda_{n,p}(\lambda(x))(1-x_{n,p}) \sum_{k\in J_p} \mu_{p,k}(x) =$$
$$= x(l_n^{(1)}e_1)(\lambda(x)) + x\{(l_n^{(1)}e_0)(\lambda(x)) - (l_n^{(1)}e_1)(\lambda(x))\} = x.$$

(ii) Following the same motivation as in the previous point and taking conditions (3) and (7) into account we can write successively

$$(L_{n,\lambda}e_2)(x) = \sum_{p\in I_n} \lambda_{n,p}(\lambda(x))x_{n,p}^2 \sum_{k\in J_p} \mu_{p,k}(x)x_{p,k}^2 + 2x\left\{ \sum_{p\in I_n} \lambda_{n,p}(\lambda(x))x_{n,p} \sum_{k\in J_p} \mu_{p,k}(x)x_{p,k} - \right.$$

$$-\sum_{p\in I_n}\lambda_{n,p}(\lambda(x))x_{n,p}^2\sum_{k\in J_p}\mu_{p,k}(x)x_{p,k}\Bigg\}+x^2\sum_{p\in I_n}\lambda_{n,p}(\lambda(x))(1-x_{n,p})^2\sum_{k\in J_p}\mu_{p,k}(x)=$$

$$=\sum_{p\in I_n}\lambda_{n,p}(\lambda(x))x_{n,p}^2\left(x^2+\frac{\psi_{2,p}(x)}{u_2(p)}\right)+2x^2\{(l_n^{(1)}e_1)(\lambda(x))-(l_n^{(2)}e_2)(\lambda(x))\}+$$

$$+x^2\{1-2(l_n^{(1)}e_1)(\lambda(x))+(l_n^{(1)}e_2)(\lambda(x))\}=x^2\left(\lambda^2(x)+\frac{\psi_{1,n}(\lambda(x))}{u_1(n)}\right)+$$

$$+\sum_{p\in I_n}\lambda_{n,p}(\lambda(x))x_{n,p}\frac{\widetilde{\psi}_n(x)}{u_1(n)}+2x^2\left(\lambda(x)-\lambda^2(x)-\frac{\psi_{1,n}(\lambda(x))}{u_1(n)}\right)+$$

$$+x^2\left(1-2\lambda(x)+\lambda^2(x)+\frac{\psi_{1,n}(\lambda(x))}{u_1(n)}\right)=x^2+\lambda(x)\frac{\widetilde{\psi}_n(x)}{u_1(n)}.$$

Remarks. (i) Every operator $L_{n,\lambda}$ maps continuously $C_B([0,\infty))$ into itself. Indeed, for $f\in C_B([0,\infty))$ and $x\geq 0$,

$$|(L_{n,\lambda}f)(x)|\leq\sum_{p\in I_n}\lambda_{n,p}(\lambda(x))\sum_{k\in J_p}\mu_{p,k}(x)\|f\|=\|f\|.$$

Moreover, $L_{n,\lambda}e_0=e_0$, hence $\|L_{n,\lambda}\|_{C_B([0,\infty))}=1$.

(ii) Let $\tau_r(L_{n,\lambda},\cdot)$ be the r^{th} central moment of $L_{n,\lambda}$, defined by

$$\tau_r(L_{n,\lambda},x)=L_{n,\lambda}((e_1-xe_0)^r,x),\quad r\in\mathbb{N}_0.$$

Theorem 1 implies

$$\tau_1(L_{n,\lambda},x)=0\quad\text{and}\quad\tau_2(L_{n,\lambda},x)=\frac{\lambda(x)\widetilde{\psi}_n(x)}{u_1(n)}.\tag{11}$$

Theorem 1 together with the theorem of Bohman-Korovkin allow us to state

Theorem 2. *Let the operator $L_{n,\lambda}$ be defined by (8) and a,b real numbers such that $0\leq a<b$.*

If $\lim\limits_{n\to\infty}\dfrac{\widetilde{\psi}_n(x)}{u_1(n)}=0$ uniformly on $[a,b]$ then $\lim\limits_{n\to\infty}(L_{n,\lambda}f)(x)=f(x)$ uniformly on $[a,b]$.

By virtue of the classical results regarding the rate of convergence (see for example [3], Theorem 5.1.2) Theorem 1 leads us to the following result.

Theorem 3. *Let the operator $L_{n,\lambda}$ be defined by (8).*

(i) If $f\in C_B([0,\infty))$ then $|(L_{n,\lambda}f)(x)-f(x)|\leq\left(1+\dfrac{1}{\delta}\sqrt{\dfrac{\lambda(x)\widetilde{\psi}_n(x)}{u_1(n)}}\right)\omega_1(f;\delta)$, for every $x\geq 0$ and $\delta>0$.

(ii) If f is differentiable on $[0,\infty)$ and $f' \in C_B([0,\infty))$ then one has

$$|(L_{n,\lambda}f)(x) - f(x)| \le \sqrt{\frac{\lambda(x)\widetilde{\psi}_n(x)}{u_1(n)}}\left(1+\frac{1}{\delta}\sqrt{\frac{\lambda(x)\widetilde{\psi}_n(x)}{u_1(n)}}\right)\omega_1(f';\delta),$$

for every $x \ge 0$ and $\delta > 0$.

Remark. In particular, for $\delta := (\widetilde{\psi}_n(x)/u_1(n))^{1/2}$ the above statements become (under the same assumptions on the function f)

$$|(L_{n,\lambda}f)(x) - f(x)| \le \left(1+\sqrt{\lambda(x)}\right)\omega_1\left(f;\sqrt{\frac{\widetilde{\psi}_n(x)}{u_1(n)}}\right),$$

respectively

$$|(L_{n,\lambda}f)(x) - f(x)| \le \sqrt{\frac{\lambda(x)\widetilde{\psi}_n(x)}{u_1(n)}}\left(1+\sqrt{\lambda(x)}\right)\omega_1\left(f';\sqrt{\frac{\widetilde{\psi}_n(x)}{u_1(n)}}\right).$$

In order to give another type of local and global estimates of the approximation error we need to introduce the weighted K-functional of second order for $f \in C([0,\infty))$ defined by

$$K_{2,\varphi}(f,t) := \inf_{g' \in AC_{loc}([0,\infty))}(\|f-g\| + t\|\varphi^2 g''\|), \quad t > 0,$$

where $g' \in AC_{loc}([0,\infty))$ means that g is differentiable and g' is absolutely continous in every compact $[a,b] \subset [0,\infty)$.

Let $x \ge 0$ be fixed and $g : [0,\infty) \to \mathbb{R}$ be arbitrary such that $g' \in AC_{loc}([0,\infty))$. Starting from Taylor's expansion

$$g(u) = g(x) + g'(x)(u-x) + \int_x^u g''(t)(u-t)dt, \quad u \ge 0,$$

and knowing that $L_{n,\lambda}$ reproduces linear functions, we have

$$(L_{n,\lambda}g)(x) - g(x) = L_{n,\lambda}\left(\int_{xe_0}^{e_1} g''(t)(e_1-t)dt, x\right).$$

We consider $\varphi : [0,\infty) \to \mathbb{R}$ an admissible step weighted function such that φ^2 is concave. For every $t = (1-\eta)u + \eta x$, $\eta \in [0,1]$, we get

$$\varphi^2(t) = \varphi^2((1-\eta)u + \eta x) \ge (1-\eta)\varphi^2(u) + \eta\varphi^2(x) \ge \eta\varphi^2(x)$$

and consequently

$$\frac{|t-u|}{\varphi^2(t)} = \frac{\eta|x-u|}{\varphi^2(t)} \le \frac{|x-u|}{\varphi^2(x)}.$$

It turns out that

$$\left|\int_x^u g''(t)(u-t)dt\right| \le \|\varphi^2 g''\| \left|\int_x^u \frac{|t-u|}{\varphi^2(t)}dt\right| \le$$

$$\le \|\varphi^2 g''\| \left|\int_x^u \frac{|x-u|}{\varphi^2(x)}dt\right| = \|\varphi^2 g''\| \frac{(x-u)^2}{\varphi^2(x)}.$$

Applying the linear and positive operator $L_{n,\lambda}$ we have

$$L_{n,\lambda}\left(\int_{xe_0}^{e_1} g''(t)(e_1-t)dt, x\right) \le \|\varphi^2 g''\| \frac{\tau_2(L_{n,\lambda},x)}{\varphi^2(x)},$$

and further

$$|(L_{n,\lambda}f)(x) - f(x)| \le |L_{n,\lambda}(f-g,x)| + |g(x)-f(x)| + |(L_{n,\lambda}g)(x) - g(x)| \le$$

$$\le 2\|f-g\| + \|\varphi^2 g''\| \frac{\tau_2(L_{n,\lambda},x)}{\varphi^2(x)}.$$

Now we take the infimum over all g with $g' \in AC_{loc}([0,\infty))$ and we get

$$|(L_{n,\lambda}f)(x) - f(x)| \le 2K_{2,\varphi}\left(f, \frac{\tau_2(L_{n,\lambda},x)}{\varphi^2(x)}\right).$$

On the other hand it is well-known that $K_{2,\varphi}(f,t^2)$ functional and Ditzian-Totik modulus of smoothness of second order $\omega_{2,\varphi}(f;t)_\infty$ are equivalent [7]. We recall

$$\omega_{2;\varphi}(f,t)_\infty = \sup_{0\le h\le t}\ \sup_{x\pm h\varphi(x)\ge 0} |f(x-\varphi(x)h) - 2f(x) + f(x+\varphi(x)h)|.$$

With the help of (11), by using the above results we get the following pointwise approximation.

Theorem 4. *Let $L_{n,\lambda}$ be defined by (8). If $\varphi : [0,\infty) \to \mathbb{R}$ is an admissible step weight function with φ^2 concave then*

$$|(L_{n,\lambda}f)(x) - f(x)| \le 2K_{2,\varphi}\left(f, \frac{\lambda(x)\widetilde{\psi}_n(x)}{u_1(n)\varphi^2(x)}\right)$$

and

$$|(L_{n,\lambda}f)(x) - f(x)| \le C_\varphi \omega_{2,\varphi}\left(f; \frac{1}{\varphi(x)}\sqrt{\frac{\lambda(x)\widetilde{\psi}_n(x)}{u_1(n)}}\right)_\infty$$

hold true for every $x \ge 0$. Here C_φ is a constant independent of f and n.

Remark. On the light of this theorem for every $\alpha \in (0,2]$ we have the following implication

$$\omega_{2,\varphi}(f;t)_\infty = \mathcal{O}(t^\alpha)(t \to 0^+) \Rightarrow |(L_{n,\lambda}f)(x) - f(x)| \le C_\varphi \left(\frac{\lambda(x)\widetilde{\psi}_n(x)}{\varphi(x)} \right)^{\alpha/2} \quad (n \to \infty).$$

Returning to Example A, for Altomare's operator $A_{n,\lambda}$ we can choose $\varphi(x) = \sqrt{x}$, $x \ge 0$, and we are able to infer

$$|(A_{n,\lambda}f)(x) - f(x)| \le 2K_{2,\varphi}\left(f, \frac{\lambda(x)}{n}\right),$$

and

$$|(A_{n,\lambda}f)(x) - f(x)| \le C_\varphi \omega_{2,\varphi}\left(f; \sqrt{\lambda(x)}n^{-1/2}\right)_\infty, \quad x \ge 0.$$

Because $\lambda \le 1$ and $\omega_{2,\varphi}(f;\cdot)_\infty$ is an increasing function, we reobtain the global estimation due to Altomare, (see [2], Theorem 2.2.(4))

$$\|A_{n,\lambda}f - f\| \le C_\varphi \omega_{2,\varphi}(f; n^{-1/2})_\infty.$$

Unfortunately, this theorem cannot be applied for Example B because in this case the step weight function $\varphi^2(x) = x^2 + x$ is not concave.

Further on, we study the particular case $\lambda(x) = c$ (c - constant), $x \ge 0$. As it will turn out below, under some additional conditions, our operators $L_{n,c}$ leave invariant the classes of monotone functions as well as of convex and Lipschitz continuous functions.

First of all we recall that a function $h : I \to \mathbb{R}$ is said to be Lipschitz continuous of order μ, $\mu \in (0,1]$, if there exists a constant $A \ge 0$ such that

$$|h(x) - h(y)| \le A|x-y|^\mu, \quad (\forall)\ (x,y) \in I \times I.$$

In this case we will write $h \in Lip_A\mu(I)$ or simply $h \in Lip_A\mu$.

Now we can state and prove the following result.

Theorem 5. *Let $L_{n,c}$ be defined by (8), where $c \in [0,1]$ is a constant. Also we assume that the hypotheses of Theorem 1 are fulfilled.*

(i) The function f is increasing if and only if $L_{n,c}f$ is increasing for each $n \in \mathbb{N}$.

(ii) If the operator $l_n^{(2)}$ preserves the convexity for any $n \in \mathbb{N}$, then f is convex if and only if $L_{n,c}f$ is convex for each $n \in \mathbb{N}$.

(iii) If $f \in Lip_A\mu$ and $l_p^{(2)} f_{n,p,x} \in Lip_{A'(n,p)}\mu$ for every $(n,p) \in \mathbb{N} \times I_n$ then

$$L_{n,c}f \in Lip_{A''}\mu \quad \textit{where} \quad A'' = A + \sum_{p\in I_n} \lambda_{n,p}(c)A'(n,p).$$

The function $f_{n,p,x}$ is defined by (9).

Proof. (i) We consider an increasing function f. Let $n \in \mathbb{N}$, $x \geq 0$, $y \geq 0$ be fixed such that $x < y$. From (9) it is clear that $f_{n,p,x} \leq f_{n,p,y}$ and $l_p^{(2)} f_{n,p,x} \leq l_p^{(2)} f_{n,p,y}$ for every $p \in I_n$. We have used the fact that $l_p^{(2)}$, being linear and positive, is monotone. Simultaneously, the function $f_{n,p,x}$ is increasing and by using (10) we can deduce successively

$$(L_{n,c}f)(x) = \sum_{p\in I_n} \lambda_{n,p}(c)(l_p^{(2)} f_{n,p,x})(x) \leq \sum_{p\in I_n} \lambda_{n,p}(c)(l_p^{(2)} f_{n,p,y})(x) \leq$$

$$\leq \sum_{p\in I_n} \lambda_{n,p}(c)(l_p^{(2)} f_{n,p,y})(y) = (L_{n,c}f)(y),$$

in other words $L_{n,c}f$ is increasing. We have also used the positivity of the functions $\lambda_{n,p}$, $(n,p) \in \mathbb{N} \times I_n$.

Under the hypothesis that $L_{n,c}f$ is increasing for every n natural, the converse implication follows by using Theorem 2.

We point out that for the operators $A_{n,\lambda}$ (Example A) a similar result was obtained by Ingrid Carbone, see [6] Proposition 2.2.

(ii) The proof of this statement follows the same steps like those established in the proof of Theorem 2.3 in [6], so we omit it.

(iii) By using (10) we have

$$|(L_{n,c}f)(y) - (L_{n,c}f)(x)| \leq \sum_{p\in I_n} \lambda_{n,p}(c)\{|l_p^{(2)} f_{n,p,y}(y) - l_p^{(2)} f_{n,p,y}(x)| +$$

$$+ |l_p^{(2)} f_{n,p,y}(x) - l_p^{(2)} f_{n,p,x}(x)|\}.$$

On the other hand, since $f \in Lip_A\mu$ and $x_{n,p} \in [0,1]$ for every $(n,p) \in \mathbb{N} \times I_n$, we have

$$|(l_p^{(2)} f_{n,p,y} - l_p^{(2)} f_{n,p,x})(x)| \leq \sum_{k\in J_p} \mu_{p,k}(x)|f(x_{n,p}x_{p,k} + (1 - x_{n,p})y) -$$

$$- f(x_{n,p}x_{p,k} + (1 - x_{n,p})x| \leq \sum_{k\in J_p} \mu_{p,k}(x)A|1 - x_{n,p}|^\mu |y - x|^\mu \leq A|y - x|^\mu.$$

By using the above inequalities and knowing that $l_p^{(2)} f_{n,p,y} \in Lip_{A'(n,p)}\mu$ we obtain

$$|(L_{n,c}f)(y) - (L_{n,c}f)(x)| \leq \left(\sum_{p \in I_n} \lambda_{n,p}(c) A'(n,p) + A \right) |y-x|^{\mu},$$

and the conclusion follows.

Remark. We return to the Examples A, B. It is known that both Favard-Szasz and Baskakov operators preserve Lipschitz constants. By using probabilistic tools, more exactly the splitting property of a random vector, M.K. Khan and M.A. Peters [8] have obtained very elegantly the mentioned results. We have: if $f \in Lip_A \mu$ then $f_{n,p,x} \in Lip_{Ax^{\mu}_{n,p}}\mu$. Taking all these facts into account, both for Example A and B we obtain

$$A'' = A + \sum_{p=0}^{n} A \binom{n}{p} c^p (1-c)^{n-p} (p/n)^{\mu} = A(1 + (b_n e_1^{\mu})(c)).$$

This way, for Example A we reobtain Theorem 3.3 in [6].

4. An asymptotic formula

The aim of this section is to establish a Voronovskaja-type formula. This result represents the answer in a certain direction regarding to the speed with which $L_{n,\lambda} f$ tends to f. Based on the approach of P.C. Sikkema ([9], page 328) we will denote by $H^{(2)}([0,\infty), \xi)$ the set of all real functions f which are defined on $[0,\infty)$ and possess the following three properties:

(i) f is two times differentiable at $x = \xi \geq 0$,

(ii) f is bounded on every subinterval $[a,b] \subset [0,\infty)$,

(iii) $f(x) = \mathcal{O}(x^2)$, $x \to \infty$.

Our operator $L_{n,\lambda}$ is well defined for every $f \in H^{(2)}([0,\infty), \xi)$. Let all the operators $L_{n,\lambda}$ be applicable to $(e_1 - \xi)^3$ and to $(e_1 - \xi)^4$. In addition, we assume that the following further requirements are fulfilled.

In relation (7) we can separate n from x; so the functions $\overline{\psi}$ and $\overline{u}_1$ exist such that

$$\widetilde{\psi}_n(x)/u_1(n) = \overline{\psi}(x)/\overline{u}_1(n), \quad \text{for every} \quad x \geq 0, \tag{12}$$

and

$$\tau_4(L_{n,\lambda}, \xi) = o\left(\frac{1}{\overline{u}_1(n)}\right) \quad (n \to \infty). \tag{13}$$

Since $f \in H^{(2)}([0,\infty),\xi)$ we can write for every $t \geq 0$

$$f(t) = f(\xi) + (t-\xi)f'(\xi) + \frac{(t-\xi)^2}{2}f''(\xi) + (t-\xi)^2 r(t-\xi), \tag{14}$$

where r is bounded on the domain and $\lim_{u\to 0} r(u) = 0$. Taking $r(0) = 0$, it becomes continuous at $u = 0$. Consequently for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that

$$|r(t-\xi)| < \varepsilon \text{ for each } t \in \mathbb{R} \text{ with } |t-\xi| < \eta_\varepsilon.$$

From the boundedness of r it follows that there exists a constant $M > 0$ such that $|r(t-\xi)| \leq M$ for all t. Using the above relations we can write

$$|r(t-\xi)| < \varepsilon + M s_{\xi,\eta_\varepsilon}(t), \quad t \in \mathbb{R}, \tag{15}$$

where $s_{\xi,\eta_\varepsilon}(t) = 0$ if $|t-\xi| < \eta_\varepsilon$ and $s_{\xi,\eta_\varepsilon}(t) = 1$ if $|t-\xi| \geq \eta_\varepsilon$. Since $L_{n,\lambda}$ is a linear and positive operator, the identity (14) implies

$$(L_{n,\lambda}f)(\xi) = f(\xi)(L_{n,\lambda}e_0)(\xi) + f'(\xi)\tau_1(L_{n,\lambda},\xi) +$$
$$+\frac{f''(\xi)}{2}\tau_2(L_{n,\lambda},\xi) + L_{n,\lambda}((\cdot-\xi)^2 r(\cdot-\xi),\xi),$$

and by using (11), (12), (15) we get

$$(L_{n,\lambda}f)(\xi) - f(\xi) = \frac{1}{2}f''(\xi)\lambda(\xi)\frac{\overline{\psi}(\xi)}{\overline{u}_1(n)} + \frac{\varepsilon\lambda(\xi)\overline{\psi}(\xi)}{\overline{u}_1(n)} + ML_{n,\lambda}((\cdot-\xi)^2 s_{\xi,\eta_\varepsilon},\xi).$$

The last term is non zero for those real t satisfying $|t-\xi| \geq \eta_\varepsilon$ which implies $1 \leq (t-\xi)^2\eta_\varepsilon^{-2}$. The monotonicity of $L_{n,\lambda}$ together with (13) guarantees that

$$ML_{n,\lambda}((\cdot-\xi)^2 s_{\xi,\eta_\varepsilon},\xi) \leq M\eta_\varepsilon^{-2}L_{n,\lambda}((\cdot-\xi)^4,\xi) = M'o\left(\frac{1}{\overline{u}_1(n)}\right) \quad (n\to\infty),$$

where M' is a constant. We have obtained

$$\left|(L_{n,\lambda}f)(\xi) - f(\xi) - \frac{1}{2}f''(\xi)\lambda(\xi)\frac{\overline{\psi}(\xi)}{\overline{u}_1(n)}\right| \leq \varepsilon\lambda(\xi)\frac{\overline{\psi}(\xi)}{\overline{u}_1(n)} + M'o\left(\frac{1}{\overline{u}_1(n)}\right) \quad (n\to\infty),$$

where $\varepsilon > 0$ is arbitrary.

This relation leads us to the following result.

Theorem 6. *Let $L_{n,\lambda}$ be defined by (8) such that the assumptions (12) and (13) hold. If $f \in H^{(2)}([0,\infty),\xi)$ then one has*

$$\lim_{n\to\infty} \overline{u}_1(n)((L_{n,\lambda}f)(\xi) - f(\xi)) = \frac{1}{2}f''(\xi)\lambda(\xi)\overline{\psi}(\xi).$$

Remark. For the Altomare's operators $M_{n,\lambda}$, $B_{n,\lambda}$ we identify $\overline{\psi}(x) = x$, $\overline{u}_1(n) = n$ respectively $\overline{\psi}(x) = x^2 + x$, $\overline{u}_1(n) = n$ where $x \geq 0$ and $n \in \mathbb{N}$. Actually, for these operators the authors have established the asymptotic behavior of the remainders as follows

(i) $\lim_{n\to\infty} n((M_{n,\lambda}f)(x) - f(x)) = \frac{x\lambda(x)}{2} f''(x)$ uniformly on $[0,\infty)$, for every $f \in E_\infty \cap C^2([0,\infty))$ such that f'' is bounded and uniformly continuous on $[0,\infty)$, see [2, Eq.(3.7)].

(ii) $\lim_{n\to\infty} n((B_{n,\lambda}f)(x) - f(x)) = \frac{x(x+1)\lambda(x)}{2} f''(x)$ in E_3 and hence in E_m for every $m \geq 3$, where $f \in C^2([0,\infty))$ such that $f'' \in C_B([0,\infty))$, see [4, Eq.(5.1)].

We conclude this paper by noticing that this stage of the research opens new perspectives of investigation. Theorem 6 can help to solve a generation problem for a certain differential operator. For a given function $\alpha \in C([0,\infty))$, the problem consists in constructing a sequence $(L_n)_{n\geq 1}$ of linear and positive operators defined on a Banach space $\mathcal{B}$ verifying the condition

$$\lim_{n\to\infty} n(L_n f - f) = \alpha f'' \text{ in } \mathcal{B},$$

for every function f defined on a subspace $\mathcal{F} \subset \mathcal{B} \cap C^2([0,\infty))$ such that $\alpha f'' \in \mathcal{B}$.

For more details on this problem [1] is a valuable work that we quote.

Final remark. Thanks to Prof. Altomare I have been able to get a copy of the paper "Some remarks on a general construction of approximation processes" by Lorenzo D'Ambrosio and Elisabetta Mangino, submitted by the authors for the same Proceedings.

In their article they start with an arbitrary positive approximation process and they consider its modification by using the Altomare type transformation. In some concerns, these new operators include my operators as a particular case but on the other hand the authors consider only particular knots rather than general knots as found in the present paper. Comparing the papers, the reader can conclude that the results are different because of the different situations.

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