



AN INEQUALITY WHICH ARISES IN THE ABSENCE OF THE MOUNTAIN PASS GEOMETRY

RADU PRECUP

r.precup@math.ubbcluj.ro

BABEȘ-BOLYAI UNIVERSITY

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

3400 CLUJ, ROMANIA.

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ABSTRACT. An integral inequality is deduced from the negation of the geometrical condition in the bounded mountain pass theorem of Schechter, in a situation where this theorem does not apply. Also two localization results of non-zero solutions to a superlinear boundary value problem are established.

Key words and phrases: Integral inequality, Mountain pass theorem, Laplacean, Boundary value problem, Sobolev space.

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1. INTRODUCTION AND PRELIMINARIES

Let $p \in [2, \infty)$, Ω be a bounded domain of \mathbb{R}^n , and let

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}.$$

We consider the quantity

$$(1.1) \quad \lambda_{p-1} = \inf \left\{ \frac{\int_{\Omega} |u|^{p-2} |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^{\frac{p^2}{2}} dx \right)^{\frac{2}{p}}} : u \in C_0^1(\overline{\Omega}) \setminus \{0\} \right\}.$$

For $p = 2$, λ_1 is the first eigenvalue of the Laplacean $-\Delta$ under the Dirichlet boundary condition, and $\frac{1}{\lambda_1}$ represents the best constant in the Wirtinger-Poincaré inequality (see [7] for the elementary Wirtinger's inequality, [8] for its extension to functions with values in an arbitrary Banach space, and [11] for Poincaré's inequality). For $p > 2$ and $n = 1$ this quantity arises in the study of compactness properties for integral operators on spaces of vector-valued functions

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The author is indebted to Professor L. Gajek for bringing reference [3] to his attention, for giving the exact value of λ_{p-1} for $n = 1$ as shows Remark 1.1, and for stimulating an improved and much more general version of this paper.

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(see [13]). Let us also note that quantities alike (1.1) arising from physics were studied by Pólya [9] and Pólya and Szegő [10] (see also [6] and its references for more recent advances).

Remark 1.1. For $n = 1$ and $\Omega = (0, T)$, where $0 < T < \infty$, the exact value of λ_{p-1} can be obtained from a result of Gajek, Kałuszka and Lenic [3] in the following way. First by change of the integration variable one has

$$\begin{aligned} \lambda_{p-1} &= \inf \left\{ \frac{\int_0^T |u|^{p-2} u'^2 dx}{\left(\int_0^T |u|^{\frac{p-2}{2}} dx \right)^{\frac{2}{p}}} : u \in C_0^1 [0, T] \setminus \{0\} \right\} \\ &= T^{-(1+\frac{2}{p})} \inf \left\{ \frac{\int_0^1 |u|^{p-2} u'^2 ds}{\left(\int_0^1 |u|^{\frac{p-2}{2}} ds \right)^{\frac{2}{p}}} : u \in C_0^1 [0, 1] \setminus \{0\} \right\} \\ &= \left(T^{1+\frac{2}{p}} \sup \left\{ \left(\int_0^1 |u|^{\frac{p-2}{2}} ds \right)^{\frac{2}{p}} : u \in C_0^1 [0, 1], \int_0^1 |u|^{p-2} u'^2 ds = 1 \right\} \right)^{-1}. \end{aligned}$$

After substituting $v = \left(\frac{2}{p}\right) |u|^{\frac{p-2}{2}}$, we obtain

$$\lambda_{p-1} = \left(T^{1+\frac{2}{p}} \left(\frac{p}{2}\right)^2 \sup \left\{ \left(\int_0^1 |v|^p ds \right)^{\frac{2}{p}} : v \in C_0^1 [0, 1], \int_0^1 v'^2 ds = 1 \right\} \right)^{-1}.$$

Notice that

$$\begin{aligned} \sup \left\{ \int_0^1 |v|^p ds : v \in C_0^1 [0, 1], \int_0^1 v'^2 ds = 1 \right\} \\ = \sup \left\{ \int_0^1 |v|^p ds : v \in C_0^1 [0, 1], \int_0^1 v'^2 ds \leq 1 \right\}. \end{aligned}$$

Now the sup in the right hand side is the quantity denoted by b in [3] and is given by

$$b = \left(\frac{p(p+2)}{\pi} \right)^{\frac{p}{2}} \frac{2^{1-p}}{p+2} \left(\frac{\Gamma\left(\frac{1}{p} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)} \right)^p.$$

As a result

$$\lambda_{p-1} = \left\{ T^{1+\frac{2}{p}} p^2 \left(\frac{p(p+2)}{\pi} \right)^{\frac{p}{2}} \frac{2^{1-p}}{p+2} \left(\frac{\Gamma\left(\frac{1}{p} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)} \right)^p \right\}^{-1}.$$

In this paper we are interested in finding upper and lower estimations for λ_{p-1} . An upper bound is obtained from the negation of the geometrical condition in Schechter's mountain pass theorem, in a situation where this theorem does not apply. To our knowledge, this is the first time that a bounded mountain pass theorem is used in order to obtain inequalities. Two localization results of non-zero solutions to a superlinear elliptic boundary value problem are also established in terms of λ_{p-1} .

1.1. Basic Results from the Theory of Linear Elliptic Equations. Here we recall some well-known results from the theory of linear elliptic boundary value problems.

(P1) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 -boundary. The Laplacean $-\Delta$ is a self-adjoint operator on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ (see [11, Theorem 3.33], or [4]). It can be regarded as a continuous operator from $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ to $L^q(\Omega)$ for each $q \in (1, \infty)$. Moreover, $-\Delta$ is invertible and $K := (-\Delta)^{-1}$ is a continuous operator from $L^q(\Omega)$ into $W^{2,q}(\Omega)$ (see [2, Theorem 9.32]). Also, K considered in $L^2(\Omega)$ is a positive self-adjoint operator.

(P2) (Sobolev embedding theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $k \in \mathbb{N}$, $1 \leq q < \infty$. Then the following holds:

(1⁰) If $kq < n$, we have

$$(1.2) \quad W^{k,q}(\Omega) \subset L^r(\Omega)$$

and the embedding is continuous for $r \in \left[1, \frac{nq}{n-kq}\right]$; the embedding is compact if

$$r \in \left[1, \frac{nq}{n-kq}\right).$$

(2⁰) If $kq = n$, then (1.2) holds with compact embedding for $r \in [1, \infty)$.

(3⁰) If $0 \leq m < k - \frac{n}{q} < m + 1$, we have

$$(1.3) \quad W^{k,q}(\Omega) \subset C^{m,\alpha}(\overline{\Omega})$$

and the embedding is continuous for $0 \leq \alpha \leq k - m - \frac{n}{q}$; the embedding is compact if $\alpha < k - m - \frac{n}{q}$.

The above results are valid for $W_0^{k,q}(\Omega)$ -spaces on arbitrary bounded domains Ω (see [1], [15, p 213] or [2, pp. 168-169]).

(P3) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 -boundary. Let $p_0 = \frac{2n}{n-2}$ if $n \geq 3$ and p_0 be any number of $(2, \infty)$ if $n = 1$ or $n = 2$. Let q_0 be the conjugate number of p_0 . Clearly, $p_0 \in (2, \infty)$ and $q_0 \in (1, 2)$. From (P1), (P2) we have that K has the following properties:

(a) $K : L^q(\Omega) \rightarrow L^p(\Omega)$ for every $q \in [q_0, 2]$, $\frac{1}{p} + \frac{1}{q} = 1$;

(b) K is continuous from $L^q(\Omega)$ to $L^p(\Omega)$ for every $q \in [q_0, 2]$, $\frac{1}{p} + \frac{1}{q} = 1$;

(c) the operator K considered in $L^2(\Omega)$ is a positive self-adjoint operator.

Indeed, K is continuous from $L^q(\Omega)$ into $W^{2,q}(\Omega)$. On the other hand $W^{2,q}(\Omega) \subset L^p(\Omega)$ with continuous embedding. This is clear if $q \geq \frac{n}{2}$. For $q < \frac{n}{2}$ and $\frac{1}{p} + \frac{1}{q} = 1$, observe that

$$p \leq \frac{2n}{n-2} \iff p \leq \frac{nq}{n-2q}.$$

According to [5, pp. 51-56], the properties (a)-(c) are sufficient for that the operator K considered from $L^q(\Omega)$ to $L^p(\Omega)$, where $p \in (2, p_0)$ and $\frac{1}{p} + \frac{1}{q} = 1$, admits a representation in the form

$$K = AA^*,$$

where

$$A : L^2(\Omega) \rightarrow L^p(\Omega), \quad Av = K^{\frac{1}{2}}v$$

and

$$A^* : L^q(\Omega) \rightarrow L^2(\Omega)$$

is the adjoint of A . Here $K^{\frac{1}{2}}$ is the square root of K considered as an operator acting from $L^2(\Omega)$ into $L^2(\Omega)$.

Throughout, by $|\cdot|_p$ we shall mean the usual norm on $L^p(\Omega)$ and by $|A|$ we shall mean

$$|A| = \sup \left\{ |Av|_p : v \in L^2(\Omega), |v|_2 = 1 \right\}.$$

1.2. Schechter's Mountain Pass Theorem. Now we present the main tool in this paper, Schechter's mountain pass theorem [14]. Let X be a real Hilbert space with inner product (\cdot, \cdot) and norm $|\cdot|$, $B_R = \{v \in X : |v| \leq R\}$ the closed ball of X of radius R , $E : X \rightarrow \mathbb{R}$ a C^1 -functional on X , $v_0, v_1 \in X$ and $r > 0$ with

$$|v_0| < r < |v_1| \leq R.$$

Let

$$\Phi = \{\varphi \in C([0, 1]; B_R) : \varphi(0) = v_0, \varphi(1) = v_1\},$$

$$c_R = \inf_{\varphi \in \Phi} \max_{t \in [0, 1]} E(\varphi(t))$$

and let

$$\mathcal{K}_{c_R} = \{v \in B_R : E(v) = c_R, E'(v) = 0\}$$

be the set of critical points of E in B_R at level c_R .

We say that E satisfies the Schechter-Palais-Smale condition on B_R ((S-P-S) $_R$ -condition) if

$$(v_k) \subset B_R, E(v_k) \text{ - bounded, } (E'(v_k), v_k) \rightarrow \nu \leq 0,$$

$$E'(v_k) - \frac{(E'(v_k), v_k)}{|v_k|^2} v_k \rightarrow 0$$

$$\implies (v_k) \text{ has a convergent subsequence.}$$

Theorem 1.2 (Schechter). *Suppose*

(i): E satisfies (S-P-S) $_R$ -condition;

(ii): there exists a constant C with $-(E'(v), v) \leq C$ for $|v| = R$;

(iii): $v \neq \lambda(v - E'(v))$ for $|v| = R$ and $\lambda \in (0, 1)$;

(iv): $\max\{E(v_0), E(v_1)\} \leq \inf\{E(v) : |v| = r\}$.

Then $\mathcal{K}_{c_R} \setminus \{v_0, v_1\} \neq \emptyset$.

We note that by the mountain pass geometry of a functional E we mean the geometrical condition (iv) in Theorem 1.2.

2. MAIN RESULTS

We first obtain a lower bound for all non-zero solutions of the superlinear problem

$$(2.1) \quad \begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 2.1. *Let Ω be a bounded domain of \mathbb{R}^n with C^2 -boundary, let $p \in (2, \frac{2n}{n-2})$ if $n \geq 3$ and $p \in (2, \infty)$ if $n = 1$ or $n = 2$, and let $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ is a non-zero solution of the problem (2.1), then the function $v = A^*(|u|^{p-2}u) = A^{-1}u$ satisfies the inequality*

$$(2.2) \quad |v|_2 \geq |A|^{-1} [(p-1) \lambda_{p-1}]^{\frac{1}{p-2}}.$$

Proof. Let us first prove that any solution of (2.1) belongs to $C^1(\overline{\Omega})$. For $n = 1$ this follows from (1.3) (choose $\alpha = 0$, $m = 1$ and $k = 2$). Suppose $n \geq 2$ and fix any number $q_0 > n(p - 1)$. If $q \geq \frac{n}{2}$, then (P2) guarantees $u \in L^{q_0}(\Omega)$. Assume $q < \frac{n}{2}$ and denote $q_1 = q$. Since $u \in W^{2,q_1}(\Omega)$ and $q_1 < \frac{n}{2}$, from (1.2) we have $u \in L^{q_1^*}(\Omega)$, where $q_1^* = \frac{nq_1}{n-2q_1}$. Then $|u|^{p-2}u \in L^{\frac{q_1^*}{p-1}}(\Omega)$. Let $q_2 = \frac{q_1^*}{p-1}$. Since $u = K(|u|^{p-2}u)$ and $|u|^{p-2}u \in L^{q_2}(\Omega)$, from (P1), we have that $u \in W^{2,q_2}(\Omega)$. If $q_2 \geq \frac{n}{2}$, as above $u \in L^{q_0}(\Omega)$; otherwise we continue this way. At the step j we find that

$$(2.3) \quad u \in W^{2,q_j}(\Omega), \quad q_j = \frac{q_{j-1}^*}{p-1}, \quad q_{j-1}^* = \frac{nq_{j-1}}{n-2q_{j-1}},$$

where $q_1, q_2, \dots, q_{j-1} < \frac{n}{2}$ ($j \geq 2$). We claim that there exists a j with $q_j \geq \frac{n}{2}$. To prove this, suppose the contrary, that is $q_j < \frac{n}{2}$ for every $j \geq 1$. Using $p < \frac{2n}{n-2}$ we can show by induction that the sequence (q_j) is increasing. Consequently, $q_j \rightarrow \bar{q} \in [q, \frac{n}{2}]$ as $j \rightarrow \infty$. Next, from (2.3) we obtain

$$q_j(n - 2q_{j-1})(p - 1) = nq_{j-1}.$$

Letting $j \rightarrow \infty$ this yields $\bar{q}(n - 2\bar{q})(p - 1) = n\bar{q}$ and so

$$\bar{q} = \frac{n(p - 2)}{2(p - 1)} \geq q = \frac{p}{p - 1}.$$

This implies $p \geq \frac{2n}{n-2}$, a contradiction. Thus our claim is proved. Therefore, $u \in L^{q_0}(\Omega)$. Furthermore $|u|^{p-2}u \in L^{q_0/(p-1)}(\Omega)$ and since $u = K(|u|^{p-2}u)$, we have $u \in W^{2,q_0/(p-1)}(\Omega)$. Since $\frac{q_0}{p-1} > n$, by (1.3) one has $W^{2,\frac{q_0}{p-1}}(\Omega) \subset C^1(\overline{\Omega})$ (choose $\alpha = 0$, $k = 2$, $m = 1$). Hence $u \in C^1(\overline{\Omega})$.

Let $\bar{u} = K(|u|^{p-1})$. Clearly, like u , $\bar{u} \in C^1(\overline{\Omega})$ and $\bar{u} = 0$ on $\partial\Omega$. By the weak maximum principle, we have

$$(2.4) \quad |u| \leq \bar{u} \text{ on } \overline{\Omega}.$$

Hence

$$-\Delta \bar{u} = |u|^{p-1} \leq |u|^{p-2} \bar{u}.$$

If we “multiply” by \bar{u}^{p-1} and “integrate” on Ω , we obtain

$$(2.5) \quad (p - 1) \int_{\Omega} \bar{u}^{p-2} |\nabla \bar{u}|^2 dx \leq \int_{\Omega} |u|^{p-2} \bar{u}^p dx.$$

Now Hölder’s inequality yields

$$(2.6) \quad \begin{aligned} \int_{\Omega} |u|^{p-2} \bar{u}^p dx &\leq \left(\int_{\Omega} \bar{u}^{\frac{p^2}{2}} dx \right)^{\frac{2}{p}} \left(\int_{\Omega} |u|^p dx \right)^{\frac{p-2}{p}} \\ &= |A(v)|_p^{p-2} \left(\int_{\Omega} \bar{u}^{\frac{p^2}{2}} dx \right)^{\frac{2}{p}}. \end{aligned}$$

Since $|Av|_p \leq |A||v|_2$ and by (2.4) one has $\bar{u} \neq 0$, from (2.5) and (2.6) we deduce that

$$(p - 1) \lambda_{p-1} \leq |A|^{p-2} |v|_2^{p-2}$$

that is (2.2). □

Our next result is the following inequality.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 -boundary. Then for every $p > 2$ one has the inequality*

$$(2.7) \quad \lambda_{p-1} \leq \frac{1}{(p-1)|A|^2}.$$

Proof. We consider the functional $E : L^2(\Omega) \rightarrow \mathbb{R}$, given by

$$(2.8) \quad E(v) = \int_{\Omega} \left(\frac{1}{2} |v(x)|^2 - \frac{1}{p} |(Av)(x)|^p \right) dx.$$

Clearly, we have

$$E(v) = \frac{|v|_2^2}{2} - \frac{|Av|_p^p}{p}.$$

For every $v, w \in L^2(\Omega)$, it is easy to compute

$$(E'(v), w) = \lim_{\lambda \rightarrow 0} \lambda^{-1} (E(v + \lambda w) - E(v))$$

and find

$$(E'(v), w) = (v - A^*FAv, w),$$

where

$$F : L^p(\Omega) \rightarrow L^q(\Omega), \quad F(u) = |u|^{p-2}u.$$

Hence

$$E'(v) = v - A^*FAv.$$

Notice if u is a solution of (2.1) then $v = A^*(|u|^{p-2}u) = A^{-1}u$ is a critical point of the functional (2.8). Conversely, if v is a critical point of the functional (2.8), then $u = Av$ is a solution of (2.1).

Our plan is as follows: we show that for every $R < R_0$, where

$$(2.9) \quad R_0 = |A|^{-1} [(p-1)\lambda_{p-1}]^{\frac{1}{p-2}}$$

(of course here we assume $\lambda_{p-1} > 0$, (2.7) being trivial if $\lambda_{p-1} = 0$), $v_0 = 0$ is the unique critical point of E in $B_R = \{v \in L^2(\Omega) : |v|_2 \leq R\}$ and that the hypotheses (i)-(iii) in Theorem 1.2 hold. Consequently, there exist no v_1 and r with $0 < r < |v_1|_2 \leq R$ such that the geometrical condition (iv) is satisfied. As a result we obtain (2.7).

(a) The (S-P-S) $_R$ -condition is satisfied for every $R > 0$. Indeed, let (v_k) be any sequence of functions in B_R with

$$(2.10) \quad (E'(v_k), v_k) \rightarrow \nu \leq 0, \quad E'(v_k) - \beta(v_k)v_k \rightarrow 0,$$

where $\beta(v_k) = \frac{(E'(v_k), v_k)}{|v_k|_2^2}$. Passing if necessarily to a subsequence, we may suppose that $|v_k|_2 \rightarrow d$ for some $d \in [0, R]$. If $d = 0$ we are done. So assume $d > 0$. Denote $w_k = E'(v_k) - \beta(v_k)v_k$. We have $w_k = (1 - \beta(v_k))v_k - A^*FAv_k$. Hence

$$(2.11) \quad v_k = (1 - \beta(v_k))^{-1} (w_k - A^*FAv_k)$$

and so

$$(2.12) \quad Av_k = (1 - \beta(v_k))^{-1} (Aw_k - KFAv_k).$$

Notice $K(L^q(\Omega)) \subset W^{2,q}(\Omega)$ and the embedding of $W^{2,q}(\Omega)$ into $L^p(\Omega)$ is compact. Indeed, from $p \in (2, \frac{2n}{n-2})$ and $\frac{1}{p} + \frac{1}{q} = 1$, we easily see that $p \in (2, \frac{nq}{n-2q})$ when $q < \frac{n}{2}$.

Hence the compact embedding is guaranteed by (P2). As a result, we may suppose that (at least for a subsequence) $(KFAv_k)$ is convergent. In addition, by (2.10), we have

$$Aw_k \rightarrow 0, \quad (1 - \beta(v_k))^{-1} \rightarrow \left(1 - \frac{\nu}{d^2}\right)^{-1} \in (0, 1].$$

Then, from (2.12), we find that (at least for a subsequence) (Av_k) is convergent. Finally (2.11) guarantees that the corresponding subsequence of (v_k) is convergent.

(b) For each $R > 0$, there exists a constant C_R such that

$$-(E'(v), v) \leq C_R \text{ for all } v \in L^2(\Omega) \text{ with } |v|_2 = R.$$

Indeed, if $|v|_2 = R$, then

$$\begin{aligned} -(E'(v), v) &= -|v|_2^2 + (A^*FAv, v) \\ &= -|v|_2^2 + (FAv, Av) \\ &= -|v|_2^2 + |Av|_p^p \\ &\leq -|v|_2^2 + |A|^p |v|_2^p \\ &= -R^2 + |A|^p R^p \\ &=: C_R. \end{aligned}$$

(c) Zero is the unique critical point of E with $|v|_2 < R_0$ (here R_0 is given by (2.9)). Indeed, if $v \in L^2(\Omega)$ is a non-zero critical point of E , then $v = A^*FAv$ and so $Av = KFAv$. Hence $u = Av$ is a non-zero solution of problem (2.1). Therefore, according to Theorem 2.1, $|v|_2 \geq R_0$.

(d) The Leray-Schauder boundary condition (iii) holds for every $R < R_0$. To prove this suppose the contrary. Then there exists a $v \in L^2(\Omega)$ with $|v|_2 = R$ and a $\lambda \in (0, 1)$ with $v = \lambda(v - E'(v))$, i.e. $v = \lambda A^*FAv$. It is easily seen that the function $\bar{v} = \lambda^{1/(p-2)}v$ satisfies $\bar{v} = A^*FA\bar{v}$, i.e. \bar{v} is a critical point of E with $|\bar{v}|_2 < R_0$. According to the conclusion of step (c), $\bar{v} = 0$ and so $v = 0$, a contradiction.

(e) Proof of (2.7). Let

$$r = |A|^{-\frac{p}{p-2}}.$$

Obviously, (2.7) can be written as $r \geq R_0$. To prove it, we shall assume the contrary, i.e. $r < R_0$. Choose any $R \in (r, R_0)$, $\lambda \in (r, R]$ and $\varepsilon > 0$ sufficiently small so that

$$(2.13) \quad \phi(\lambda) + p^{-1}\lambda^p\varepsilon \leq \phi(r),$$

where

$$\phi(\sigma) = \frac{\sigma^2}{2} - p^{-1}\sigma^p |A|^p \quad (\sigma \geq 0).$$

Notice r is the maximum point of ϕ , ϕ is increasing on $[0, r]$ and decreasing on $[r, \infty)$. Now we choose a function $v_2 \in L^2(\Omega)$ with

$$|v_2|_2 = 1 \quad \text{and} \quad |Av_2|_p^p \geq |A|^p - \varepsilon.$$

We claim that condition (iv) in Theorem 1.2 holds for $v_0 = 0$ and $v_1 = \lambda v_2$. Indeed

$$(2.14) \quad \begin{aligned} E(v_1) &= E(\lambda v_2) \\ &= \frac{\lambda^2}{2} - p^{-1}\lambda^p |Av_2|_p^p \\ &\leq \frac{\lambda^2}{2} - p^{-1}\lambda^p |A|^p + p^{-1}\lambda^p\varepsilon \\ &= \phi(\lambda) + p^{-1}\lambda^p\varepsilon. \end{aligned}$$

Also, for every $v \in L^2(\Omega)$ with $|v|_2 = r$, we have

$$(2.15) \quad E(v) = \frac{r^2}{2} - p^{-1}r^p |A(r^{-1}v)|_p^p \geq \frac{r^2}{2} - p^{-1}r^p |A|^p = \phi(r).$$

Now (2.13), (2.14) and (2.15) guarantee (iv). From Theorem 1.2 it follows that E has a non-zero critical point in the closed ball B_R of $L^2(\Omega)$. This contradiction to the conclusion at step (c) proves (2.7). □

We note that Theorems 2.1-2.2 were previously announced in [12].

The next inequality of Poincaré type shows that $\lambda_{p-1} > 0$ for $p \in [2, \frac{2n}{n-2}]$ if $n \geq 3$ and for $p \in [2, \infty)$ if $n = 2$. Moreover, its proof connects λ_{p-1} to the embedding constant of $W_0^{1,2}(\Omega)$ into $L^p(\Omega)$.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded open and let $p \in [2, \frac{2n}{n-2}]$ if $n \geq 3$, $p \in [2, \infty)$ for $n = 2$. Then there exists a constant $c > 0$ depending only on p and Ω , such that*

$$(2.16) \quad \left(\int_{\Omega} |u|^{\frac{p^2}{2}} dx \right)^{\frac{2}{p}} \leq c \int_{\Omega} |u|^{p-2} |\nabla u|^2 dx$$

for all $u \in C_0^1(\overline{\Omega})$.

Proof. According to (P2), we have $W_0^{1,2}(\Omega) \subset L^p(\Omega)$ with continuous embedding. Hence there exists a constant $c_0 > 0$ with

$$|v|_p \leq c_0 |v|_{W_0^{1,2}(\Omega)} \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

Here

$$|v|_{W_0^{1,2}(\Omega)} = \left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}}.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$, we may suppose that

$$c_0 = \sup \left\{ |v|_p : v \in C_0^\infty(\Omega), |v|_{W_0^{1,2}(\Omega)} = 1 \right\}.$$

The space $C_0^\infty(\Omega)$ is also dense in $C_0^1(\overline{\Omega})$, and so

$$\lambda_{p-1} = \left(\sup \left\{ \left(\int_{\Omega} |u|^{p^2/2} dx \right)^{\frac{2}{p}} : u \in C_0^\infty(\Omega), \int_{\Omega} |u|^{p-2} |\nabla u|^2 dx = 1 \right\} \right)^{-1}.$$

After substituting $v = \left(\frac{2}{p}\right) |u|^{\frac{p}{2}}$, we obtain

$$\begin{aligned} \lambda_{p-1} &= \left(\frac{2}{p}\right)^2 \left(\sup \left\{ |v|_p^2 : v \in C_0^\infty(\Omega), |v|_{W_0^{1,2}(\Omega)} = 1 \right\} \right)^{-1} \\ &= \left(\frac{2}{p c_0}\right)^2. \end{aligned}$$

Thus (2.16) holds with the smallest constant

$$c = \lambda_{p-1}^{-1} = \left(p \frac{c_0}{2}\right)^2. \quad \square$$

Finally we establish a localization result for a non-zero solution to the problem (2.1).

Theorem 2.4. Let Ω be a bounded domain of \mathbb{R}^n with C^2 -boundary and let $p \in (2, \frac{2n}{n-2})$ if $n \geq 3$ and $p \in (2, \infty)$ if $n = 1$ or $n = 2$. Then the problem (2.1) has a solution u with

$$(2.17) \quad |A|^{-1} [(p-1) \lambda_{p-1}]^{\frac{1}{p-2}} \leq |A^{-1}u|_2 \leq |A|^{-\frac{p}{p-2}}.$$

Proof. First notice the left inequality in (2.17) is true for all non-zero solutions of (2.1) according to Theorem 2.1.

Next we prove that for each $R > r = |A|^{-\frac{p}{p-2}}$, (2.1) has a solution u such that

$$(2.18) \quad |A^{-1}u|_2 \leq R.$$

Indeed, two cases are possible:

- (1) The Leray-Schauder boundary condition (iii) in Theorem 1.2 does not hold. Then, there are $v \in L^2(\Omega)$ and $\lambda \in (0, 1)$ such that $|v|_2 = R$ and $v = \lambda A^* F A v$. It is easy to see that the function $\bar{v} = \lambda^{1/(p-2)} v$ satisfies $\bar{v} = A^* F A \bar{v}$, i.e. \bar{v} is a critical point of E , and $0 < |\bar{v}|_2 < |v|_2 = R$. Hence $u := A \bar{v}$ is a solution of (2.1) and satisfies (2.18).
- (2) Condition (iii) in Theorem 1.2 holds. Then, as follows from the proof of Theorem 2.2, all the assumptions of Theorem 1.2 are satisfied. Now the existence of a solution u of (2.1) satisfying (2.18) is guaranteed by Theorem 1.2.

Finally, for each positive integer k we put $R = r + \frac{1}{k}$ to obtain a solution u_k with $|A^{-1}u_k|_2 \leq r + \frac{1}{k}$, and the result will follow via a limit argument. \square

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