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On the rate of convergence for semigroups and processes of Feller type

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Abstract

Based on the probabilistic theory, the paper contains local estimates of the rate of convergence for a contraction C_0-semigroup. Simultaneously a class of linear positive operators of Feller-Stancu type is introduced, and the local and global rate of convergence for continuous functions is studied.
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1 Introduction

Probabilistic methods have proved quite useful in the theory of approximation operators. The purpose of this note is twofold. The first part is concerned with C_0-semigroup of linear contractions, named $(T(t))_{t\geq 0}$. Motivated by the results of P.L. Butzer and L. Hahn [4], we present a large class of representation formulae in semigroup theory with the emphasis upon rates of convergence. The basic aspect is that the order of approximation is expressed in terms of the second modulus of smoothness. Our estimations are valid in the pointwise sense for each $t \geq 0$ and hold uniformly in any compact t-interval $[0, T]$. The probabilistic concepts have the advantage of supplying short proofs.

The last section is devoted to the creation of an approximation process $(\Lambda_n)_{n\geq 1}$ by manipulating sequences of independent random variables. Establishing the behaviour of our operators for the test functions of Korovkin type, we indicate the rate of convergence in the space of continuous function $C(I)$, $I \subset R$. Sufficient conditions are provided to guarantee that $\lim_{n\to\infty} \|\Lambda_n h - h\|_\infty = 0$ for every $h \in C(I)$, where $\|\cdot\|_\infty$ represents the usual sup-norm.

2 The order of approximation for a contraction C_0-semigroup

For the convenience of the reader we briefly recall the basic elements of the theory of C_0-semigroups. Let $\mathcal{E}$ be a real Banach space and $\mathcal{L}_b(\mathcal{E})$ be the space of all bounded linear operators from $\mathcal{E}$ into $\mathcal{E}$ endowed with the norm $\|\cdot\|$ defined by $\|S\| = \sup\limits_{\|f\|\leq 1} \|Sf\|$, where f belongs to $\mathcal{E}$.

A *one-parameter semigroup* of bounded linear operators on $\mathcal{E}$ is a family $(T(t))_{t\geq 0}$ of elements of $\mathcal{L}_b(\mathcal{E})$ such that

$$(i)\ T(0) = I_{\mathcal{E}}, \quad (ii)\ T(s+t) = T(s)T(t) \text{ for every } s,t \geq 0, \tag{1}$$

where $I_{\mathcal{E}}$ denotes the identity operator on $\mathcal{E}$.

A semigroup $(T(t))_{t\geq 0}$ on $\mathcal{E}$ is said to be *strongly continuous* if for every $t_0 \geq 0$ and $f \in \mathcal{E}$ one has

$$\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0. \tag{2}$$

Taking into account the second property in (1), relation (2) holds if and only if $\lim\limits_{t\to 0+} \|T(t)f - f\| = 0$ for every $f \in \mathcal{E}$. A strongly continuous semigroup is also called a *C_0-semigroup*.

A *contraction C_0-semigroup* is a C_0-semigroup $(T(t))_{t\geq 0}$ of linear contractions, which means that $\|T(t)\| \leq 1$ for every $t \geq 0$.

As usual we denote the *infinitesimal generator* of the semigroup $(T(t))_{t\geq 0}$ by A. This linear operator is defined on the linear subspace $D(A) := \{f \in \mathcal{E} |$ there exists $\lim\limits_{t\to 0+} (T(t)f - f)t^{-1} \in \mathcal{E}\}$ as follows

$$Af := \lim_{t\to 0+} \frac{T(t)f - f}{t} \text{ for every } f \in D(A). \tag{3}$$

Note that if the semigroup $(T(t))_{t\geq 0}$ verifies $\lim\limits_{t\to 0+} \|T(t) - I_{\mathcal{E}}\| = 0$ then $D(A) = \mathcal{E}$ and A is bounded. More details about this rich mathematical theory can be found e.g., in the monograph [2; *Chapter 1, §1.6*] or in [6; *pages 36-47*].

On the other hand, throughout the paper let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space with distribution function F_Z of the random variable $Z \in R^{\Omega}$, defined by $F_Z(x) = P(\{\omega \in \Omega : Z(\omega) \leq x\})$ for every $x \in R$. Furthermore, let $(X_i)_{i\in N}$ be a sequence of non-negative independent random variables and φ a positive normalizing function,

$$\varphi : N \to R_+, \quad \varphi(n) = o(1) \quad (n \to \infty), \tag{4}$$

o being the Landau symbol. Many of the limit theorems of probability theory may be formulated as theorems concerning the convergence of the normalized sum

$$U_n := \varphi(n)S_n, \text{ where } S_n = \sum_{i=1}^{n} X_i. \tag{5}$$

In what follows, let $E(X_i)$ and $Var(X_i)$ be the expectation respectively the variance of the variable X_i, $i \in N$.

Remark. If $(T(t))_{t\geq 0}$ is a C_0-semigroup then $T(Z)f$ is strongly measurable for any non-negative random variable Z. Since

$$\int_\Omega \|T(Z)f\|dP \leq \int_\Omega \|f\|dP = \|f\|, \quad f \in \mathcal{E},$$

by contractiveness, $T(Z)f$ is integrable and $E[T(Z)f]$ is well defined.

As a first step we establish a Jackson-type inequality.

THEOREM 1. *Let $(X_i)_{i\in N}$ be a sequence of non-negative independent random variables with $E(X_i) = t \in [0, \infty)$ and $Var(X_i) := v_{i,t}^2 < \infty$, $i \in N$. Let $(T(t))_{t\geq 0}$ be a contraction C_0-semigroup. For every f belonging to $D(A^2)$ and $t \geq 0$ one has*

$$\|E[T(U_n)f] - T(t)f\| \leq |n\varphi(n) - 1|\|T(t)f\| + n\varphi^2(n)\left(t^2 + \frac{1}{2n}\sum_{i=1}^{n} v_{i,t}^2\right)\|A^2 f\|,$$

where U_n, $n \in N$, are defined by (5).

Proof. We consider $(Y_i)_{i\in N}$ a second sequence of independent identically distributed random variable, distributed as X^t, i.e., $F_{Y_i} = F_{X^t}$, $i \in N$, where X^t is a random variable with distribution $F_{X^t} = \varepsilon_t$. We recall that ε_t represents the *unit mass* defined by $\varepsilon_t(B) = 1$ for $t \in B$ and $\varepsilon_t(B) = 0$ for $t \notin B$ $(t \in \Omega, B \in \mathcal{A})$.

Further on, we define $V_n := \sum_{i=1}^{n} Y_i$ and $R_{n,i} := \sum_{j=1}^{i-1} X_j + \sum_{j=i+1}^{n} Y_j$, $n \in N$, $1 \leq i \leq n$, with the convention $X_0 = Y_{n+1} = 0$.

For every $f \in D(A^2) := \{h |\ h \in D(A) \text{ and } Ah \in D(A)\}$ we get

$$\sum_{i=1}^{n} [T(\varphi(n)(R_{n,i} + X_i))f - T(\varphi(n)(R_{n,i} + Y_i))f] = T(\varphi(n)S_n)f - T(\varphi(n)V_n)f.$$

Since $\varphi(n)V_n$ has the distribution $n\varphi(n)\varepsilon_t$, clearly

$$E[T(\varphi(n)V_n)f] = n\varphi(n)E[T(X^t)f] = n\varphi(n)T(t)f \tag{6}$$

and applying the operator E to identity (6) we obtain

$$E[T(\varphi(n)S_n)f]-n\varphi(n)T(t)f=\sum_{i=1}^{n}E[T(\varphi(n)(R_{n,i}+X_i))f-T(\varphi(n)(R_{n,i}+Y_i))f]$$

In accordance with Butzer and Berens result [3; *page 11*], if f belongs to $D(A^2)$ then for every $s\geq 0$ one has

$$T(s)f=f+sAf+s^2\int_0^1(1-u)T(us)A^2f\,du.$$

Substituting $f\in D(A^2)$ by $T(s_1)f\in D(A^2)$, $s_1\geq 0$, this yields

$$T(s_1+s)f=T(s_1)f+sT(s_1)Af+s^2\int_0^1(1-u)T(s_1+us)A^2f\,du. \quad (8)$$

For every $n\in N$ and $1\leq i\leq n$ the random variables $R_{n,i}$, X_i, Y_i are independent and consequently, for each $f\in\mathcal{E}$, $T(R_{n,i})f$, X_i, Y_i have the same property.

Choosing successively in (8) $s=\varphi(n)X_i$, $s_1=\varphi(n)R_{n,i}$ respectively $s=\varphi(n)Y_i$, $s_1=\varphi(n)R_{n,i}$ and knowing that $E(Y_i)=E(X_i)=t$ for every $1<i<n$, we can write

$$E[T(\varphi(n)(R_{n,i}+X_i))f]-E[T(\varphi(n)(R_{n,i}+Y_i))f]$$

$$=\varphi^2(n)\int_0^1(1-u)\{T(\varphi(n)(R_{n,i}+uX_i))A^2fE(X_i^2)$$

$$-T(\varphi(n)(R_{n,i}+uY_i))A^2fE(Y_i^2)\}du:=\Delta_{\varphi(n),i}(X_i,Y_i;A^2f).$$

We totalize the above identities with respect to i and taking into account (6) we get

$$\|E[T(\varphi(n)S_n)f]-E[T(\varphi(n)V_n)f]=\left\|\sum_{i=1}^{n}\Delta_{\varphi(n),i}(X_i,Y_i;A^2f)\right\|$$

$$\leq\varphi^2(n)\|A^2f\|\sum_{i=1}^{n}(E(X_i^2)+E(Y_i^2))\int_0^1(1-u)du$$

$$=\varphi^2(n)\|A^2f\|\left(nt^2+\frac{1}{2}\sum_{i=1}^{n}v_{i,t}^2\right). \quad (7)$$

We have used the contraction property of the semigroup as well as the following true relations implied by our assumptions

$$E(X_i^2)=Var(X_i)+E^2(X_i)=v_{i,t}^2+t^2,\quad E(Y_i^2)=E[(X^t)^2]=t^2.$$

At the same time, based on (7) we get

$$\|E[T(U_n)f] - T(t)f\|$$
$$\leq \|E[T(\varphi(n)S_n)f - n\varphi(n)T(t)f\| + |n\varphi(n) - 1|\|T(t)f\|,$$

and with the help of (9) we obtain the claimed result.

In order to evaluate the rate of convergence by using the second modulus of smoothness of the semigroup, we take advantage that this modulus is equivalent to the K-functional of Peetre [3; *page 192*], i.e. there exist positive constants M_1, M_2 such that for all $f \in \mathcal{E}$ and $t > 0$

$$M_1\omega_2(t, f) \leq K(t^2, f; \mathcal{E}, \mathcal{G}) \leq M_2\omega_2(t, f).$$

We recall

$$\omega_2(t, f) := \sup_{0 \leq s \leq t} \|(T(s) - I_{\mathcal{E}})^2 f\|, \quad K(t, f; \mathcal{E}, \mathcal{G}) := \inf_{g \in \mathcal{G}}(\|f - g\| + t|g|),$$

where $\mathcal{G}$ is any normal Banach subspace of $(\mathcal{E}, \|\cdot\|)$, which means that there is a seminorm $|\cdot|$ on $\mathcal{G}$ which makes $\mathcal{G}$ become Banach space with respect to the norm $\|\cdot\|_{\mathcal{G}} = \|\cdot\| + |\cdot|$. In our case $\mathcal{G} = D(A^2)$ and we consider $K(t^2, f) = \inf_{g \in D(A^2)}(\|f - g\| + t^2\|A^2 g\|)$.

Gathering the above facts, Theorem 1 allows us to state the main result of this section.

THEOREM 2. *Let $(X_i)_{i \in N}$ be a sequence of non-negative independent random variables with $E(X_i) = t \in [0, \infty)$ and $Var(X_i) := v_{i,t}^2 < \infty$, $i \in N$. Let $(T(t))_{t \geq 0}$ be a contraction C_0-semigroup. For all $f \in \mathcal{E}$ and $t \geq 0$ there holds*

$$\|E[T(U_n)f] - T(t)f\| \leq |n\varphi(n) - 1|\|T(t)f\|$$
$$+M\omega_2\left(\varphi(n)\sqrt{nt^2 + \frac{1}{2}\sum_{i=1}^{n} v_{i,t}^2}, f\right),$$

where $U_n, n \in N$, are defined by (5) and M is a constant independent of n, and f. In particular, if X_i are independent identically distributed random variables with $Var(X_i) := v_t^2$, $i \in N$, then

$$\|E[T(\varphi(n)S_n)f] - T(t)f\| \leq |n\varphi(n) - 1|\|T(t)f\|$$
$$+M\omega_2(\sqrt{n}\varphi(n)\sqrt{t^2 + v_t^2/2}, f).$$

Particular cases. (i) Under the general assumptions of Theorem 2 if we consider $\varphi(n) = n^{-1}$, $n \in N$, then we reobtain a known result due to P.L. Butzer and L. Hahn, see [4; *Theorem 1*].

(ii) If we require $\lim_{n\to\infty} n\varphi(n) = 1$, then we get an approximation process in the Banach space $\mathcal{E}$, in other words we have

$$\lim_{n\to\infty} \|E[T(\varphi(n)S_n)f] - T(t)f\| = 0, \quad f \in \mathcal{E}.$$

3 An extension of Feller type

To define the Feller operator, let $(X_n)_{n\geq 1}$ be a sequence of random variables having the distribution function $F^*_{n,x}$ with the expectation $E(X_n) = x$ and the variance $\sigma_n^2(x)$ where x is a real continuous parameter. For a function $f \in C(R)$ we define the linear operator

$$(L_n f)(x) = E[f(X_n)] = \int_R f(t)dF^*_{n,x}(t) \quad \text{if} \quad E[|f(X_n)|] < \infty. \tag{8}$$

An important step for constructing operators useful in the theory of uniform approximation of continuous functions was made by D.D. Stancu [7] who considered X_n, $n \in N$, independent and identically distributed random variables with the mean x and the variance $\sigma^2(x)$, where the parameter x takes values in an interval I, possibly unbounded. If S_n is defined by (5) then (10) is equivalent to

$$(L_n f)(x) = E[f(S_n/n)] = \int_R f\left(\frac{t}{n}\right) dF_{n,x}(t), \tag{9}$$

where $F_{n,x}$ is the distribution function of S_n.

By Stancu's method one derives the well-known operators of Bernstein, Mirakjan-Favard-Szasz, Baskakov, Weierstrass, a variant of Meyer-König and Zeller operators. We should remark that in the same paper, the author considered an example of random variables which are not independent and based on Markov-Pólya urn scheme, he studied an operator of discrete type which it is known in literature as Stancu operator.

In what follows, let $\{X_{n,j} \;:\; j = 1, 2, \ldots, n;\; n \in N\}$ be a triangular array of independent random variables such that for each fixed n, $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$ are identically distributed with $E(X_{n,j}) = e_n(x)$ and finite variance $Var(X_{n,j}) = \sigma_n^2(x) > 0$, $j = \overline{1,n}$, where $x \in I \subset R$ is a parameter. We also consider $\{\lambda_{n,j} \;:\; j = 1, 2, \ldots, n;\; n \in N\}$ a triangular array of positive numbers. With these elements we construct the following sequence of operators given by

$$(\Lambda_n h)(x) = E[h(Z_n)] = \int_R h\Big(\Big(\sum_{j=1}^{n} \lambda_{n,j}\Big) u\Big) dF_{n,x}(u),\ h \in \mathcal{L}, \tag{10}$$

where $Z_n = \sum_{j=1}^{n} \lambda_{n,j} X_{n,j}$, $F_{n,x}$ is the distribution function of Z_n and $\mathcal{L}$ stands for the domain of Λ_n containing the set of all well-defined measurable functions on R for which the right-hand side in (11) makes sense, in other words $E[h(Z_n)] < \infty$. Obviously $\mathcal{L}$ includes all real-measurable bounded functions on R.

Remark. To specialize (12), for every $n \geq 1$ we choose $\lambda_{n,1} = \lambda_{n,2} = \ldots = \lambda_{n,n} := \varphi(n)$. In concordance with (5) one has $Z_n = U_n$ and $(\Lambda_n)_{n\geq 1}$ becomes a sequence introduced and studied by Mohammad Kazim Khan [5]. Furthermore, if $X_{n,j}$, $j = \overline{1,n}$, are identically distributed for all n, $e_n(x) = x$, $\sigma_n^2(x) = \sigma^2(x) > 0$ and $\varphi(n) = n^{-1}$ then Λ_n reduces to the Feller operator presented in (11).

Setting m_j for the j-th monomial, $m_j(t) = t^j$, $t \in I$, $j \in N_0 := N \cup \{0\}$, by simple computations we obtain

LEMMA 1. *The operators $\Lambda_n, n \in N$, defined by (12) verify*

$$\Lambda_n m_0 = m_0, \quad \Lambda_n m_1 = \Big(\sum_{j=1}^{n} \lambda_{n,j}\Big) e_n,$$

$$\Lambda_n m_2 = \Big(\sum_{j=1}^{n} \lambda_{n,j}^2\Big) \sigma_n^2 + \Big(\sum_{j=1}^{n} \lambda_{n,j}\Big)^2 e_n^2,$$

and consequently the central moment of second order $\mu_2(\Lambda_n, \cdot)$ is given by

$$\mu_2(\Lambda_n, x) = \Big(\Big(\sum_{j=1}^{n} \lambda_{n,j}\Big) e_n(x) - x\Big)^2 + \Big(\sum_{j=1}^{n} \lambda_{n,j}^2\Big) \sigma_n^2(x),\ x \in I. \tag{11}$$

The main goal of this section is to obtain the rate of convergence of (12) for continuous functions. In this aim we involve the first modulus of smoothness $\omega_1(h, \cdot)$ associated to any bounded function h. We ennunciate the following

THEOREM 3. *Let the operators $\Lambda_n, n \in N$, be defined by (12). Then for every $h \in C(I)$ and $\alpha > 0$ holds true*

$$|(\Lambda_n h)(x) - h(x)| \leq (1 + n^{\alpha} \mu_2(\Lambda_n, x)) \omega_1(h, n^{-\alpha/2}), x \in I, \tag{12}$$

where $\mu_2(\Lambda_n, x)$ is given at (13).

Proof. Among the well-known properties of $\omega_1(h,\cdot)$ we recall that for every $\delta>0$, $|h(t)-h(x)|\le(1+\delta^{-2}(t-x)^2)\omega_1(h,\delta)$, $(t,x)\in I\times I$, see e.g. [2; *Chapter 5, §1*] or [1; *Section 1.2*]. With the help of both this inequality and Lemma 1 we get

$$|(\Lambda_n h)(x)-h(x)|\le E[|h(Z_n)-h(x)|]$$

$$\le(1+\delta^{-2}E[(Z_n-x)^2])\omega_1(h,\delta)=(1+\delta^{-2}\mu_2(\Lambda_n,x))\omega_1(h,\delta).$$

By taking $\delta=n^{-\alpha/2}$ the result follows.

Examining (14) we deduce easily

Corollary. Let $\lambda_n, n\in N$ be as defined by (12). If there exists $\alpha_0>0$ with the properties

$$\begin{cases}\Big(\sum_{j=1}^{n}\lambda_{n,j}\Big)e_n(x)-x=o(n^{-\alpha_0/2})\ (n\to\infty),\\[2ex] \Big(\sum_{j=1}^{n}\lambda_{n,j}^2\Big)\sigma_n^2(x)=o(n^{-\alpha_0})\ (n\to\infty),\end{cases}$$

then $\lim_{n\to\infty}\|\Lambda_n h-h\|_\infty=0$, *for every* $h\in C(I)$.

In the particular case $Z_n=U_n$ (see (5)) the requirements (15) can be restated $n\varphi(n)e_n(x)-x=o(n^{-\alpha_0/2})$ $(n\to\infty)$ and $\sqrt{n}\varphi(n)\sigma_n(x)=o(n^{-\alpha_0/2})$ $(n\to\infty)$, where $\sigma_n(x)$ is the standard deviation of $X_{n,j}$. Furthermore, choosing $\varphi(n)=n^{-1}$, for $\alpha_0=1$ we reobtain the classical result regarding the Feller operators.

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