Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 4, 843–861

Fixed Point Theorems for Decomposable Multi-Valued Maps and Applications

R. Precup

Abstract. We present fixed point theorems for weakly sequentially upper semicontinuous decomposable non–convex-valued maps. They are based on an extension of the Arino-Gautier-Penot Fixed Point Theorem for weakly sequentially upper semicontinuous maps with convex values. Applications are given to abstract operator inclusions in L^p spaces. An example is included to illustrate the theory.

Keywords: *Multi-valued map, operator inclusion, functional-differential inclusion, fixed point, continuation principle, measure of non-compactness, weak topology*

AMS subject classification: Primary 54H25, secondary 47H10, 47J35

1. Introduction

Various types of boundary value problems for differential inclusions, integro-differential inclusions or, more generally, functional-differential inclusions can be equivalently reformulated as operator inclusions of the form

$$u \in \Psi\Phi u \tag{1.1}$$

in an appropriate space of functions, where by $\Psi\Phi$ we mean the composition $\Psi \circ \Phi$. Most frequently Ψ is an "integral type" map, the inverse of a differential operator, while Φ is a multi-valued map associated with the right-hand side of the functional-differential inclusion.

For the theory of differential inclusions and its applications we refer the reader to the books of Deimling [9], Górniewicz [12], Hu and Papageorgiou [14, 15] and Kamenskii, Obukhovskii and Zecca [16].

Radu Precup: Babeş-Bolyai University, Fac. Math. & Comp. Sci., 3400 Cluj, Romania; r.precup@math.ubbcluj.ro

ISSN 0232-2064 / $ 2.50

Using a fixed point approach to problem (1.1), we may first try to apply fixed point theorems to the composite multi-valued map $\mathcal{F} = \Psi\Phi$. Several difficulties arise when treating such multi-valued compositions this way. One of them consists in guaranteeing continuity properties for the maps; another one concerns the geometric properties of their values. For example, even if the values of Φ are convex and Ψ is single-valued (but nonlinear), the values of $\mathcal{F} = \Psi\Phi$ can be non-convex. In this connection we may think to use fixed point theorems for non–convex-valued maps, for example, the Eilenberg-Montgomery Theorem (see Couchouron and Precup [5, 6]). However, it is expectable that one can take advantage from the representation of $\mathcal{F}$ as $\Psi\Phi$. Several authors have done this under various aspects (see Andres and Bader [1], Bader [3] and Górniewicz [12]). The main purpose of the present paper is to develop a fixed point theory for maps which are decomposable into $\Psi\Phi$, with both Φ and Ψ convex-valued maps between Banach spaces. We shall succeed this by considering the Cartezian product map

$$\Pi(x, y) = \Psi y \times \Phi x$$

whose values are convex in the corresponding product space $X \times Y$ endowed with the weak topologies on X and Y.

The abstract results established in this paper can be used to prove elementarily that the hypothesis of contractibility asked in Couchouron and Kamenskii [4] and that one of acyclicity from Couchouron and Precup [5, 6] are not necessary (for [4] this was previously shown by Bader [3] by means of a topological fixed-point index theory for decomposable maps, under a stronger compactness condition on Ψ). In Section 3 the abstract continuation principle established in Section 2 is applied to discuss operator inclusions in L^p spaces, under general assumptions which were inspired by those in Couchouron and Kamenskii [4] and in Couchouron and Precup [5]. Finally, we present a simple example concerning functional-differential inclusions.

The main contributions of this paper are as follows:

1) A fixed point theory for non–convex-valued maps which can be represented as compositions of two convex-valued maps. This theory improves and extends the results from Couchouron and Kamenskii [4] and from Couchouron and Precup [5, 6]. Also, our theory represents a fixed point alternative to the index theory presented in Bader [3] under some more restrictive conditions (for example, in [3] only Φ is multi-valued).

2) A continuation principle accompanying the Arino-Gautier-Penot Fixed Point Theorem [2] for weakly sequentially upper semicontinuous maps.

3) Theorems of Mönch type for set-valued maps with conditions expressed with respect to the strong or the weak topology. These results complement those in Mönch [17], O'Regan [18] and in O'Regan and Precup [19].

For the remainder of this section we gather together some definitions and results which we will need in what follows.

For any Hausdorff topological space X we define

$$P_f(X) = \Big\{ A \subset X : \, A \text{ is non-empty, closed} \Big\}$$
$$P_k(X) = \Big\{ A \subset X : \, A \text{ is non-empty, compact} \Big\}.$$

If X is a closed convex subset of a Banach space, then we define

$$P_{fc}(X) = \Big\{ A \subset X : \, A \text{ is non-empty, closed, convex} \Big\}$$
$$P_{k^w c}(X) = \Big\{ A \subset X : \, A \text{ is non-empty, weakly compact, convex} \Big\}.$$

A multi-valued map $\Phi : X \to 2^Y$, where X and Y are Hausdorff topological spaces, is said to be *upper semicontinuous* if for every closed subset A of Y the set

$$\Phi^-(A) = \big\{ x \in X : \, A \cap \Phi x \neq \emptyset \big\}$$

is closed in X.

Throughout this paper we shall consider multi-valued maps $\Phi : \, X \to 2^Y$ where X and Y are subsets of two Banach spaces. We shall use the following terminology:

- Φ is *u.s.c.* if Φ is upper semicontinuous with respect to the strong topologies of X and Y.
- Φ is *w-u.s.c.* if Φ is upper semicontinuous with respect to the weak topologies of X and Y.
- Φ is *sequentially w-u.s.c.* if for every weakly closed subset $A \subset Y$ the set $\Phi^-(A)$ is sequentially closed for the weak topology on X.

We recall the following two known fixed point theorems: **Theorem 1.1** (Bohnenblust-Karlin). *If X is a Banach space, C is a non-empty compact convex subset of X and $\Phi : C \to P_{fc}(C)$ is u.s.c., then there exists an $x \in C$ with $x \in \Phi x$.*

Theorem 1.2 (Arino-Gautier-Penot). *If X is a Banach space (or, more generally, a metrizable locally convex linear topological space), C is a non-empty weakly compact convex subset of X and $\Phi : C \to P_{fc}(C)$ is sequentially w-u.s.c., then there exists an $x \in C$ with $x \in \Phi x$.*

Notice that Theorem 1.2 is an immediate consequence of Ky Fan's Fixed Point Theorem (see Deimling [8: pp. 310 – 315]) and of the following lemma (Arino, Gautier and Penot [2], O'Regan [18]) whose proof is based upon the Eberlein-Šmulian Theorem (see Dunford and Schwartz [11: pp. 430]). **Lemma 1.1.** *Let X, Y be Banach spaces (or, more generally, locally convex linear topological spaces, and X metrizable) and let C be a weakly compact subset of X. Then any sequentially w-u.s.c. map $\Phi : C \to 2^Y$ is w-u.s.c.*

Remark 1.1. For a map $\Phi : C \to 2^C$ with C a compact subset of a Banach space, the notions of u.s.c., w-u.s.c. and sequentially w-u.s.c. are identical. Thus in Theorem 1.1 Φ can be equivalently assumed to be sequentially w-u.s.c. So Theorem 1.2 appears as a generalization of Theorem 1.1.

Next we recall the definitions of measures of non-compactness and weak non-compactness. By a *measure of non-compactness* in a closed convex subset C of a Banach space X we mean a real function μ defined on the collection of all non-empty bounded subsets of C, such that

$$\begin{aligned} \mu(A) &= \mu(\overline{\mathrm{co}}\, A) \\ \mu(A) = 0 &\iff A \text{ is relatively compact} \\ A \subset B &\implies \mu(A) \le \mu(B). \end{aligned}$$

We shall denote by β_X the *ball measure of non-compactness* in X,

$$\beta_X(A) = \inf \Big\{ \varepsilon > 0 : A \text{ admits a finite cover by balls of radius } \varepsilon \Big\}.$$

By a *measure of weak non-compactness* in a closed convex subset C of a Banach space we mean a real function χ defined on the collection of all non-empty bounded subsets of C, such that

$$\begin{aligned} \chi(A) &= \chi(\overline{\mathrm{co}}\, A) \\ \chi(A) = 0 &\iff A \text{ is relatively weakly compact} \\ A \subset B &\implies \chi(A) \le \chi(B). \end{aligned}$$

For an example of a measure of weak non-compactness see De Blasi [7].

We conclude this section with two well-known compactness criteria in $L^p(0,T;E)$ (see Guo, Lakshmikantham and Liu [13: pp. 15 – 18] and Diestel, Ruess and Schachermayer [10], respectively). Here $0 < T < \infty$, $p \in [1,\infty]$ and E is a Banach space with norm $|\cdot|_E$. For a function $u : [0,T] \to E$ we define the *translation* by h $(0 < h < T)$ to be the function $\tau_h u : [0, T-h] \to E$ given by $\tau_h u(t) = u(t+h)$. **Theorem 1.3.** *Let $p \in [1,\infty]$. Let $M \subset L^p(0,T;E)$ be countable and assume that there exists a function $\nu \in L^p(0,T;\mathbb{R}_+)$ with $|u(t)|_E \le \nu(t)$ a.e. on $[0,T]$, for all $u \in M$. In addition, assume that $M \subset C([0,T];E)$ if $p = \infty$. Then M is relatively compact in $L^p(0,T;E)$ if and only if*

(i) $\sup_{u\in M} |\tau_h u - u|_{L^p(0,T-h;E)} \to 0$ *as* $h \downarrow 0$

(ii) $M(t) = \{u(t) : u \in M\}$ *is relatively compact in* E *for a.e.* $t \in [0,T]$.

Theorem 1.4. *Let* $p \in [1,\infty)$. *Let* $M \subset L^p(0,T;E)$ *be countable and assume that there exists a function* $\nu \in L^p(0,T;\mathbb{R}_+)$ *with* $|u(t)|_E \leq \nu(t)$ *a.e. on* $[0,T]$, *for all* $u \in M$. *If* $M(t)$ *is relatively compact in* E *for a.e.* $t \in [0,T]$, *then* M *is weakly relatively compact in* $L^p(0,T;E)$.

2. Fixed point theory First we give an extension of the Arino-Gautier-Penot Fixed Point Theorem [2] to decomposable non–convex-valued maps. Theorem 2.1. ***Let X and Y be Banach spaces (or, more generally, metrizable locally convex linear topological spaces), let A and B be non-empty weakly compact convex subsets of X and Y, respectively, and let***

$$\Phi : A \to P_{fc}(B)$$
$$\Psi : B \to P_{fc}(A)$$

be two multi-valued maps. Assume Φ ***and*** Ψ ***are sequentially w-u.s.c. Then there exists at least one*** $x \in A$ ***with*** $x \in \Psi\Phi x$ ***and, equivalently, there exists at least one*** $y \in B$ ***with*** $y \in \Phi\Psi y$.

Proof. Let $X \times Y$ be endowed with the product topology. In this way, $X \times Y$ is a Banach space (respectively, a metrizable locally convex linear topological space). Consider the multi-valued map acting in $X \times Y$, $\Pi : A \times B \to P_{fc}(A \times B)$, given by

$$\Pi(x,y) = \Psi y \times \Phi x.$$

We have that $A \times B$ is a weakly compact convex subset of $X \times Y$. In addition, Π is sequentially w-u.s.c. (see Kamenskii, Obukhovskii and Zecca [16: Theorem 1.2.12]). Thus we may apply the Arino-Gautier-Penot Fixed Point Theorem. Therefore, there exists a $(x,y) \in A \times B$ with $(x,y) \in \Pi(x,y)$. We have $x \in \Psi y$ and $y \in \Phi x$. Consequently, $x \in \Psi\Phi x$ and $y \in \Phi\Psi y$ ∎

Remark 2.1. The Arino–Gautier–Penot Theorem appears as a particular case of Theorem 2.1, when $X = Y$, $A = B$ and Φ or Ψ is the identity map of A.

Theorem 2.2. *Let X, Y be Banach spaces, let C be a closed convex subset of X, and let*

$$\Phi : C \to P_{kwc}(Y)$$
$$\Psi : \overline{co}\,\Phi(C) \to P_{fc}(C)$$

be two multi-valued maps. Assume that, for every weakly compact convex subset A of C, Φ and Ψ are sequentially w-u.s.c. on A and on $\overline{co}\,\Phi(A)$, respectively. In addition, assume that there exists an $x_0 \in C$ such that the condition

$$\left.\begin{aligned} A \subset C \\ A = \overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(A))\Big) \end{aligned}\right\} \quad \Longrightarrow \quad A \text{ is weakly compact} \tag{2.1}$$

is satisfied. Then there exists at least one $x \in C$ with $x \in \Psi\Phi x$.

Proof. Let $\mathcal{M}$ be the collection of all non-empty closed convex subsets M of C with

$$\overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(M))\Big) \subset M.$$

Clearly, $C \in \mathcal{M}$ and $x_0 \in M$ for every $M \in \mathcal{M}$. Moreover, it is easy to see that

$$M \in \mathcal{M} \quad \Longrightarrow \quad \overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(M))\Big) \in \mathcal{M}. \tag{2.2}$$

Define the set

$$A = \cap\{M : M \in \mathcal{M}\}.$$

We have $A \in \mathcal{M}$. Also, (2.2) implies

$$A = \overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(A))\Big).$$

Then (2.1) guarantees that A is weakly compact. Now Theorem 2.1 applies to A and $B = \overline{co}\,\Phi(A)$. Notice (see Kamenskii, Obukhovskii and Zecca [16: Theorem 1.1.7]) that $\Phi(A)$ is weakly compact since Φ is w-u.s.c. on A (from Lemma 1.1) and has weakly compact values. Then the Krein-Šmulian Theorem (Dunford and Schwartz [11: pp. 434]) implies that $\overline{co}\,\Phi(A)$ is weakly compact ∎

Remark 2.2. If in addition C is weakly compact, then condition (2.1) trivially holds and Theorem 2.2 becomes Theorem 2.1.

Theorem 2.2 yields in particular the following result for convex-valued self-maps of a closed convex subset of a Banach space (compare Theorem 4.3 in O'Regan [18] and Theorem 2.1 for single-valued maps in Mönch [17]), an alternative result to Theorem 3.1 in O'Regan and Precup [19]. Corollary 2.1. *Let X be a Banach space, C a closed convex subset of X and $\Phi : C \to P_{k^w c}(C)$. Assume Φ is sequentially w-u.s.c. and that there is an $x_0 \in C$ such that*

$$\left.\begin{array}{r} A \subset C \\ A = \overline{co}\Big(\{x_0\} \cup \Phi(A)\Big) \end{array}\right\} \quad \Longrightarrow \quad A \text{ is weakly compact.} \tag{2.3}$$

Then there exists at least one $x \in C$ with $x \in \Phi x$.

Proof. We apply Theorem 2.2 to $Y = X$ and $\Psi = I_X$, the identity map of X. Note that

$$\overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(A))\Big) = \overline{co}\Big(\{x_0\} \cup \overline{co}\,\Phi(A)\Big) = \overline{co}\Big(\{x_0\} \cup \Phi(A)\Big)$$

and the assertion is proved ∎

Remark 2.3. If in addition C is weakly compact, then condition (2.3) trivially holds and Corollary 2.1 becomes the Arino-Gautier-Penot Theorem.

Under a stronger condition than (2.1) and a weaker one on Φ, we have the following result. Theorem 2.3. *Let*

X and Y be Banach spaces, let C be a closed convex subset of X, and let

$$\Phi : C \to P_{k^w c}(Y)$$
$$\Psi : \overline{co}\,\Phi(C) \to P_{fc}(C)$$

be two multi-valued maps. Assume that, for every compact convex subset A of C, Φ and Ψ are sequentially w-u.s.c. on A and $\overline{co}\,\Phi(A)$, respectively. In addition, assume that there exists an $x_0 \in C$ such that the condition

$$\left.\begin{aligned} A \subset C \\ A = \overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(A))\Big) \end{aligned}\right\} \quad \Longrightarrow \quad A \text{ is compact} \tag{2.4}$$

is satisfied. Then there exists at least one $x \in C$ with $x \in \Psi\Phi x$.

Next we present a fixed point theorem of Leray-Schauder type (a continuation principle) for decomposable non–convex-valued maps. Theorem 2.4. *Let X and Y be Banach spaces, K a closed convex subset of X, U a convex relatively open subset of K, $x_0 \in U$ and let*

$$\Phi : \overline{U} \to P_{k^w c}(Y)$$
$$\Psi : \overline{co}\,\Phi(\overline{U}) \to P_{fc}(K)$$

be two multi-valued maps. Assume that, for every compact convex subset A of $\overline{U}$, Φ and Ψ are sequentially w-u.s.c. on A and $\overline{co}\,\Phi(A)$, respectively. In addition, assume that the two conditions

$$\left.\begin{aligned} A \subset \overline{U} \\ A \text{ closed convex} \\ A \subset \overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(A))\Big) \end{aligned}\right\} \quad \Longrightarrow \quad A \text{ is compact} \tag{2.5}$$

and

$$x \notin (1-\lambda)x_0 + \lambda\Psi\Phi x \qquad \forall\; x \in \overline{U}\setminus U, \lambda \in (0,1) \quad (2.6)$$

are satisfied. Then there exists at least one $x \in \overline{U}$ with $x \in \Psi\Phi x$.

Proof. If $U = K$, then $\overline{U}\setminus U = \emptyset$, so (2.6) is superfluous and the result follows from Theorem 2.3, where $C = K$. Assume $U \neq K$. Let

$$C = \overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(\overline{U}))\Big).$$

It clear that $x_0 \in C \subset K$ and C is closed convex. Since U is open in K, convex, and $x_0 \in U$, we can define a single-valued operator $P : K \to \overline{U}$ by

$$Px = \begin{cases} x & \text{if } x \in \overline{U} \\ (1-\lambda)x_0 + \lambda x & \text{if } x \notin \overline{U} \end{cases}$$

where $\lambda \in (0,1)$ is such that $(1-\lambda)x_0 + \lambda x \in \overline{U}\setminus U$. Clearly, P is continuous.

Consider

$$\begin{aligned} \widehat{\Phi} : C \to P_{k^w c}(Y), &\qquad \widehat{\Phi}x = \Phi P x \quad (x \in C) \\ \widehat{\Psi} : \overline{co}\,\widehat{\Phi}(C) \to P_{fc}(C), &\qquad \widehat{\Psi}y = \Psi y \qquad (y \in \overline{co}\,\widehat{\Phi}(C). \end{aligned}$$

We first check that $\widehat{\Phi}$ is sequentially w-u.s.c. on any compact convex subset A of C. Indeed, we can see that it suffices to prove this for compact convex sets A with $x_0 \in A$. In this situation, $P(A) = A \cap \overline{U}$, so $P(A)$ is compact and convex. Now let $B \subset Y$ be weakly closed. We have to show that the set

$$M = \Big\{x \in A :\; \widehat{\Phi}x \cap B \neq \emptyset\Big\}$$

is weakly sequentially closed. Assume $x_k \in A$, $\widehat{\Phi}x_k \cap B \neq \emptyset$ and $x_k \to x$ weakly. Since A is compact, there is a subsequence $(x_{k'})$ of (x_k) with $x_{k'} \to x$ strongly. Then $Px_{k'} \to Px$ strongly. Since $P(A)$ is compact convex, Φ is sequentially w-u.s.c. on $P(A)$. Consequently, the set

$$N = \left\{ y \in P(A) : \Phi y \cap B \neq \emptyset \right\}$$

is weakly sequentially closed. Since $Px_{k'}$ belongs to N for all k', we have $Px \in N$, too. Thus $\Phi Px \cap B \neq \emptyset$ with $x \in A$. Therefore, $x \in M$ as desired. It is easy to see that $\widehat{\Psi}$ is sequentially w-u.s.c. on $\overline{co}\,\widehat{\Phi}(A)$.

Next we show that (2.4) holds for the couple $(\widehat{\Phi}, \widehat{\Psi})$. Let $A \subset C$ be such that

$$A = \overline{co}\Big(\{x_0\} \cup \widehat{\Psi}(\overline{co}\,\widehat{\Phi}(A))\Big).$$

Clearly,

$$A = \overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi P(A))\Big).$$

We have

$$P(A) = A \cap \overline{U} \subset \overline{co}\Big(\{x_0\} \cup \Psi(\overline{co}\,\Phi P(A))\Big)$$

where $P(A)$ is a closed convex subset of $\overline{U}$. Then (2.5) guarantees that $P(A)$ is compact. Let (x_k) be any sequence in A. Since $P(A)$ is compact, there exists a subsequence $(x_{k'})$ of (x_k) with $Px_{k'} \to y$ strongly for some $y \in P(A)$. We have $Px_{k'} = (1-\lambda_{k'})x_0 + \lambda_{k'}x_{k'}$ for some $\lambda_{k'} \in [0,1]$. Passing eventually to a new subsequence we may assume that $\lambda_{k'} \to \lambda$ for some $\lambda \in [0,1]$. If $\lambda > 0$, we immediately find that $(x_{k'})$ is strongly convergent. Assume $\lambda = 0$. Then $y = x_0$ and so $Px_{k'} = x_{k'}$ for all $k' \geq k_0$. Hence $(x_{k'})$ is strongly convergent as well. Hence A is compact.

Thus all the assumptions of Theorem 2.3 are satisfied for the couple $(\widehat{\Phi}, \widehat{\Psi})$. Therefore, there exists $x \in C$ with $x \in \widehat{\Psi}\widehat{\Phi}x$. Clearly, $x \in \Psi\Phi Px$. We claim that

$x \in \overline{U}$. Assume the contrary, that is $x \notin \overline{U}$. Then $Px = (1-\lambda)x_0 + \lambda x$ for some $\lambda \in (0,1)$ and $Px \in \overline{U} \setminus U$. From $x \in \Psi\Phi Px$ we deduce

$$Px = (1-\lambda)x_0 + \lambda x \in (1-\lambda)x_0 + \lambda\Psi\Phi Px$$

which contradicts (2.6). Hence $x \in \overline{U}$, so $Px = x$ and $x \in \Psi\Phi x$ ∎

Theorem 2.4 yields in particular the following continuation principle for convex-valued maps (compare Theorem 2.2 for single-valued maps in Mönch [17]), an alternative result to Theorem 3.2 in O'Regan and Precup [19]. Corollary 2.2. *Let X be a Banach space, K a closed convex subset of X, U a convex relatively open subset of K, $x_0 \in U$ and let*

$$\Phi : \overline{U} \to P_{k^w c}(K)$$

be a multi-valued map. Assume that Φ is sequentially w-u.s.c. on each compact convex subset of $\overline{U}$. In addition, assume that the two conditions

$$\left.\begin{array}{r} A \subset \overline{U} \\ A \text{ closed convex} \\ A \subset \overline{co}\Big(\{x_0\} \cup \Phi(A)\Big) \end{array}\right\} \implies A \text{ is compact}$$

and

$$x \notin (1-\lambda)x_0 + \lambda\Phi x \qquad \forall\ x \in \overline{U} \setminus U, \lambda \in (0,1)$$

are satisfied. Then there exists at least one $x \in \overline{U}$ with $x \in \Phi x$.

Remark 2.4. Let U be bounded, and let Φ and Ψ send bounded sets into bounded sets. If μ is a measure of strong non-compactness in K, χ is a measure of weak

non-compactness on $\overline{co}\,\Phi(\overline{U})$, and there are functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ with ψ non-decreasing such that

$$\begin{aligned} &\psi\phi(\tau) < \tau \qquad (\tau > 0) \\ &\chi(\Phi(M)) \le \phi(\mu(M)) \quad (M \subset \overline{U}) \\ &\mu(\Psi(M)) \le \psi(\chi(M)) \quad (M \subset \overline{co}\,\Phi(\overline{U}), \end{aligned} \tag{2.7}$$

then condition (2.5) holds. Indeed, if $A \subset \overline{U}$ and $A \subset \overline{co}\big(\{x_0\} \cup \Psi(\overline{co}\,\Phi(A))\big)$, then

$$\mu(A) \le \mu\Big(\Psi(\overline{co}\,\Phi(A))\Big) \le \psi\Big(\chi(\overline{co}\,\Phi(A))\Big) = \psi\Big(\chi(\Phi(A))\Big) \le \psi\phi(\mu(A)). \blacksquare$$

Then (2.7) implies $\mu(A) = 0$, i.e. A is compact.

3. Operator inclusions in $\mathbf{L}^p$ spaces

In this section we are concerned with the abstract operator inclusion

$$w \in \Psi\Phi w \qquad (w \in K) \tag{3.1}$$

in a closed convex subset K of $L^p(0,T;F)$, where

$\Phi :\, K \to 2^{L^q(0,T;E)}$ is a multi-valued map

$\Psi :\, L^q(0,T;E) \to K$ is a single-valued operator.

Here $0 < T < \infty$, $p \in [1,\infty]$, $q \in [1,\infty)$, and E and F are Banach spaces. We shall denote by r the conjugate exponent of q, i.e. $\frac{1}{q} + \frac{1}{r} = 1$. By $|\cdot|_q$ we shall denote the norm of $L^q(0,T;E)$ and by $\|\cdot\|$ an equivalent norm on the closed subspace of $L^p(0,T;F)$ generated by K.

We now state our assumptions:

(Ψ1) There exists a function $\eta :\, [0,T] \times L^q(0,T;\mathbb{R}_+) \to \mathbb{R}_+$, non-decreasing in its second variable such that, for every $t \in [0,T]$,

$$\sup_{g \in L^q(0,T;\mathbb{R}_+)} |\eta(t,g+h) - \eta(t,g)| \to 0 \qquad (|h|_q \to 0) \tag{3.2}$$

and

$$|(\Psi f_1 - \Psi f_2)(t)|_F \le \eta(t, |(f_1 - f_2)(\cdot)|_E)$$

a.e. on $[0,T]$, for all $f_1, f_2 \in L^q(0,T;E)$.

(Ψ2) There exists a constant $L > 0$ with $\|\Psi f_1 - \Psi f_2\| \le L|f_1 - f_2|_q$ for all $f_1, f_2 \in L^q(0,T;E)$.

(Ψ3) For any compact $C \subset E$ and any sequence (f_k) of $L^q(0,T;E)$ with $\{f_k(t)\}_{k\ge1} \subset C$ for a.e. $t \in [0,T]$, the weak convergence $f_k \to f$ implies $\Psi f_k \to \Psi f$ strongly in $L^p(0,T;F)$.

(Φ1) The values of Φ are non-empty, weakly compact, convex, and Φ is sequentially w-u.s.c. on any compact convex subset A of K.

(Φ2) For every $a > 0$ there exists a $\nu_a \in L^q(0,T;\mathbb{R}_+)$ such that $|f(t)|_E \le \nu_a(t)$ a.e. on $[0,T]$, for all $f \in \Phi w$ and all $w \in K$ satisfying $\|w\| \le a$.

(Φ3) For every separable closed subspaces E_0 and F_0 of E and F, respectively, there exists a map $\zeta : L^p(0,T;\mathbb{R}_+) \to L^q(0,T;\mathbb{R}_+)$ such that $\zeta(0) = 0$ and

$$\beta_{E_0}\big(\Phi(M)(t) \cap E_0\big) \le \zeta\big(\beta_{F_0}(M(\cdot))\big)(t) \tag{3.3}$$

a.e. on $[0,T]$, for every countable set $M \subset K$ with $M(t) \subset F_0$ a.e. on $[0,T]$, for which there exists $\nu \in L^p(0,T;\mathbb{R}_+)$ with $|w(t)|_F \le \nu(t)$ a.e. on $[0,T]$ for any $w \in M$. In addition, $\varphi = 0$ is the unique solution in $L^p(0,T;\mathbb{R}_+)$ to the inequality

$$\varphi(t) \le \eta(t, \zeta(\varphi)) \qquad \text{a.e. on } [0,T]. \tag{3.4}$$

(L-S) There exists a bounded convex subset U of K, open in K, and a $w_0 \in U$ such that $w \notin (1-\lambda)w_0 + \lambda\Psi\Phi w$ for all $w \in \overline{U} \setminus U$ and $\lambda \in (0,1)$. **Theorem 3.1.** *Let assumptions* (Ψ1) – (Ψ3), (Φ1) – (Φ3) *and* (L-S) *hold. Then inclusion problem* (3.1) *has at least one solution in* $\overline{U}$.

For the proof we need the following Lemmas 3.1 and 3.2. **Lemma 3.1.** *Let assumptions* (Ψ1) *and* (Ψ3) *hold. Further, let $B \subset L^q(0,T;E)$ be countable with*

$$|f(t)|_E \le \nu(t) \tag{3.5}$$

a.e. on $[0,T]$ for all $f \in B$, where $\nu \in L^q(0,T;\mathbb{R}_+)$. At last, let E_0 and F_0 be separable closed subspaces of E and F, respectively, with $f(t) \in E_0$ and $\Psi f(t) \in F_0$ a.e. on $[0,T]$ for every $f \in B$. Then the function φ defined by $\varphi(t) = \beta_{E_0}(B(t))$ belongs to $L^q(0,T;\mathbb{R}_+)$ and satisfies

$$\beta_{F_0}(\Psi(B)(t)) \le \eta(t, \varphi) \tag{3.6}$$

a.e. on $[0,T]$.

Proof. Let $B = \{f_n\}_{n\geq 1}$. The space E_0 being separable, we may represent it as $\overline{\cup_{k\geq 1} E_k}$ where, for each k, E_k is a k-dimensional subspace of E_0 with $E_k \subset E_{k+1}$. The fact that φ is measurable follows from the formula of representation of the ball measure of non-compactness for separable spaces which yields

$$\varphi(t) = \lim_{k\to\infty} \sup_{n\geq 1} d(f_n(t), E_k). \tag{3.7}$$

From $d(f_n(t), E_k) \leq |f_n(t)|_E$, (3.5) and (3.7) we have $\varphi(t) \leq \nu(t)$ a.e. on $[0,T]$. Consequently, $\varphi \in L^q(0,T;\mathbb{R}_+)$.

Since B is countable, we may suppose that (3.5) holds for all $t \in [0,T]$ and $f \in B$. To prove (3.6), let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|\Theta| \leq \delta \quad \Longrightarrow \quad \int_\Theta \nu(t)^q dt \leq \varepsilon^q. \tag{3.8}$$

Here $|\Theta|$ is the Lebesgue measure of Θ. Also, choose a constant $\rho > 0$ such that $|\Theta_1| < \frac{\delta}{2}$ for $\Theta_1 = \{t \in [0,T] : \nu(t) > \rho\}$. So we have $d(f_n(t), E_k) \leq |f_n(t)|_E \leq \rho$ for $t \in I \setminus \Theta_1$ and $n, k \geq 1$. Consequently, $d(f_n(t), E_k) = d(f_n(t), \overline{C}_k)$ with $\overline{C}_k = \{x \in E_k : |x|_E \leq \rho\}$.

From (3.7) and Egoroff's Theorem (see Dunford and Schwartz [11: pp. 149]) there is a set $\Theta_2 \subset [0,T] \setminus \Theta_1$ with $|\Theta_2| \leq \frac{\delta}{2}$ and an integer k_0 such that

$$\sup_{n\geq 1} d(f_n(t), \overline{C}_k) \leq \varphi(t) + \varepsilon \tag{3.9}$$

for $t \in [0,T] \setminus (\Theta_1 \cup \Theta_2)$ and $k \geq k_0$. Since B is a countable set of strongly measurable functions, we may find a set $\Theta_3 \subset [0,T]$ with $|\Theta_3| = 0$ and a countable set $\widetilde{B} = \{\widetilde{f}_n\}_{n\geq 1}$ of finitely-valued functions from $[0,T]$ to E with

$$|f_n(t) - \widetilde{f}_n(t)|_E \leq \varepsilon \tag{3.10}$$

for $t \in [0,T] \setminus \Theta_3$ and $n \geq 1$. From (3.9) and (3.10) we obtain

$$d(\widetilde{f}_n(t), \overline{C}_k) \leq \varphi(t) + 2\varepsilon$$

for $n \geq 1$, $k \geq k_0$ and $t \in [0,T] \setminus \Theta$ with $\Theta = \Theta_1 \cup \Theta_2 \cup \Theta_3$. Then there exists a finitely-valued function $\widehat{f}_{n,k}$ from $[0,T]$ to $\overline{C}_k$ with

$$|f_n(t) - \widehat{f}_{n,k}(t)|_E \leq \varphi(t) + 3\varepsilon \tag{3.11}$$

for $n \geq 1$, $k \geq k_0$ and $t \in [0,T] \setminus \Theta$. We put $\widehat{f}_{n,k}(t) = 0$ for $t \in \Theta$. Notice that $|\Theta| \leq \delta$.

For each fixed $k \geq k_0$, Theorem 1.4 guarantees that the set $\{\widehat{f}_{n,k}\}_{n\geq 1}$ is weakly relatively compact in $L^q(0,T;E)$. Then, from assumption (Ψ3), the set $\{\Psi \widehat{f}_{n,k}\}_{n\geq 1}$ is relatively compact in $L^p(0,T;F)$. Therefore, by Theorem 1.3, the set $\{\Psi \widehat{f}_{n,k}(t)\}_{n\geq 1}$ is relatively compact in F for all $t \in [0,T]$ except a subset of measure zero. Since an at most countable union of sets of measure zero also has measure zero, we may assume that $\{\Psi \widehat{f}_{n,k}(t)\}_{n\geq 1}$ is relatively compact for all $k \geq k_0$ and $t \in [0,T] \setminus \Theta_0$, where $|\Theta_0| = 0$. Let $t_0 \in [0,T] \setminus \Theta_0$ be arbitrary. Using assumption (Ψ1) and (3.11), we obtain

$$\begin{aligned} \big|\Psi f_n(t_0) - \Psi \widehat{f}_{n,k}(t_0)\big|_F &\leq \eta\big(t_0, |f_n(\cdot) - \widehat{f}_{n,k}(\cdot)|_E\big) \\ &\leq \eta(t_0,\varphi) + \big|\eta(t_0,\varphi+h) - \eta(t_0,\varphi)\big| \end{aligned} \tag{3.12}$$

where

$$h(t) = \begin{cases} 3\varepsilon & \text{for } t \in [0,T] \setminus \Theta \\ \nu(t) & \text{for } t \in \Theta. \end{cases}$$

Writing

$$h = h_1 + h_2$$

with

$$h_1(t) = \begin{cases} 3\varepsilon & \text{for } t \in [0,T] \setminus \Theta \\ 0 & \text{for } t \in \Theta \end{cases}$$

$$h_2(t) = \begin{cases} 0 & \text{for } t \in [0,T] \setminus \Theta \\ \nu(t) & \text{for } t \in \Theta \end{cases}$$

and using (3.8), we find that

$$|h|_q \leq |h_1|_q + |h_2|_q \leq 3\varepsilon T^{\frac{1}{q}} + \varepsilon.$$

Now (3.12) and (3.2) shows that the set $\{\Psi f_n(t_0)\}_{n\geq 1}$ admits a relatively compact ϵ-net of the form $\{\Psi \widehat{f}_{n,k}(t_0)\}_{n\geq 1}$ for every $\epsilon > \eta(t_0,\varphi)$. Letting $\epsilon \downarrow \eta(t_0,\varphi)$ we obtain (3.6) ∎

Lemma 3.2. *Let assumptions* (Ψ2) *and* (Ψ3) *hold. Further, let B be a countable subset of $L^q(0,T;E)$ such that $B(t)$ is relatively compact for a.e. $t \in [0,T]$ and there exists a function $\nu \in L^q(0,T;\mathbb{R}_+)$ with $|f(t)|_E \leq \nu(t)$ a.e. on $[0,T]$, for all $f \in B$. Then the set $\Psi(B)$ is relatively compact in $L^p(0,T;F)$. In addition, Ψ is continuous from B equipped with the relative weak topology of $L^q(0,T;E)$ to $L^p(0,T;F)$ equipped with its strong topology.*

Proof. Let $B = \{f_n\}_{n\geq 1}$ and let $\varepsilon > 0$ be arbitrary. As in the proof of Lemma 3.1 we can find functions $\widehat{f}_{n,k}$ with values in a compact $\overline{C}_k \subset E$ ($\overline{C}_k$ being a closed ball of a k-dimensional subspace of E) such that $|f_n - \widehat{f}_{n,k}|_q \leq \varepsilon$ for every $n \geq 1$. Then assumption (Ψ2) implies

$$\|\Psi f_n - \Psi \widehat{f}_{n,k}\| \leq L|f_n - \widehat{f}_{n,k}|_q \leq \varepsilon L. \tag{3.13}$$

On the other hand, the set $\{\widehat{f}_{n,k}\}_{n\geq 1} \subset L^q(0,T;E)$ is weakly relatively compact in $L^q(0,T;E)$. Next, assumption $(\Psi 3)$ guarantees that $\{\Psi\widehat{f}_{n,k}\}_{n\geq 1}$ is relatively compact in $L^p(0,T;F)$. Hence from (3.13) we see that $\{\Psi\widehat{f}_{n,k}\}_{n\geq 1}$ is a relatively compact εL-net of $\Psi(B)$ with respect to the norm $\|\cdot\|$. Since ε was arbitrary, we conclude that $\Psi(B)$ is relatively compact in $L^p(0,T;F)$.

Next we show that the graph

$$\Lambda = \big\{(f,w) : \, f \in B, \, w = \Psi f\big\}$$

is weakly-strongly sequentially closed in $L^q(0,T;E) \times L^p(0,T;F)$. To this end, assume (f_k) and (w_k) are sequences with $f_k \in B$ and $w_k = \Psi f_k$, $f_k \rightharpoonup f$ weakly and $w_k \to w$ strongly for some $f \in B$ and $w \in L^p(0,T;F)$. We shall prove that $w = \Psi f$. For an arbitrary number $\varepsilon > 0$, we have already seen that the proof of Lemma 3.1 provides a compact set P_ε and a sequence (f_k^ε) of P_ε-valued functions satisfying

$$|f_k - f_k^\varepsilon|_q \leq \varepsilon \tag{3.14}$$

for every k. The set $\{f_k^\varepsilon\}_{k\geq 1}$ being weakly relatively compact in $L^q(0,T,E)$, a suitable subsequence $(f_{k'}^\varepsilon)$ must be weakly convergent in $L^q(0,T,E)$ towards some f^ε. Consequently, $\Psi f_{k'}^\varepsilon \to \Psi f^\varepsilon$ strongly in $L^p(0,T;F)$. Also, Mazur's Lemma and (3.14) imply

$$|f - f^\varepsilon|_q \leq \varepsilon. \tag{3.15}$$

Now assumption $(\Psi 2)$ and the triangle inequality yields

$$\begin{aligned}
&\|w - \Psi f\| \\
&\quad \leq \|w - \Psi f_{k'}\| + \|\Psi f_{k'} - \Psi f_{k'}^\varepsilon\| + \|\Psi f_{k'}^\varepsilon - \Psi f^\varepsilon\| + \|\Psi f^\varepsilon - \Psi f\| \\
&\quad \leq \|w - w_{k'}\| + L|f_{k'} - f_{k'}^\varepsilon|_q + \|\Psi f_{k'}^\varepsilon - \Psi f^\varepsilon\| + L|f^\varepsilon - f|_q.
\end{aligned}$$

Using (3.14), (3.15) and $\|w - w_{k'}\| \to 0$ and $\|\Psi f_{k'}^\varepsilon - \Psi f^\varepsilon\| \to 0$ as $k' \to \infty$ we deduce that

$$\|w - \Psi f\| \leq 2\varepsilon L. \tag{3.16}$$

Since ε was arbitrary, (3.16) gives $w = \Psi f$ and the proof of Lemma 3.2 is complete ∎

Proof of Theorem 3.1. We apply Theorem 2.4 with $x_0 := w_0$, X the closed subspace of $L^p(0,T;F)$ generated by K, and $Y := L^q(0,T;E)$. Notice that, since $\overline{U}$ is bounded in K, there exists $a > 0$ such that $\|w\| \leq a$ for all $w \in \overline{U}$. Then from assumption $(\Phi 2)$ one has $|f(t)|_E \leq \nu_a(t)$ a.e. on $[0,T]$ for all $f \in \Phi w$ and $w \in \overline{U}$. It follows that the same inequality is true for all $f \in \overline{\text{co}}\,\Phi(\overline{U})$.

To guarantee that Ψ is sequentially w-u.s.c. on $\overline{\mathrm{co}}\,\Phi(A)$ for any compact convex subset A of $\overline{U}$ we have to show that

$$f_k \to f \text{ weakly, } f_k \in \overline{\mathrm{co}}\,\Phi(A) \quad \Longrightarrow \quad \Psi f_k \to \Psi f \text{ strongly.}$$

Let $A_c \subset A$ be countable such that $\{f_k\}_{k\geq 1} \subset \overline{\mathrm{co}}\,\Phi(A_c)$. In virtue of Theorem 1.3, $A_c(t)$ is relatively compact in F for a.e. $t \in [0,T]$. Then from (3.3) we deduce that $\beta_{E_0}(\Phi(A_c)(t) \cap E_0) = 0$ a.e. on $[0,T]$, for every separable closed subspace E_0 of E. As a result the set $\{f_k(t)\}_{k\geq 1}$ is relatively compact in E for a.e. $t \in [0,T]$. Now Lemma 3.2 guarantees that $\Psi f_k \to \Psi f$ strongly.

It remains to check condition (2.5) for the couple $[\Phi, \Psi]$. Let $A \subset \overline{U}$ be a closed convex set with

$$A \subset \overline{\mathrm{co}}\big(\{w_0\} \cup \Psi(\overline{\mathrm{co}}\,\Phi(A))\big).$$

To prove that A is compact it suffices that every sequence (w_n^0) of A has a convergent subsequence. Let $A_0 = \{w_n^0\}_{n\geq 1}$. Clearly, there exists a countable subset

$$A_1 = \{w_n^1\}_{n\geq 1}$$

of A, $f_n^1 \in \overline{\mathrm{co}}\,\Phi(A_1)$ and $v_n^1 = \Psi f_n^1$ with $A_0 \subset \overline{\mathrm{co}}(\{w_0\} \cup V^1)$, where $V^1 = \{v_n^1\}_{n\geq 1}$. Furthermore, there exists a countable subset

$$A_2 = \{w_n^2\}_{n\geq 1}$$

of A, $f_n^2 \in \overline{\mathrm{co}}\,\Phi(A_2)$ and $v_n^2 = \Psi f_n^2$ with $A_1 \subset \overline{\mathrm{co}}(\{w_0\} \cup V^2)$, where $V^2 = \{v_n^2\}_{n\geq 1}$, and so on. Hence for every $k \geq 1$ we find a countable subset

$$A_k = \{w_n^k\}_{n\geq 1}$$

of A and correspondingly $f_n^k \in \overline{\mathrm{co}}\,\Phi(A_k)$ and $v_n^k = \Psi f_n^k$ such that $A_{k-1} \subset \overline{\mathrm{co}}(\{w_0\} \cup V^k)$ and $V^k = \{v_n^k\}_{n\geq 1}$. Let

$$A^* = \cup_{k\geq 0} A_k.$$

It is clear that A^* is countable, $A_0 \subset A^* \subset A$ and $A^* \subset \overline{\mathrm{co}}(\{w_0\} \cup V^*)$, where $V^* = \cup_{k\geq 1} V^k$. Let $W^* := \{f_n^k\}_{n,k\geq 1}$. Since A^*, V^* and W^* are countable sets of strongly measurable functions, we may suppose that their values belong to a separable closed subspace F_0 of F in the case of A^* and V^*, respectively E_0 of E in the case of W^*. Since $|f_n^k(t)| \leq \nu_a(t)$ a.e. on $[0,T]$, Lemma 3.1 guarantees

$$\beta_{F_0}(A^*(t)) \leq \beta_{F_0}(V^*(t)) \leq \eta\big(t, \beta_{E_0}(W^*(\cdot))\big) \tag{3.17}$$

while assumption $(\Phi 3)$ gives

$$\beta_{E_0}(W^*(s)) \le \beta_{E_0}\big(\Phi(A^*)(s) \cap E_0\big) \le \zeta\big(\beta_{F_0}(A^*(\cdot))\big)(s). \qquad (3.18)$$

Since η is non-decreasing in its second variable, from (3.17) and (3.18) it follows that

$$\beta_{F_0}(A^*(t)) \le \eta\big(t, \zeta\big(\beta_{F_0}(A^*(\cdot))\big)\big).$$

Moreover, the function φ given by $\varphi(t) = \beta_{F_0}(A^*(t))$ belongs to $L^p(0,T;\mathbb{R}_+)$. Consequently, $\varphi \equiv 0$, and so $\varphi(t) = \beta_{F_0}(A^*(t)) = 0$ a.e. on $[0,T]$. Then (3.18) and $\zeta(0) = 0$ guarantee

$$\beta_{E_0}(W^*(t)) = 0 \qquad \text{a.e. on } [0,T]. \qquad (3.19)$$

Let (v_i^*) be any sequence of V^* and let (f_i^*) be the corresponding sequence of W^* with $v_i^* = \Psi f_i^*$ for all $i \ge 1$. Using (3.19) we have that (f_i^*) has a weakly convergent subsequence in $L^q(0,T;E)$, say converging to f. Then the corresponding subsequence of (v_i^*) converges to $v = \Psi f$ in $L^p(0,T;F)$. Hence V^* is relatively compact. Now Mazur's Lemma guarantees that the set $\overline{\text{co}}(\{w_0\} \cup V^*)$ is compact and so its subset A^* is relatively compact. Thus A_0 possesses a convergent subsequence as we wished. Now the result follows from Theorem 2.4 ∎

Remark 3.1.

(a) If the values of Ψ are in $C(0,T;F)$, then any solution of inclusion problem (3.1) in $K \subset L^p(0,T;F)$ $(1 \le p \le \infty)$ belongs to $C(0,T;F)$.

(b) The existence theory in $C(0,T;F)$ appears as a particular case, where $p = \infty$ and $K \subseteq C(0,T;F)$.

Remark 3.2.

(a) The typical example of a function η in assumption $(\Psi 1)$ which occurs in applications is the one defined by $\eta(t,\varphi) = \int_0^T k(t,s)\varphi(s)\,ds$ where $k : [0,T]^2 \to \mathbb{R}_+$ and $k(t,\cdot) \in L^r(0,T)$ for a.e. $t \in [0,T]$ (see Couchouron and Precup [5, 6], and O'Regan and Precup [21]). In this case condition $(\Psi 2)$ is a consequence of condition $(\Psi 1)$.

(b) For $k(t,s) = \begin{cases} 0 & \text{if } t < s \\ m & \text{if } s \le t \end{cases}$, where $m > 0$ is a constant, the function η is defined as $\eta(t,\varphi) = m\int_0^t \varphi(s)\,ds$ and occurs when Ψ is the mild solution operator of the Cauchy problem associated to abstract evolution equations (see Couchouron and Kamenskii [4], and Kamenskii, Obukhovskii and Zecca [16]). In this case, and if

$$\zeta(\varphi)(t) = m_0\varphi(t) + \int_0^t \delta(s)\varphi(s)\,ds \qquad (3.20)$$

where $m_0 > 0$ and $\delta \in L^{r'}(0,T;\mathbb{R}_+)$ with $r' > 2$, the null function is the unique solution of inequality (3.4). Indeed, if $\varphi(t) \le \eta(t,\zeta(\varphi))$, then

$$\begin{aligned}\varphi(t) &\le m\int_0^t \left(m_0\varphi(s) + \int_0^s \delta(\tau)\varphi(\tau)\,d\tau\right)ds \\ &= m\int_0^t \left(m_0 e^{\theta s}\varphi(s)e^{-\theta s} + \int_0^s e^{\theta\tau}\delta(\tau)\varphi(\tau)e^{-\theta\tau}d\tau\right)ds \\ &\le m m_0 |e^{\theta s}|_{L^2(0,t)}|\varphi(s)e^{-\theta s}|_{L^2(0,T)} \\ &\quad + mT|e^{\theta s}|_{L^r(0,t)}|\delta|_{L^{r'}(0,T)}|\varphi(s)e^{-\theta s}|_{L^2(0,T)}\end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} + \frac{1}{2} = 1$. It follows that

$$\varphi(t) \le m e^{\theta t}|\varphi(s)e^{-\theta s}|_{L^2(0,T)}\Big(\frac{m_0}{\sqrt{2\theta}} + \frac{T|\delta|_{L^{r'}(0,T)}}{(\theta r)^{1/r}}\Big).$$

Divide by $e^{\theta t}$ and take the L^2-norm to obtain

$$|\varphi(s)e^{-\theta s}|_{L^2(0,T)} \le m\sqrt{T}|\varphi(s)e^{-\theta s}|_{L^2(0,T)}\Big(\frac{m_0}{\sqrt{2\theta}} + \frac{T|\delta|_{L^{r'}(0,T)}}{(\theta r)^{1/r}}\Big).$$

Clearly, if θ is sufficiently large, this implies $|\varphi(s)e^{-\theta s}|_{L^2(0,T)} = 0$. Thus $\varphi = 0$.

Remark 3.3. Let Ψ satisfy the following stronger compactness condition:

(Ψ4) If B is any bounded subset of $L^q(0,T;E)$ for which there exists a function $\nu \in L^q(0,T;\mathbb{R}_+)$ such that $|f(t)|_E \le \nu(t)$ a.e. on $[0,T]$, for all $f \in B$, then $\{\Psi f\}_{f\in B}$ is relatively compact in $L^p(0,T;F)$.

Then the conclusion of Theorem 4.1 is true without assumptions (Ψ1) and (Φ3). Indeed, under assumption (Ψ4) the compactness of the set A satisfying (2.5) is immediate since $\Psi(\overline{\text{co}}\,\Phi(A))$ is relatively compact in $L^p(0,T;F)$.

Condition (Ψ4) has been required in Bader [3]. For a discussion on this condition, when Ψ is the mild solution operator for the initial value problem associated to an m-accretive map, see Vrabie [23].

Example. Let us consider the initial value problem for a functional-differential inclusion

$$\left.\begin{aligned} &u'(t) \in (\Phi u)(t) \quad \text{a.e. on } [0,T] \\ &u(0) = u_0 \end{aligned}\right\}. \tag{3.21}$$

Theorem 3.2. *Let E be a Banach space and let $\Phi : C([0,T];E) \to 2^{L^1(0,T;E)}$. Let assumptions* ($\Phi$1) – ($\Phi$3) *hold with* $p = \infty, q = 1, E = F, K =$

$C([0,T];E)$ *and* ζ *given by* (3.20). *In addition, assume that there exists* $a \in L^1(0,T;\mathbb{R}_+)$ *and a non-decreasing function* $b : \mathbb{R}_+ \to (0,\infty)$ *such that*

$$|f(t)|_E \le a(t)b(|u(t)|_E)$$

a.e. on $[0,T]$, *for all* $u \in C([0,T];E)$ *and* $f \in \Phi u$, *and*

$$\int_0^T a(s)\,ds < \int_{|u_0|_E}^{\infty} \frac{d\tau}{b(\tau)}.$$

Then problem (3.21) *has a solution in* $W^{1,1}(0,T;E)$.

Proof. Let $\Psi : L^1(0,T;E) \to C([0,T];E)$ be defined by

$$(\Psi f)(t) = u_0 + \int_0^t f(s)\,ds.$$

We can easily see that Ψ satisfies assumptions $(\Psi 1) - (\Psi 3)$ with $\eta(t,\varphi) = \int_0^t \varphi(s)\,ds$. Then recall Remark 3.2. On the other hand, a standard argument (see O'Regan and Precup [20: pp. 29] and Precup [22: pp. 74]) guarantees the existence of a number $R > 0$ with $|u(t)|_E < R$ for all $t \in [0,T]$ and any solution u of $u \in \lambda\Psi\Phi u$, for $\lambda \in [0,1]$. Hence $\|u\| := \max_{t\in[0,T]} |u(t)|_E < R$ and so condition (L-S) holds with $U = \{u \in C([0,T];E) : \ \|u\| < R\}$. Now the result follows from Theorem 3.1 ∎

References

[1] Andres, J. and R. Bader: *Asymptotic boundary value problems in Banach spaces.* J. Math. Anal. Appl. 274 (2002), 437 – 457.

[2] Arino, O., Gautier, S. and J. P. Penot: *A fixed point theorem for sequentially continuous maps with application to ordinary differential equations.* Funkcial. Ekvac. 27 (1984), 273 – 279.

[3] Bader, R.: *A topological fixed-point index theory for evolution inclusions.* Z. Anal. Anw. 20 (2001), 3 – 15.

[4] Couchouron, J.-F. and M. Kamenskii: *An abstract topological point of view and a general averaging principle in the theory of differential inclusions.* Nonlin. Anal. 42 (2000), 1101 – 1129.

[5] Couchouron, J.-F. and R. Precup: *Existence principles for inclusions of Hammerstein type involving noncompact acyclic multivalued maps.* Electron. J. Diff. Equ. 2002 (2002), No. 04, 1 – 21.

[6] Couchouron, J.-F. and R. Precup: *Anti-periodic solutions for second order differential inclusions* (to appear).

[7] De Blasi, F. S.: *On a property of the unit sphere in Banach spaces.* Bull. Math. Soc. Sci. Math. Roum. 21 (1977), 259 – 262.

[8] Deimling, K.: *Nonlinear Functional Analysis.* Berlin et al.: Springer-Verlag 1985.

[9] Deimling, K.: *Multivalued Differential Equations.* Berlin - New York: Walter De Gruyter 1992.

[10] Diestel, J., Ruess, W. M. and W. Schachermayer: *Weak compactness in* $L^1(\mu, X)$*.* Proc. Amer. Math. Soc. 118 (1993), 447 – 453.

[11] Dunford, N. and J. T. Schwartz: *Linear Operators.* Part I: *General Theory.* New York: Intersci. 1957.

[12] Górniewicz, L.: *Topological approach to differential inclusions.* In: Topological Methods in Differential Equations and Inclusions (NATO ASI Series C 472; eds.: A. Granas and M. Frigon). Dordrecht: Kluwer Acad. Publ. 1995, pp. 129 – 190.

[13] Guo, D., Lakshmikantham, V. and X. Liu: *Nonlinear Integral Equations in Abstract Spaces.* Dordrecht - Boston - London: Kluwer Acad. Publ. 1996.

[14] Hu, S. and N. S. Papageorgiou: *Handbook of Multivalued Analysis.* Vol. I: *Theory.* Dordrecht - Boston - London: Kluwer Acad. Publ. 1997.

[15] Hu, S. and N. S. Papageorgiou: *Handbook of Multivalued Analysis.* Vol. II: *Applications.* Dordrecht - Boston - London: Kluwer Acad. Publ. 2000.

[16] Kamenskii, M., Obukhovskii, V. and P. Zecca: *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces.* Berlin - New York: Walter de Gruyter 2001.

[17] Mönch, H.: *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces.* Nonlin. Anal. 4 (1980), 985 – 999.

[18] O'Regan, D.: *Fixed point theory of Mönch type for weakly sequentially upper semicontinuous maps.* Bull. Austral. Math. Soc. 61 (2000), 439 – 449.

[19] O'Regan, D. and R. Precup: *Fixed point theorems for set-valued maps and existence principles for integral inclusions.* J. Math. Anal. Appl. 245 (2000), 594 – 612.

[20] O'Regan, D. and R. Precup: *Theorems of Leray-Schauder Type and Applications.* Amsterdam: Gordon and Breach Sci. Publ. 2001.

[21] O'Regan, D. and R. Precup: *Existence theory for nonlinear operator equations of Hammerstein type in Banach spaces.* J. Dyn. Syst. Appl. (to appear).

[22] Precup, R.: *Methods in Nonlinear Integral Equations.* Dordrecht - Boston - London: Kluwer Acad. Publ. 2002.

[23] Vrabie, I. I.: *Compactness Methods for Nonlinear Evolutions.* Harlow: Longman Sci. & Techn. 1987.

Received 13.02.2003