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ON SOME BERNSTEIN TYPE OPERATORS: ITERATES AND GENERALIZATIONS

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The present paper focuses on two approaches. Firstly, by using the contraction principle, we give a method for obtaining the limit of iterates of a class of linear positive operators. This general method is applied in studying three sequences of modified Bernstein type operators. Secondly, we define a generalization of Goodman-Sharma operators. We investigate the degree of approximation obtaining pointwise and global estimates in the framework of various function spaces.

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1. Introduction

The Bernstein type approximation processes have been the object of many investigations serving as a guide for theorems that can be proved for a large class of positive linear operators on a bounded interval. The purpose of this note is twofold.

Based on the theory of weakly Picard operators, see e.g. [6], our first aim is to present a general method for obtaining the limit of iterates of a given linear operator. This approach will be given in Section 2. In order to illustrate the method we apply it to three different classes of modified Bernstein operators. These sequences of operators were introduced respectively by Cheney and Sharma [2], Stancu [7], Goodman and Sharma [4], and studied in time intensively by many authors.

Further on, in Section 3, we generalize the Goodman and Sharma operators by replacing the binomial coefficients with other general coefficients satisfying a suitable recursive relation with the help of a given sequence. These new polynomials P_nf, $n \in \mathbb{N}$, are associated to any function f belonging to the space $L_1([0,1])$ and, under some additional conditions, they define an approximation process. This way, the second aim of the paper is to investigate the new class of linear positive operators. We enlarge upon this study in Section 4, proving that $(P_n)_{n\geq 1}$ converges to the identity operator multiplied by an analytic function φ and estimating the degree of convergence by using the modulus of smoothness of first order. We establish both pointwise and global estimates of the rate of convergence of our operators. Special cases of the function φ are presented in the final part of this section.

2. Operator iterates and contraction principle

Let (X,d) be a metric space and $A : X \to X$ an operator. In the sequel $\mathcal{F}_A$ stands for the fixed point set of A, $\mathcal{F}_A := \{x \in X | A(x) = x\}$. As usual, we set $A^0 = I_X$, $A^1 = A$, $A^{m+1} = A \circ A^m$, $m \in \mathbb{N}$, where I_X indicates the identity operator of the space X.

Following [6] closely we recall the notion of weakly Picard operators (briefly, WPO) and a characterization of them.

Definition 1. Let (X,d) be a metric space. An operator $A : X \to X$ is weakly Picard operator if the sequence of iterates $(A^m(x))_{m\geq 1}$ converges for every $x \in X$ and the limit is a fixed point of A.

If the operator A is WPO and $\mathcal{F}_A$ has a unique element, say x^*, then A is called a Picard operator (PO).

Theorem 1. ([6]) *Let (X,d) be a metric space. The operator $A : X \to X$ is WPO if and only if a partition of X exists, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that for every $\lambda \in \Lambda$ one has*

(i) $X_\lambda \in \mathcal{I}(A)$,
(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,*

where $\mathcal{I}(A) := \{Y \subset X | Y \neq \emptyset,\ A(Y) \subset Y\}$ represents the family of the non-empty invariant subsets of A.

Further on, if A is WPO we consider $A^\infty : X \to X$ defined by

$$A^\infty(x) := \lim_{m\to\infty} A^m(x), \quad x \in X. \tag{1}$$

If A is WPO, then one has $\mathcal{F}_{A^m} = \mathcal{F}_A \neq \emptyset$ for all $m \in \mathbb{N}$. We deduce that

$$A^\infty(X) = \mathcal{F}_A. \tag{2}$$

Based on these results, in what follows we consider $X = C([0,1])$ or $X = L_1([0,1])$. These spaces are respectively endowed with the sup-norm

$\|\cdot\|_\infty$, $\|f\|_\infty = \sup\limits_{x\in[0,1]} |f(x)|$, and L_1-norm $\|\cdot\|_1$, $\|f\|_1 = \int_0^1 |f(t)|\,dt$. Defining the sets

$$X_{\alpha,\beta} := \{f \in X \mid f(0) = \alpha,\ f(1) = \beta\}, \quad (\alpha,\beta) \in \mathbb{R}\times\mathbb{R}, \tag{3}$$

we remark that every $X_{\alpha,\beta}$ is a closed subset of X. At the same time, the system $\{X_{\alpha,\beta} : (\alpha,\beta) \in \mathbb{R}\times\mathbb{R}\}$ makes a partition of this space.

Let $L : X \to X$ be a linear operator satisfying the following conditions:

(4) (C_1) L has a degree of exactness 1,

this meaning that $Le_j = e_j$, $j \in \{0,1\}$, where $e_j \in \mathbb{R}^{[0,1]}$ stands, as usual, for the monomial test-function: $e_0(x) = 1$, $e_j(x) = x^j$, $j \in \mathbb{N}$, $x \in [0,1]$.

(5) (C_2) L is interpolatory at both ends of the interval $[0,1]$,

this meaning $(Lf)(0) = f(0)$ and $(Lf)(1) = f(1)$, for any $f \in X$. Consequently, for all $(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}$, $X_{\alpha,\beta}$ defined by (3) is an invariant set of L. In this respect, we can formulate the third condition.

(6) (C_3) $L|_{X_{\alpha,\beta}} : X_{\alpha,\beta} \to X_{\alpha,\beta}$ is a contraction for every $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$.

At this moment we define $p_{\alpha,\beta}(x) := \alpha + (\beta-\alpha)x$, $x \in [0,1]$.

Clearly $p_{\alpha,\beta} \in X_{\alpha,\beta}$. Since L is linear and relation (4) holds true, L reproduces the affine functions, and consequently, $p_{\alpha,\beta}$ is a fixed point of L. For any $f \in X$ one has $f \in X_{f(0),f(1)}$ and, by using Theorem 1 combined with relations (1) and (2), we conclude that

$$\lim_{m\to\infty} (L^m f)(x) = f(0) + (f(1) - f(0))x, \quad x \in [0,1], \tag{7}$$

in the norm of the space X.

By using this approach we are able to obtain the limit of iterates of some sequences of Bernstein type operators. At the beginning, we recall the fundamental Bernstein polynomials $(b_{n,k})_{k=\overline{0,n}}$ of degree n, given by

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1]. \tag{8}$$

Also we need

$$\min_{x\in[0,1]} ((1-x)^q + x^q) = 2^{1-q}, \text{ for every } q \in \mathbb{N}. \tag{9}$$

Example 1. Based on a combinatorial identity of Abel–Jensen, Cheney and Sharma [2] have introduced and investigated the operators

$$(Q_n f)(x;\beta) = \sum_{k=0}^{n} q_{n,k}(x;\beta) f\left(\frac{k}{n}\right),\ f \in C([0,1]),\ x \in [0,1],\ n \in \mathbb{N}, \tag{10}$$

where $q_{n,k}(x;\beta) = (1+n\beta)^{1-n}\binom{n}{k}x(x+k\beta)^{k-1}(1-x)[1-x+(n-k)\beta]^{n-1-k}$ and β is a non-negative real parameter.

After calculation of some integrals involved, the authors proved that Q_n preserves the constant functions. But these operators seem to be more rewarding than the authors imagined. Indeed, in [8] was shown that Q_n reproduces the linear functions, thus (4) is fulfilled. Also, it is easy to see that (5) holds true. Moreover, for any $f, g \in X_{\alpha,\beta}$, see (3), we get

$$\begin{aligned}
&|(Q_n f)(x) - (Q_n g)(x)| \\
&\quad \le (1 - q_{n,0}(x;\beta) - q_{n,n}(x;\beta)) \sup_{x\in[0,1]} |f(x) - g(x)| \\
&\quad = \left(1 - \frac{(1-x)(1-x+n\beta)^{n-1} + x(x+n\beta)^{n-1}}{(1+n\beta)^{n-1}}\right) \|f-g\|_\infty \\
&\quad \le \left(1 - \frac{(1-x)^n + x^n}{(1+n\beta)^{n-1}}\right) \|f-g\|_\infty \\
&\quad \le \left(1 - \frac{1}{2^{n-1}(1+n\beta)^{n-1}}\right) \|f-g\|_\infty,
\end{aligned}$$

since $\beta \ge 0$; also we used (9). Consequently, (6) is fulfilled and, for every $f \in C([0,1])$, relation (7) holds true, where $L = Q_n$, $n \in \mathbb{N}$.

Example 2. By using a probabilistic method, Stancu [7, Eq. (3.2)] constructed the operators

$$(11) \qquad (L_{n,r}f)(x) = \sum_{k=0}^{n-r} b_{n-r,k}(x)\left\{(1-x)f\left(\frac{k}{n}\right) + xf\left(\frac{k+r}{n}\right)\right\},$$

$f \in C([0,1])$, $x \in [0,1]$, r is a fixed non-negative integer such that $n > 2r$ and $b_{n-r,k}$, $k = \overline{0, n-r}$, are given by (8). By a straightforward calculation one can verify both (4) and (5). Like in the previous example, for f, g belonging to the space $X_{\alpha,\beta}$, we have

$$\begin{aligned}
&|(L_{n,r}f)(x) - (L_{n,r}g)(x)| \\
&\quad \le \left\{(1-x)\sum_{k=1}^{n-r} p_{n-r,k}(x) + x\sum_{k=0}^{n-r-1} p_{n-r,k}(x)\right\}\|f-g\|_\infty \\
&\quad = \{(1-x)(1-p_{n-r,0}(x)) + x(1-p_{n-r,n-r}(x))\}\|f-g\|_\infty \\
&\quad = \{1 - ((1-x)^{n-r+1} + x^{n-r+1})\}\|f-g\|_\infty \\
&\quad \le (1-2^{r-n})\|f-g\|_\infty,
\end{aligned}$$

see (9). Hence, the condition (C_3) is verified and the limit of iterates of any polynomial $L_{n,r}f$, $f \in C([0,1])$, is given in (7).

We mention that choosing $\beta = 0$ in (10) respectively $r = 0$ in (11) both classes become the Bernstein polynomials $(B_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f(k/n)$, where $(b_{n,k})_{k=\overline{0,n}}$ is given in (8). Regarding this sequence, the result from (7) was obtained long time ago by Kelisky and Rivlin [5].

Example 3. Now we are dealing with the operators introduced and studied by Goodman and Sharma in [4]. For each $n > 2$, these operators are given by

$$
\begin{aligned}
(U_n f)(x) &= f(0) b_{n,0}(x) + f(1) b_{n,n}(x) \\
&\quad + (n-1) \sum_{k=1}^{n-1} b_{n,k}(x) \int_0^1 b_{n-2,k-1}(t) f(t)\, dt,
\end{aligned}
\tag{12}
$$

where $f \in L_1([0,1])$ and $x \in [0,1]$. They share, in some sense, the advantages both of Bernstein discrete operators B_n and Durrmeyer integral operators M_n studied by Derriennic [3],

$$
(M_n f)(x) = (n+1) \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 b_{n,k}(t) f(t)\, dt, \quad f \in L_1([0,1]).
$$

Like the Bernstein operator, U_n reproduces linear functions and $U_n f$ interpolates f at the vertices of the interval $[0,1]$. Consequently, the conditions (C_1), (C_2) are fulfilled. Since

$$
\int_0^1 b_{n-2,k-1}(t)\, dt = (n-1)^{-1}, \quad k = \overline{1, n-1},
\tag{13}
$$

for every f and g belonging to $X_{\alpha,\beta}$ defined in (3), one has

$$
|(U_n f)(x) - (U_n g)(x)| \le (1 - b_{n,0}(x) - b_{n,n}(x)) \|f-g\|_\infty \le \left(1 - \frac{1}{2^{n-1}}\right) \|f-g\|_\infty,
$$

and this implies that (C_3) is satisfied. Thus, (7) takes place for $L = U_n$.

Starting from (12) and keeping in mind a generalization of the classical Bernstein polynomials $B_n f$ due to Campiti and Metafune [1], we present

3. A generalization of Goodman–Sharma operators

We consider a given sequence $\widetilde{a} = (a_n)_{n \ge 1}$ of real positive numbers and define the coefficients $c_{n,k}$, $0 \le k \le n$, as follows

$$
c_{n,0} = c_{n,n} := a_n, \quad c_{n+1,k+1} = c_{n,k+1} + c_{n,k}, \quad 0 \le k \le n-1.
\tag{14}
$$

For convenience of the reader, we indicate some particular values.

For $n=3$: $c_{3,3}=c_{3,0}=a_3$ and $c_{3,1}=c_{3,2}=2a_1+a_2$.

For $n=4$: $c_{4,4}=c_{4,0}=a_4$, $c_{4,1}=c_{4,3}=2a_1+a_2+a_3$ and $c_{4,2}=2a_2+4a_1$.

For $n=5$: $c_{5,0}=c_{5,5}=a_5$, $c_{5,1}=c_{5,4}=2a_1+a_2+a_3+a_4$ and $c_{5,2}=c_{5,3}=6a_1+3a_2+a_3$.

Remarks. For each $n>2$ formula (14) implies the following properties.

(1) $c_{n,1}=c_{n,n-1}=2a_1+a_2+\cdots+a_{n-1}$.

(2) If the sequence $\widetilde{a}$ consists of 1's, then $c_{n,k}=\binom{n}{k}$, $0\le k\le n$.

(3) If the sequence $\widetilde{a}$ is bounded ($0\le a_i\le M$, $i\in\mathbb{N}$), then $c_{n,k}\le M\binom{n}{k}$, $0\le k\le n$.

For every $n\in\mathbb{N}$ and $k\in\{0,1,\ldots,n\}$ we define the polynomials

$$q_{n,k}(x):=c_{n,k}p_{n,k}(x),\ \text{ where } p_{n,k}(x):=x^k(1-x)^{n-k},\ \ x\in[0,1]. \tag{15}$$

Lemma 1. *If $q_{n,k}$, $0\le k\le n$, are defined by (15), then*

$$\sum_{k=1}^{n-1} q_{n,k}(x)=\sum_{m=1}^{n-1} a_m\{x(1-x)^m+(1-x)x^m\},\quad n\ge 2. \tag{16}$$

Proof. It is simple and runs by induction. For the sake of brevity we set $X_m:=x(1-x)^m+(1-x)x^m$, $1\le m\le n-1$.

For $n=2$ the common value of the two numbers is $2a_1x(1-x)$. Combining (14), (15) and taking into account that (16) holds true, for $n+1$ we get

$$\begin{aligned}
\sum_{k=1}^{n} q_{n+1,k}(x) &= \sum_{k=1}^{n} c_{n,k}p_{n+1,k}(x)+\sum_{k=1}^{n} c_{n,k-1}p_{n+1,k}(x)\\
&= (1-x)\sum_{k=1}^{n-1} c_{n,k}p_{n,k}(x)+c_{n,n}(1-x)x^n\\
&\qquad +c_{n,0}(1-x)^n x+x\sum_{j=1}^{n-1} c_{n,j}p_{n,j}(x)\\
&= (1-x)\sum_{m=1}^{n-1} a_mX_m+a_nX_n+x\sum_{j=1}^{n-1} a_jX_j\\
&= \sum_{m=1}^{n} a_mX_m,
\end{aligned}$$

and this completes the proof. □

With the help of (16) and (8), for each $n > 2$ we define the polynomials

$$(17)\quad \begin{aligned}(P_n f)(x) &= f(0)(1-x)^n + f(1)x^n \\ &+ (n-1)\sum_{k=1}^{n-1} q_{n,k}(x)\int_0^1 b_{n-2,k-1}(t)f(t)\,dt,\end{aligned}$$

where $f \in L_1([0,1])$ and $x \in [0,1]$.

It is clear that the polynomial $P_n f$ is determined uniquely by the sequence $\widetilde{a}$. We can emphasize on this fact by using a more precise notation named $P_n^{\langle \widetilde{a}\rangle} f$. Throughout the paper we shall use one or another of the notations as required by the context.

Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree less than or equal to n. Examining (17) and taking into account our Remarks, in the next theorem we gathered some evident properties of these operators.

Theorem 2. *Let P_n, $n > 2$, be defined by (17).*

(i) *P_n is a linear positive operator and maps the space $L_1([0,1])$ into $\mathcal{P}_n$.*

(ii) *$(P_n f)(x_0) = f(x_0)$, for $x_0 \in \{0,1\}$.*

(iii) *If the sequences $\widetilde{a} = (a_n)_{n\ge 1}$, $\widetilde{a}' = (a'_n)_{n\ge 1}$ satisfy the condition $a_n \le a'_n$ for every n, then $P_n^{\langle \widetilde{a}\rangle} f \le P_n^{\langle \widetilde{a}'\rangle} f$ for every positive function $f \in L_1([0,1])$.*

(iv) *If the sequence $\widetilde{a}$ consists of 1's, then $P_n^{\langle 1\rangle}$ reduces to the operator U_n defined by (12).*

(v) *If M is an upper bound for the elements of $\widetilde{a}$, then for every $f \in C([0,1])$ one has $\|P_n^{\langle \widetilde{a}\rangle} f\|_\infty \le M\|f\|_\infty$.*

Next, we shall study the convergence of our sequence of operators, giving estimates of the rate of convergence.

4. Approximation properties of P_n operators

We need the following two technical results.

Lemma 2. *If P_n, $n > 2$, are defined by (17) then, for $x \in [0,1]$,*

$$(18)\quad \text{(i)}\quad (P_n e_0)(x) = (1-x)^n + x^n + \sum_{m=1}^{n-1} a_m\left(x(1-x)^m + (1-x)x^m\right).$$

$$(19)\quad \text{(ii)}\quad |(P_n f)(x)| \le \beta(n)(U_n|f|)(x), \text{ where}$$

$$(20)\quad \beta(n) := \max\{1, \alpha(n)\}, \quad \alpha(n) := \max_{1\le m\le n-1} a_m$$

and U_n is given in (12).

Proof. The first statement is implied by Lemma 1.

(ii) The operator P_n being linear and positive, we have $|P_n f| \le P_n|f|$. On the other hand, since $c_{n,k} \le \alpha(n)\binom{n}{k}$ for $k = \overline{1, n-1}$, we can write

$$\begin{aligned}(P_n|f|)(x) &\le |f(0)|(1-x)^n + |f(1)|x^n \\ &\quad +\alpha(n)(n-1)\sum_{k=1}^{n-1} b_{n,k}(x)\int_0^1 b_{n-2,k-1}(t)|f(t)|\,dt \\ &= \alpha(n)(U_n|f|)(x) + (1-\alpha(n))(|f(0)|(1-x)^n + |f(1)|x^n).\end{aligned}$$

The above relation implies: if $\alpha(n) \ge 1$, then $P_n|f| \le \alpha(n)U_n|f|$; if $\alpha(n) < 1$, then $P_n|f| \le U_n|f|$. Thus, (19) holds true and the proof is finished. □

The quantity $\mathcal{M}_2(L, x) := L((e_1 - xe_0)^2, x)$ is the second order central moment of the operator L. Using the properties proved in [4], we find

$$\mathcal{M}_2(U_n, x) = \frac{2x(1-x)}{n+1}, \quad x \in [0,1]. \tag{21}$$

Theorem 3. *Let P_n, $n > 2$, be defined by* (17). *For any bounded function f from $L_1([0,1])$ holds the inequality*

$$\begin{aligned}&|(P_n f)(x) - f(x)(P_n e_0)(x)| \\ &\quad \le \beta(n)\{((1-x)^n + x^n)\omega_f(1) + (1 + 2x(1-x))\omega_f(1/\sqrt{n+1})\},\end{aligned}$$

$x \in [0,1]$, where ω_f is the first modulus of smoothness of f and $\beta(n)$ is given in (20).

Proof. Based on Theorem 2, see (ii), for $x \in \{0,1\}$ the left side of the inequality vanishes, thus we concentrate only on $x \in (0,1)$. We set

$$S_n(f,x) := (n-1)\sum_{k=1}^{n-1} b_{n,k}(x)\int_0^1 b_{n-2,k-1}(t)|f(t) - f(x)|\,dt.$$

Writing (19) for the function $f - f(x)e_0$ and using the linearity of P_n, we get

$$\begin{aligned}&|(P_n f)(x) - f(x)(P_n e_0)(x)| \\ &\quad \le \beta(n)\{|f(0) - f(x)|(1-x)^n + |f(1) - f(x)|x^n + S_n(f,x)\} \\ &\quad \le \beta(n)\{((1-x)^n + x^n)\omega_f(1) + S_n(f,x)\}.\end{aligned} \tag{22}$$

The well-known estimate

$$|f(t) - f(x)| \le (1 + \delta^{-2}(t-x)^2)\omega_f(\delta), \quad (t,x) \in [0,1]\times[0,1],\ \delta > 0,$$

combined with (13) and (21) as well, allow us to write

$$\begin{aligned} S_n(f,x) & \\ & \le \Big\{\sum_{k=1}^{n-1} b_{n,k}(x) + \frac{1}{\delta^2}(\mathcal{M}_2(U_n,x) - x^2(1-x)^n - (1-x)^2x^n)\Big\}\omega_f(\delta) \\ & \le \left(1 + \frac{2x(1-x)}{(n+1)\delta^2}\right)\omega_f(\delta). \end{aligned}$$

Choosing $\delta = (n+1)^{-1/2}$ and returning to (22) we obtain the desired result. □

Under the assumption that the sequence $\widetilde{a}$ is bounded, we can define the numbers $\gamma(n)$, $n > 2$, respectively the function $\varphi \in \mathbb{R}^{[0,1]}$ as follows

$$\begin{aligned} \gamma(n) &:= \sup_{m\ge n} a_m, \\ \varphi(0) = \varphi(1) = 1,\ \varphi(x) &= \sum_{m=1}^{\infty} a_m(x(1-x)^m \\ &\quad +(1-x)x^m),\ x \in (0,1). \end{aligned} \tag{23}$$

The definition of φ was suggested by formula (18). Also, we mention that the boundedness of $\widetilde{a}$ guarantees that the power series which appears in (23) has a radius of convergence greater than or equal to 1.

In the particular case $\widetilde{a} = \{1\}$, we obtain $\varphi(x) = 1$, $x \in [0,1]$.

Lemma 3. *If $\gamma(n)$ and φ are given by (23), then P_n, $n > 2$, satisfy*

$$|(P_n e_0)(x) - \varphi(x)| \le \max\{1, \gamma(n)\}((1-x)^n + x^n), \quad x \in [0,1].$$

Proof. The inequality is clear for $x = 0$ and $x = 1$. Assume that $x \in (0,1)$. Taking into account (18) we can write

$$\begin{aligned} |(P_n e_0)(x) - \varphi(x)| & \\ &= (1-x)^n + x^n + \sum_{m=n}^{\infty} a_m(x(1-x)^m + (1-x)x^m) \\ &\le (1-x)^n + x^n + \gamma(n)\Big\{x\sum_{m=n}^{\infty}(1-x)^m + (1-x)\sum_{m=n}^{\infty} x^m\Big\} \\ &= (1-x)^n + x^n + \gamma(n)((1-x)^n + x^n), \end{aligned}$$

and the conclusion of our lemma follows. □

Theorem 4. *Let P_n, $n > 2$, be defined by (17) with a bounded sequence $\widetilde{a}$. For any bounded function $f \in L_1([0,1])$ holds the inequality*

$$\begin{aligned} |(P_n f)(x) - \varphi(x) f(x)| & \\ &\le \widetilde{\beta}(n)\Big\{\frac{3}{2}\omega_f(1/\sqrt{n+1}) + ((1-x)^n + x^n)(\omega_f(1) + |f(x)|)\Big\}, \end{aligned}$$

$x \in [0,1]$, *where* $\widetilde{\beta}(n) := \max\{1, \max_{1\le m\le n-1} a_m, \sup_{m\ge n} a_m\}$.

Proof. Evidently,

$$(24)\qquad \begin{aligned} |(P_n f)(x) - \varphi(x)f(x)| &\le |(P_n f)(x) - f(x)(P_n e_0)(x)| \\ &\quad + |f(x)||(P_n e_0)(x) - \varphi(x)|. \end{aligned}$$

With the help of Theorem 3, Lemma 3, the definition of $\beta(n)$, $\gamma(n)$ and the elementary result $\max_{x\in[0,1]} x(1-x) = 1/4$, the announced inequality is proved.

□

As usual, we denote by $B([0,1])$ the Banach space of all real-valued bounded functions defined on $[0,1]$, endowed with the uniform norm $\|\cdot\|_\infty$.

Clearly, $\max_{x\in[0,1]}((1-x)^n + x^n) = 1$, $\omega_f(1) \le 2\|f\|_\infty$. Taking into account these facts, Theorem 4 leads us to the main result of this section establishing that $(P_n)_n$ defines an approximation process and giving a global estimate of the rate of convergence.

Theorem 5. *Let* P_n, $n > 2$, *be defined by* (17) *with a bounded sequence* $\widetilde{a}$. *For every* $f \in B([0,1]) \cap L_1([0,1])$, *we have*

(i) $\lim_{n\to\infty} (P_n f)(x) = \varphi(x)f(x)$, *uniformly on every compact* $K \subset (0,1)$ *and pointwise on* $[0,1]$.

(ii) $\|P_n f - f\|_\infty \le 3\widetilde{\beta}(n)(\|f\|_\infty + 2^{-1}\omega_f(1/\sqrt{n+1}))$.

In order to express the rate of convergence in the L_1-norm we need

Lemma 4. *For every* f *from the Hilbert space* $(L_2([0,1]), \|\cdot\|_2)$, *holds the estimate*

$$(25)\qquad \|u_n f\|_1 \le \frac{2\|f\|_2}{\sqrt{2n+1}},\ \text{where } u_n(x) = (1-x)^n + x^n,\ x \in [0,1].$$

Proof. Using Cauchy's inequality, we obtain

$$\begin{aligned} \|u_n f\|_1 &\le \left(\int_0^1 u_n^2(t)\,dt\right)^{1/2} \|f\|_2 \\ &\le \left(2\int_0^1 ((1-t)^{2n} + t^{2n})\,dt\right)^{1/2} \|f\|_2 = \frac{2\|f\|_2}{\sqrt{2n+1}}. \end{aligned}$$

□

Theorem 6. *Let* P_n, $n > 2$, *be defined by* (17) *with a bounded sequence* $\widetilde{a}$. *For every* $f \in L_1([0,1])$,

$$(26)\qquad \|P_n f - \varphi f\|_1 \le \widetilde{\beta}(n)\left\{\frac{4}{3}\omega_f\left(\frac{1}{\sqrt{n+1}}\right) + \|u_n f\|_1 + \frac{2\omega_f(1)}{n+1}\right\},$$

where u_n is given by (25).

Moreover, if $f \in L_2([0,1])$, then $\lim_{n\to\infty} P_n f = \varphi f$ in the L_1-norm.

Proof. For the first statement we are going to (24) and apply $\int_0^1 dx$. Since $\int_0^1 u_n(x)dx = 2/(n+1)$, Theorem 3, combined with Lemma 3, implies (26).

The second statement is a consequence of (26) and Lemma 4. □

Particular case. It is obvious that the function φ is strongly dependent on the sequence $\widetilde{a}$. Thus, manipulating $\widetilde{a}$ we can construct different φ functions. For examples, we can build up $\varphi|_{(0,1)}$ as a polynomial of degree less than or equal to q, $q \geq 2$. To this purpose, we consider a sequence $\widetilde{a}$ with $a_n = \overline{a}$ for every $n \geq q$. Relation (23) implies

$$\varphi(x) = \overline{a}u_q(x) + x(1-x)\sum_{m=1}^{q-1} a_m u_{m-1}(x), \quad x \in (0,1), \tag{27}$$

where u_k, $k \geq 0$, are given in (25).

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