On the nonlocal initial value problem for first order differential equations

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Abstract. The paper is devoted to the existence of solutions of initial value problems for nonlinear first order differential equations with nonlocal conditions. We shall rely on the Leray–Schauder fixed point principle to prove the main result. The novelty is a growth condition which is splitted into two parts, one for the subinterval containing the points involved by the nonlocal condition, and other for the rest of the interval.
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1 Introduction

Consider the nonlocal initial value problem for first order differential equations

$$\begin{cases} x'(t) = f(t, x(t)), & \text{a.p.t. on } [0,1] \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0. \end{cases} \tag{1.1}$$

Here $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function, t_k are given points with $0 \leq t_1 \leq t_2 \leq ... \leq t_m < 1$ and a_k are real numbers with $1 + \sum_{k=1}^{m} a_k \neq 0$. We seek solutions x in $W^{1,1}(0,1)$.

Notice the non-homogeneous nonlocal initial condition

$$x\,(0) + \sum_{k=1}^{m} a_k x\,(t_k) = x_0$$

can always be reduced to the homogeneous one (with $x_0 = 0$) by the change of variable $y\,(t) = x\,(t) - \left(1 + \sum_{k=1}^{m} a_k\right)^{-1} x_0$.

Problem (1.1) is equivalent (see Boucherif [2]) to the following integral equation in $C\,[0,1]$

$$x\,(t) = -a \sum_{k=1}^{m} a_k \int_0^{t_k} f\,(s, x\,(s))\,ds + \int_0^t f\,(s, x\,(s))\,ds, \quad t \in [0,1]\,.$$

This can be viewed as a fixed point problem in $C\,[0,1]$ for the completely continuous operator $T : C\,[0,1] \to C\,[0,1]\,,$ given by

$$T\,(x)\,(t) = -a \sum_{k=1}^{m} a_k \int_0^{t_k} f\,(s, x\,(s))\,ds + \int_0^t f\,(s, x\,(s))\,ds.$$

Notice that T appears as sum of two integral operators, one, say of Fredholm type, whose values depend only on the restrictions of functions to $[0, t_m]\,,$

$$T_F\,(x)\,(t) = \begin{cases} -a \sum_{k=1}^{m} a_k \int_0^{t_k} f\,(s, x\,(s))\,ds + + \int_0^t f\,(s, x\,(s))\,ds & \text{if } t < t_m \\ -a \sum_{k=1}^{m} a_k \int_0^{t_k} f\,(s, x\,(s))\,ds + \int_0^{t_m} f\,(s, x\,(s))\,ds & \text{if } t \geq t_m \end{cases}$$

and the other one, a Volterra type operator,

$$T_V\,(x)\,(t) = \begin{cases} 0 \ \text{ if } t < t_m \\ \int_{t_m}^t f\,(s, x\,(s))\,ds \ \text{ if } t \geq t_m \end{cases}$$

depending on the restrictions of functions to $[t_m, 1]\,.$ This allows us to split the growth condition on the nonlinear term $f\,(t, x)$ into two parts, one for $t \in [0, t_m]$ and an other one for $t \in [t_m, 1]$ in a such way that one reobtains the classical growth condition (see Frigon [4] and O'Regan and Precup [5]) when $t_m = 0$, that is for the local initial condition $x\,(0) = 0$. This split of conditions on two subintervals appears to be the novelty of the present paper compared to previous papers on nonlocal Cauchy and boundary value problems (see Byszewski [3] and Boucherif [1], [2]).

In what follows the notation $|x|_{C[a,b]}$ stands for the max-norm on $C\,[a,b]\,,$ $|x|_{C[a,b]} = \max_{t\in[a,b]} |x\,(t)|\,,$ while $|x|_{L^\infty(a,b)}$ is the ess sup-norm on $L^\infty\,(a,b)\,,$ i.e., $|x|_{L^\infty(a,b)} = \inf\{c \in \mathbf{R}_+ : |x\,(t)| \leq c \text{ for a.e. } t \in [a,b]\}\,.$

2 Main result

Here are our hypotheses:

(H1) $1+\sum_{k=1}^{m}a_k\neq 0.$

(H2) $f:[0,1]\times\mathbf{R}\to\mathbf{R}$ is such that $f(.,x)$ is measurable for each $x\in\mathbf{R}$ and $f(t,.)$ is continuous for a.e. $t\in[0,1]$.

(H3) There exists $\omega\in\mathrm{Car}_{\mathrm{loc}}([0,t_m]\times\mathbf{R}_+;\mathbf{R}_+)$ nondecreasing in its second argument, $\alpha\in L^1(t_m,1)$, $\beta:\mathbf{R}_+\to\mathbf{R}_+$ nondecreasing with $1/\beta\in L^1_{\mathrm{loc}}(\mathbf{R}_+)$, and $R_0>0$ such that

$$|f(t,x)|\leq\begin{cases}\omega(t,|x|) & \text{for a.e. } t\in[0,t_m]\\ \alpha(t)\beta(|x|) & \text{for a.e. } t\in[t_m,1]\end{cases}$$

for all $x\in\mathbf{R}$,

$$\rho>R_0 \text{ implies } \frac{1}{\rho}\int_0^{t_m}\omega(t,\rho)\,dt<\frac{1}{A} \tag{2.1}$$

and

$$\int_{t_m}^{1}\alpha(t)\,dt<\int_{R_0^*}^{\infty}\frac{1}{\beta(\rho)}d\rho \tag{2.2}$$

where $A=1+|a|\sum_{k=1}^{m}|a_k|$ and $R_0^*=A\int_0^{t_m}\omega(t,R_0)\,dt$.

Remark 2.1 If $\omega(t,\rho)=\alpha_0(t)\beta_0(\rho)$, then from (2.1) we deduce that $\beta_0(\rho)\leq c\rho+c'$ for all $\rho\in\mathbf{R}_+$ and some constants c and c', i.e., β_0 has at most a linear growth. However, β may have a superlinear growth. Thus, we may say that by (H3) the growth of $f(t,x)$ in x is at most linear for $t\in[0,t_m]$ and can be superlinear for $t\in[t_m,1]$.

Theorem 2.1 *If the assumptions (H1), (H2) and (H3) are satisfied, then the initial problem (1.1) has at least one solution.*

Proof. The result will follows from the Leray–Schauder fixed point theorem (see Precup [6]) once we have proved the boundedness of the set of all solutions to equations $x=\lambda T(x)$ for $\lambda\in[0,1]$.

Let x be such a solution. Then for $t\in[0,t_m]$, we have

$$\begin{aligned}|x(t)| &= \lambda\left|-a\sum_{k=1}^{m}a_k\int_0^{t_k}f(s,x(s))\,ds+\int_0^{t}f(s,x(s))\,ds\right|\\ &\leq \left(1+|a|\sum_{k=1}^{m}|a_k|\right)\int_0^{t_m}|f(s,x(s))|\,ds\\ &\leq A\int_0^{t_m}\omega\left(t,|x|_{C[0,t_m]}\right)dt.\end{aligned}$$

Now we take the supremum over $t \in [0, t_m]$ to obtain

$$|x|_{C[0,t_m]} \le A \int_0^{t_m} \omega \left(t, |x|_{C[0,t_m]} \right) dt.$$

This, according to (2.1), guarantees that

$$|x|_{C[0,t_m]} \le R_0. \tag{2.3}$$

Next we let $t \in [t_m, 1]$. Then

$$\begin{aligned} |x(t)| &= \lambda \left| -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s))\, ds + \int_0^t f(s, x(s))\, ds \right| \\ &\le A \int_0^{t_m} \omega(t, R_0)\, ds + \int_{t_m}^t |f(s, x(s))|\, ds \\ &\le A \int_0^{t_m} \omega(t, R_0)\, dt + \int_{t_m}^t \alpha(s)\, \beta(|x(s)|)\, ds \\ &= : \phi(t). \end{aligned}$$

We have

$$\phi'(t) = \alpha(t)\, \beta(|x(t)|) \le \alpha(t)\, \beta(\phi(t)) \quad \text{for a.e. } t \in [t_m, 1].$$

This implies

$$\int_{\phi(t_m)}^{\phi(t)} \frac{1}{\beta(\rho)} d\rho = \int_{t_m}^t \frac{\phi'(s)}{\beta(\phi(s))} ds \le \int_{t_m}^t \alpha(s)\, ds.$$

This together with (2.2) guarantees $\phi(t) \le R_1$ for all $t \in [t_m, 1]$ and some $R_1 > 0$. Then $|x(t)| \le R_1$ for all $t \in [t_m, 1]$, whence

$$|x|_{C[t_m,1]} \le R_1. \tag{2.4}$$

Let $R = \max\{R_0, R_1\}$. Estimates (2.3) and (2.4) yield $|x|_{C[0,1]} \le R$ as we wished. ■

Remark 2.2 In particular, when $t_m = 0$, that is for the local initial condition $x(0) = 0$, (H3) reduces to the classical condition:

(H3*) There exists $\alpha \in L^1(0, 1)$ and $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ nondecreasing with $1/\beta \in L^1_{\text{loc}}(\mathbf{R}_+)$ such that

$$|f(t, x)| \le \alpha(t)\, \beta(|x|) \quad \text{for a.e. } t \in [0, 1] \text{ and all } x \in \mathbf{R}$$

and

$$\int_0^1 \alpha(t)\, dt < \int_0^\infty \frac{1}{\beta(\rho)} d\rho.$$

3 Some particular cases

3.1 Nonlinearities with growth at most linear

Here we show that the existence of solutions to problem (1.1) follows directely from the Schauder fixed point theoren in case that f satisfies (H1), (H2) and the growth condition in x

$$|f(t,x)| \leq \begin{cases} b\,|x| + d & \text{for a.e. } t \in [0,t_m] \\ c\,|x| + d & \text{for a.e. } t \in [t_m,1] \end{cases} \tag{3.1}$$

and all $x \in \mathbf{R}$, provided that

$$b\,t_m A < 1.$$

In this case, (H3) holds with $\omega(t,\rho) = b\rho + d$, $\alpha(t) = 1$, $\beta(\rho) = c\rho + d$ and $R_0 = b\,t_m A/(1 - b\,t_m A)$.

To this end, we look for a nonempty, bounded, closed and convex subset B of $C[0,1]$ with $T(B) \subset B$.

Let x be any element of $C[0,1]$. For $t \in [0,t_m]$ we have

$$\begin{aligned} |T(x)(t)| &= \left| -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s,x(s))\,ds + \int_0^t f(s,x(s))\,ds \right| \\ &\leq A \int_0^{t_m} |f(s,x(s))|\,ds \leq b\,t_m A\,|x|_{C[0,t_m]} + d\,t_m A. \end{aligned}$$

Hence

$$|T(x)|_{C[0,t_m]} \leq b\,t_m A\,|x|_{C[0,t_m]} + d\,t_m A. \tag{3.2}$$

For $t \in [t_m,1]$ and any $\theta > 0$, we have

$$\begin{aligned} |T(x)(t)| &= \left| -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s,x(s))\,ds + \int_0^t f(s,x(s))\,ds \right| \\ &\leq b\,t_m A\,|x|_{C[0,t_m]} + d\,t_m A + \int_{t_m}^t (c\,|x(s)| + d)\,ds \\ &\leq b\,t_m A\,|x|_{C[0,t_m]} + d\,(t_m A + 1 - t_m) \\ &\quad + c \int_{t_m}^t e^{-\theta(\sigma - t_m)}\,|x(s)|\,e^{\theta(s-t_m)} ds \\ &\leq bt_m A\,|x|_{C[0,t_m]} + c_0 + \frac{c}{\theta} e^{\theta(t-t_m)}\,\|x\|_{C[t_m,1]} \end{aligned}$$

where $c_0 = d\,(t_m A + 1 - t_m)$ and $\|x\|_{C[t_m,1]} = \max_{t\in[t_m,1]} |x(t)|\, e^{-\theta(t-t_m)}$. Deviding by $e^{\theta(t-t_m)}$ and taking the supremum we obtain

$$\|T(x)\|_{C[t_m,1]} \le b\, t_m A\, |x|_{C[0,t_m]} + \frac{c}{\theta}\, \|x\|_{C[t_m,1]} + c_0. \tag{3.3}$$

Now we consider an equivalent norm on $C[0,1]$ defined by

$$\|x\| = \max\left\{ |x|_{C[0,T_m]},\, \|x\|_{C[t_m,1]} \right\}.$$

From (3.2) and (3.3) we have

$$\|T(x)\| \le \left(b\, t_m A + \frac{c}{\theta}\right) \|x\| + c_1 \tag{3.4}$$

where $c_1 = \max\{d\, t_m A, c_0\}$. Since $b\, t_m A < 1$, we may find a $\theta > 0$ such that $b\, t_m A + c/\theta < 1$. Then there exists a number $R > 0$ with

$$\left(b\, t_m A + \frac{c}{\theta}\right) R + c_1 \le R. \tag{3.5}$$

Now we let $B = \{x \in C[0,1] : \|x\| \le R\}$. Inequalities (3.4) and (3.5) guarantee that $T(B) \subset B$ and thus the Schauder fixed point theorem can be applied.

3.2 Lipschitz nonlinearities

We deal in this subsection with problem (1.1) when f satisfies a Lipschitz condition in x of the form

$$|f(t,x) - f(t,y)| \le \gamma(t)\, |x - y| \quad \text{for a.e. } t \in [0,1] \text{ and all } x, y \in \mathbf{R} \tag{3.6}$$

where $\gamma \in L^\infty(0,1)$. A certain upper bound is required for Lipschitz constant $\gamma(t)$ only for $t \in [0, t_m]$. The existence and uniqueness result is based on Banach contraction principle.

Theorem 3.1 *Assume (H1) and that $f(.,x)$ is measurable for all $x \in \mathbf{R}$. In addition assume that $f(.,0) \in L^\infty(0,1)$ and there exists a function $\gamma \in L^\infty(0,1)$ such that (3.6) holds and*

$$|\gamma|_{L^\infty(0,t_m)}\, t_m A < 1.$$

Then (1.1) has a unique solution $x^* \in W^{1,1}(0,1)$ and $|x_n - x^*|_{C[0,1]} \to 0$ as $n \to \infty$, where x_0 is any function in $C[0,1]$ and $x_n = T(x_{n-1})$ for $n = 1, 2, \ldots$.

Proof. Let $x, y \in C[0,1]$. For $t \in [0, t_m]$ we have

$$\begin{aligned} |T(x)(t) - T(y)(t)| &\leq |\gamma|_{L^\infty(0,t_m)} A \int_0^{t_m} |x(s) - y(s)| \, ds \\ &\leq |\gamma|_{L^\infty(0,t_m)} t_m A |x - y|_{C[0,t_m]} . \end{aligned}$$

It follows that

$$|T(x) - T(y)|_{C[0,t_m]} \leq |\gamma|_{L^\infty(0,t_m)} t_m A |x - y|_{C[0,t_m]} . \tag{3.7}$$

For $t \in [t_m, 1]$, we have

$$\begin{aligned} |T(x)(t) - T(y)(t)| &\leq |\gamma|_{L^\infty(0,t_m)} t_m A |x - y|_{C[0,t_m]} \\ &\quad + |\gamma|_{L^\infty(t_m,1)} \int_{t_m}^{t} |x(s) - y(s)| \, ds \\ &\leq |\gamma|_{L^\infty(0,t_m)} t_m A |x - y|_{C[0,t_m]} \\ &\quad + \frac{|\gamma|_{L^\infty(t_m,1)}}{\theta} \|x - y\|_{C[t_m,1]} e^{\theta(t - t_m)} . \end{aligned}$$

Then

$$\begin{aligned} \|T(x) - T(y)\|_{C[t_m,1]} &\leq |\gamma|_{L^\infty(0,t_m)} t_m A |x - y|_{C[0,t_m]} \\ &\quad + \frac{|\gamma|_{L^\infty(t_m,1)}}{\theta} \|x - y\|_{C[t_m,1]} . \end{aligned}$$

Consequently

$$\|T(x) - T(y)\| \leq \left(|\gamma|_{L^\infty(0,t_m)} t_m A + \frac{|\gamma|_{L^\infty(t_m,1)}}{\theta} \right) \|x - y\| .$$

Finally we choose any $\theta > 0$ such that $|\gamma|_{L^\infty(0,t_m)} t_m A + |\gamma|_{L^\infty(t_m,1)} / \theta < 1$ and we apply Banach contraction principle. ■

Remark 3.1 Under the assumptions of Theorem 2, condition (3.1) is satisfied with $b = |\gamma|_{L^\infty(0,t_m)}$, , $c = |\gamma|_{L^\infty(t_m,1)}$ and $d = |f(.,0)|_{L^\infty(0,1)}$.

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