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ITERATES OF SOME BIVARIATE APPROXIMATION PROCESS VIA WEAKLY PICARD OPERATORS

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Abstract. In the present paper we introduce a general class of positive operators of discrete type acting on the space of real valued functions defined on a plane domain. Based on the weakly Picard operators and the contraction principle as well, our aim is to study the convergence of the iterates of our defined operators. Also, some approximation properties of this process are revealed and concrete examples of our approach are given.

1. Introduction

The fixed point theory has proved quite useful in the theory of approximation operators. More precisely, we refer here to the theory of weakly Picard operators, approached by the second author, see e.g. [4, 5]. In section 2 we briefly present the basic elements of this research direction which will be used in establishing our main result. Simultaneously, some useful notations are given. Further on, in section 3 we construct a sequence of operators of discrete type associated to any real valued function defined on a rectangular domain. Under some additional conditions imposed to our operators, they become an approximation process. By using the modulus of smoothness of the first order, we estimate the degree of convergence of our sequence to the identity operator. At the same time, in order to obtain the limit of the iterates, we prove that the conditions formulated in the previous section are fulfilled by our general class of operators.

The focus of section 4 is to present some concrete examples. Applying our method, results regarding the iterates of Bernstein respectively Stancu bivariate operators are obtained.

2. Weakly Picard operators on metric spaces

We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and, at first, we give the following informal definition.

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Definition 1 [4]. Let (X,ρ) be a metric space. The operator $T : X \to X$ is weakly Picard operator (WPO) if the sequence of iterates $(T^m(x))_{m\in\mathbb{N}}$ converges for every x belonging to X and the limit is a fixed point of T.

Throughout the paper $\mathcal{F}_T := \{x \in X |\ T(x) = x\}$ stands for the fixed point set of the operator T and, as usually, we put $T^0 = I_X$, $T^1 = T$, $T^{m+1} = T \circ T^m$, $m \in \mathbb{N}$. Here I_X denotes the identity operator on X.

We recall: if T is a weakly Picard operator and $\mathcal{F}_T$ has a unique element, then, by definition, T is a Picard operator (PO).

In [4] was given the following characterization of a weakly Picard operator.

Theorem 1. *Let (X,ρ) be a metric space. The operator $T : X \to X$ is weakly Picard if and only if a partition of X exists, $X = \bigcup_{s\in S} X_s$ such that for every $s \in S$ one has*

(i) $X_s \in \mathcal{I}(T)$,

(ii) $T|_{X_s} : X_s \to X_s$ *is a Picard operator,*

where $\mathcal{I}(T) := \{Y \subset X |\ Y \neq \emptyset$ and $T(Y) \subset Y\}$ represents the family of the non-empty invariant subsets of T.

Further on, if T is WPO we introduce $T^\infty \in X^X$ defined by

$$T^\infty(x) := \lim_{m\to\infty} T^m(x), \quad x \in X. \tag{1}$$

We remark that $T^\infty(X) = \mathcal{F}_T$.

Also, if T is WPO, then the following relations holds true:

$$\mathcal{F}_{T^m} = \mathcal{F}_T \neq \emptyset, \text{ for every } m \in \mathbb{N}.$$

Moreover, if T is PO, then $\mathcal{F}_{T^m} = \mathcal{F}_T = \{x^*\}$ for every $m \in \mathbb{N}$, in other words T is a Bessaga operator.

For the convenience of the reader, we accompany this brief exposition with a generic example of weakly Picard operator.

Example 1. Let (X_i, ρ_i), $i \in I$, be a family of metric spaces, $T_i : X_i \to X_i$, a family of Picard operators and x_i^* the unique fixed point of T_i for every $i \in I$. Let $X := \bigcup_{i\in I} X_i$ be the disjoint union of the family $(X_i)_{i\in I}$. We define $\rho : X \times X \to \mathbb{R}_+$,

$$\rho(x,y) = \begin{cases} \rho_i(x,y), & \text{if } (x,y) \in X_i \times X_i,\ i \in I, \\ \rho_i(x,x_i^*) + \rho_j(y,x_j^*) + 1, & \text{if } (x,y) \in X_i \times X_j,\ i \neq j. \end{cases}$$

Clearly ρ is a metric on X and T is WPO, where $T(x) := T_i(x)$ for $x \in X_i$, $i \in I$.

Finally we indicate some notations used in the sequel. We shall need the so called *test functions* of two variables $e_{i,j}$; we recall $e_{i,j} : D \to \mathbb{R}$, $e_{i,j}(x,y) = x^i y^j$ for every $(x,y) \in D \subset \mathbb{R}\times\mathbb{R}$, where $i \in \mathbb{N}_0$, $j \in \mathbb{N}_0$, $i + j \leq 2$.

For a set S, we shall denote by $B(S)$ the Banach space of all real-valued bounded functions defined on S, endowed with the norm of the uniform convergence, briefly the sup-norm, defined by

$$\|f\|_\infty := \sup_{x\in S} |f(x)| \text{ for every } f \in B(S).$$

If S is a topological space, $C(S)$ denotes the space of all real-valued continuous functions on S. Furthermore, setting $C_B(S) := C(S) \cap B(S)$, this space endowed with the sup-norm is a Banach space. Clearly, if S is compact, then $C_B(S) = C(S)$.

Actually, if S is compact then $C(S)$ is a Banach lattice, consequently every positive linear operator T acting on this space is continuous and one has

$$\|T\| = \|T(\mathbf{1})\|_\infty, \tag{2}$$

where $\mathbf{1}$ denotes the constant function 1.

3. On a general sequence of operators

For $a_1 < b_1$ and $a_2 < b_2$ we consider $I_1 := [a_1, b_1]$, $I_2 := [a_2, b_2]$, $D := I_1 \times I_2$ and

$$\mathcal{V}_D = \{(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2)\}, \tag{3}$$

the vertices set of the domain D.

For every $(m, n) \in \mathbb{N} \times \mathbb{N}$ we define the bi-dimensional net

$$\begin{cases} \Delta_{1,m}(a_1 = x_{m,0} < x_{m,1} < \cdots < x_{m,m} = b_1) \\ \Delta_{2,n}(a_2 = y_{n,0} < y_{n,1} < \cdots < y_{n,n} = b_2) \end{cases} \tag{4}$$

and the following systems of non-negative real-valued functions

$$0 \le \psi_{1,m,i} \in C(I_1),\ \ 0 \le i \le m,\ \ 0 \le \psi_{2,n,j} \in C(I_2),\ \ 0 \le j \le n. \tag{5}$$

As regards these functions, the following properties will be fulfilled:

$$(P_1)\ \ \sum_{i=0}^{m} \psi_{1,m,i}(x) = \sum_{j=0}^{n} \psi_{2,n,j}(y) = 1,\ \ (x, y) \in I_1 \times I_2, \tag{6}$$

$$(P_2)\ \ \sum_{i=0}^{m} x_{m,i}\psi_{1,m,i}(x) = x\ (x \in I_1),\ \ \sum_{j=0}^{n} y_{n,j}\psi_{2,n,j}(y) = y\ (y \in I_2), \tag{7}$$

$$(P_3)\ \ \psi_{1,m,0}(a_1) = \psi_{1,m,m}(b_1) = \psi_{2,n,0}(a_2) = \psi_{2,n,n}(b_2) = 1. \tag{8}$$

Using the above data, for every function $f \in \mathbb{R}^D$ we define the operators

$$(L_{m,n}f)(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \psi_{1,m,i}(x)\psi_{2,n,j}(y)f(x_{m,i}, y_{n,j}),\ \ (x, y) \in D. \tag{9}$$

Examining (6), (7) and (2) (with $S = D$ and $e_{0,0} = \mathbf{1}$) we can state

Lemma 2. *The operators $L_{m,n}$, $(m, n) \in \mathbb{N} \times \mathbb{N}$, defined by (9) verify*

(i) $L_{m,n}e_{i,j} = e_{i,j}$, $(i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$*;*

(ii) *If $f \in C(D)$ then $\|L_{m,n}\| = 1$.*

The above result implies that $L_{m,n}$ has the degree of exactness equal with 1.

In what follows we are concerned in establishing the rate of convergence of this class of operators. In this respect we use the modulus of smoothness of the first order defined as follows

$$\begin{aligned}\omega_f(\delta_1,\delta_2) := \sup\big\{&|f(x_1,y_1)-f(x_2,y_2)| : \\ &x_1,x_2\in I_1,\ y_1,y_2\in I_2, |x_1-x_2|\le\delta_1,\ |y_1-y_2|\le\delta_2\big\},\end{aligned}$$

for every $f\in B(D)$, $\delta_1>0$ and $\delta_2>0$. Other used notation for it: $\omega(f;\delta_1,\delta_2)$. Clearly, ω_f is an increasing function and $\omega_f(0,0)=0$. The method used by us seems to be simple and it is based on the properties of ω_f investigated by A. F. Ipatov [3]; among these properties we recall

$$\omega_f(\lambda_1\delta_1,\lambda_2\delta_2)\le(1+\lambda_1+\lambda_2)\omega_f(\delta_1,\delta_2),\ \ \lambda_1>0,\ \ \lambda_2>0. \tag{10}$$

Let $\delta_1>0$, $\delta_2>0$ be two real numbers which are independent of i and j. Taking into account both the identities (6) and formula (10) we can write

$$\begin{aligned}&|(L_{m,n}f)(x,y)-f(x,y)|\\ &\le\sum_{i=0}^{m}\sum_{j=0}^{n}\psi_{1,m,i}(x)\psi_{2,n,j}(y)|f(x_{m,i},y_{n,j})-f(x,y)|\\ &\le\sum_{i=0}^{m}\sum_{j=0}^{n}\psi_{1,m,i}(x)\psi_{2,n,j}(y)\omega_f\left(\frac{1}{\delta_1}|x_{m,i}-x|\delta_1,\frac{1}{\delta_2}|y_{n,j}-y|\delta_2\right)\\ &\le\Big(1+\frac{1}{\delta_1}\sum_{i=0}^{m}\psi_{1,m,i}(x)|x_{m,i}-x|+\frac{1}{\delta_2}\sum_{j=0}^{n}\psi_{2,n,j}(y)|y_{n,j}-y|\Big)\omega_f(\delta_1,\delta_2).\end{aligned} \tag{11}$$

On the other hand, Cauchy's inequality and the properties (6), (7) imply

$$\begin{aligned}\sum_{i=0}^{m}\psi_{1,m,i}(x)|x_{m,i}-x| &\le\left(\sum_{i=0}^{m}\psi_{1,m,i}(x)\right)^{1/2}\left(\sum_{i=0}^{m}\psi_{1,m,i}(x)(x_{m,i}-x)^2\right)^{1/2}\\ &=\left(\sum_{i=0}^{m}\psi_{1,m,i}(x)x_{m,i}^2-x^2\right)^{1/2},\end{aligned}$$

and respectively

$$\sum_{j=0}^{n}\psi_{2,n,j}(y)|y_{n,j}-y|\le\left(\sum_{j=0}^{n}\psi_{2,n,j}(y)y_{n,j}^2-y^2\right)^{1/2}.$$

By using (11) we end off to evaluate the order of approximation as follows.

Theorem 3. *The operators $L_{m,n}$, $(m,n)\in\mathbb{N}\times\mathbb{N}$, defined by (9) verify*

$$|(L_{m,n}f)(x,y)-f(x,y)|\le\left(1+\frac{1}{\delta_1}\sqrt{{}_1\widetilde{\Psi}_m(x)}+\frac{1}{\delta_2}\sqrt{{}_2\widetilde{\Psi}_n(y)}\right)\omega_f(\delta_1,\delta_2),\qquad(12)$$

for every $f\in B(D)$, $\delta_1>0$, $\delta_2>0$, where

$$\begin{cases}{}_1\widetilde{\Psi}_m(x)=\displaystyle\sum_{i=0}^{m}\psi_{1,m,i}(x)x_{m,i}^2-x^2,\ \ x\in I_1,\\ {}_2\widetilde{\Psi}_n(y)=\displaystyle\sum_{j=0}^{n}\psi_{2,n,j}(y)x_{n,j}^2-y^2,\ \ y\in I_2.\end{cases}\qquad(13)$$

Endowing $\mathbb{R}\times\mathbb{R}$ with the metric ρ, $\rho(v_1,v_2)=|x_1-x_2|+|y_1-y_2|$ for $v_k=(x_k,y_k)$, $k=1,2$, we could have estimated the rate of convergence using another type of modulus of smoothness given by

$$\omega_\rho(f;\delta):=\sup\left\{|f(v_1)-f(v_2)|:\ v_1\in D,v_2\in D,\rho(v_1,v_2)\le\delta\right\},$$

for every $f\in B(D)$ and $\delta>0$. It is easy to see that $\omega_f(\delta_1,\delta_2)\le\omega_\rho(f;\delta_1+\delta_2)$ and, of course, formula (12) will have another structure.

Returning to Theorem 3 we are in position to indicate the necessary and sufficient condition which offers to the sequence $(L_{m,n})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ the attribute of approximation process.

Theorem 4. *Let $L_{m,n}$, $(m,n)\in\mathbb{N}\times\mathbb{N}$, be given by (9) and let the functions ${}_1\widetilde{\Psi}_m$, ${}_2\widetilde{\Psi}_n$ be defined by (13).*

If $\lim\limits_{m\to\infty}{}_1\widetilde{\Psi}_m=0$ uniformly on I_1 and $\lim\limits_{n\to\infty}{}_2\widetilde{\Psi}_n=0$ uniformly on I_2, then for every $f\in C(D)$ one has

$$\lim_{(m,n)}L_{m,n}f=f\ \ \textit{uniformly on}\ \ D.$$

Actually, this result can be motivated directly by using the celebrated Bohman-Korovkin theorem for bidimensional case. The hypotheses of Theorem 4 guarantee that $L_{m,n}e_{2,0}$, $L_{m,n}e_{0,2}$ converge to $e_{2,0}$ respectively to $e_{0,2}$. Adding this fact to Lemma 2, see (i), we just obtained the uniform convergence of our operators on the all six test functions of the mentioned theorem.

Conditions (8) and (6) combined with (5) ensure

$$\psi_{1,m,i}(a_1)=\psi_{1,m,i-1}(b_1)=0,\ \ \psi_{2,n,j}(a_2)=\psi_{2,n,j-1}(b_2)=0,$$

for $1\le i\le m$ and $1\le j\le n$. Consequently we have

Lemma 5. *The operators $L_{m,n}$, $(m,n) \in \mathbb{N} \times \mathbb{N}$, defined by (9) verify*

$$(L_{m,n}f)(u_0, v_0) = f(u_0, v_0), \ \ \textit{for every} \ \ (u_0, v_0) \in \mathcal{V}_D,$$

where $\mathcal{V}_D$ is given at (3).

At this moment we introduce the matrices

$$[f; \mathcal{V}_D] := \begin{pmatrix} f(a_1, a_2) & f(b_1, a_2) \\ f(a_1, b_2) & f(b_1, b_2) \end{pmatrix}, \ \Lambda := \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix}, \tag{14}$$

where $\mathcal{V}_D$ appears at (3) and $\alpha_{i,j}$ $(1 \le i, j \le 2)$ are real numbers arbitrarily fixed. Let us define the set

$$X_\Lambda := \{f \in C(D) : \ [f; \mathcal{V}_D] = \Lambda\}. \tag{15}$$

In what follows we give some basic properties regarding both to X_Λ and $L_{m,n}$.

Lemma 6. *Let X_Λ and $L_{m,n}$ be given by (15) and (9) respectively.*

(i) *For each matrix $\Lambda \in \mathcal{M}_2(\mathbb{R})$, X_Λ is a closed subset of $C(D)$.*

(ii) $C(D) = \bigcup\limits_{\Lambda \in \mathcal{M}_2(\mathbb{R})} X_\Lambda$ *is a partition of the space $C(D)$.*

(iii) *For each matrix $\Lambda \in \mathcal{M}_2(\mathbb{R})$, X_Λ is an invariant subset by every $L_{m,n}$, $(m,n) \in \mathbb{N} \times \mathbb{N}$, in other words $L_{m,n}(X_\Lambda) \subset X_\Lambda$.*

The first two statements are obvious and the third of them is implied by Lemma 5.

In order to present the limit of iterates of our operators, we define the following polynomial function associated to matrix Λ

$$p^*_\Lambda(x,y) = \alpha_{1,1} + A_{1,0}(x - a_1) + A_{0,1}(y - a_2) + A_{1,1}(x - a_1)(y - a_2), \tag{16}$$

where $(x,y) \in D$ and

$$A_{1,0} = \frac{\alpha_{1,2} - \alpha_{1,1}}{b_1 - a_1}, \ \ A_{0,1} = \frac{\alpha_{2,1} - \alpha_{1,1}}{b_2 - a_2}, \ \ A_{1,1} = \frac{\alpha_{2,2} - \alpha_{1,2} + \alpha_{1,1} - \alpha_{2,1}}{(b_1 - a_1)(b_2 - a_2)}.$$

Lemma 7. *For the function p^*_Λ defined by (16) the following relations*

(i) $p^*_\Lambda \in X_\Lambda$,

(ii) $L_{m,n} p^*_\Lambda = p^*_\Lambda$,

hold true.

Proof. By a straightforward calculation of the set $p^*_\Lambda(\mathcal{V}_D)$, see (3), one has $[p^*_\Lambda; \mathcal{V}_D] = \Lambda$. The second statement of this lemma is implied by Lemma 2 and the linearity of our operators.

Lemma 8. *Let $L_{m,n}$ be the operator defined by (9). For every $(m,n) \in \mathbb{N} \times \mathbb{N}$ and $\Lambda \in \mathcal{M}_2(\mathbb{R})$, the operator $L_{m,n}|_{X_\Lambda} : X_\Lambda \to X_\Lambda$ is a contraction.*

Proof. Setting

$$\mathcal{K} := \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\} \text{ and } \partial\mathcal{K} := \{(0,0), (0,n), (m,0), (m,n)\},$$

for each f and g belonging to X_Λ, in view of Lemma 5, we get

$$\begin{aligned}
&|(L_{m,n}f)(x,y) - (L_{m,n}g)(x,y)| \\
&= \left| \sum_{(i,j)\in\mathcal{K}\setminus\partial\mathcal{K}} \psi_{1,m,i}(x)\psi_{2,n,j}(y)(f-g)(x_{m,i}, y_{n,j}) \right| \\
&\le \sum_{(i,j)\in\mathcal{K}\setminus\partial\mathcal{K}} \psi_{1,m,i}(x)\psi_{2,n,j}(y)\|f-g\|_\infty \\
&= (1 - u_{m,n}(x,y))\|f-g\|_\infty \\
&\le (1-\lambda_{m,n})\|f-g\|_\infty,
\end{aligned}$$

where

$$u_{m,n}(x,y) := \sum_{(i,j)\in\partial\mathcal{K}} \psi_{1,m,i}(x)\psi_{2,n,j}(y) \text{ and } \lambda_{m,n} := \inf_{(x,y)\in D} u_{m,n}(x,y). \tag{17}$$

Thus $\|L_{m,n}f - L_{m,n}g\|_\infty \le (1-\lambda_{m,n})\|f-g\|_\infty$ and the conclusion follows.

Proceeding along the lines of section 2, we are able to present our main result. Lemmas 6, 7 and 8 lead us to the following property of the iterates of our operators.

Theorem 9. *Let $L_{m,n}$, $(m,n) \in \mathbb{N}\times\mathbb{N}$, be defined by (9) and $\lambda_{m,n}$ be given at (17). If $\lambda_{m,n} \neq 0$ then one has*

$$\lim_{k\to\infty} L^k_{m,n} f = p^*_{[f;\mathcal{V}_D]}, \quad f \in C(D),$$

uniformly on D, where $p^*_{[f;\mathcal{V}_D]}$ is defined at (16) via (14).

Remarks.

1) In the particular case of the square $D = [0,1]\times[0,1]$ we have

$$\begin{aligned}
p^*_{[f,\mathcal{V}_D]}(x,y) &\equiv p^*_f(x,y) \\
&= f(0,0) + (f(1,0) - f(0,0))x + (f(0,1) - f(0,0))y \\
&\quad + (f(1,1) - f(1,0) + f(0,0) - f(0,1))xy.
\end{aligned} \tag{18}$$

2) If we consider the univariate case of our operators described by the below form

$$(L_m f)(x) = \sum_{i=0}^{m} \psi_{m,i}(x) f(x_{m,i}), \quad f \in \mathbb{R}^{I_1}, \quad x \in I_1,$$

with $0 \le \psi_{m,i} \in C(I_1)$ $(i = \overline{0,m})$, $\sum_{i=0}^m \psi_{m,i}(x) = 1$, $\sum_{i=0}^m \psi_{m,i}(x)x_{m,i} = x$, then Theorem 9 can be read as follows.
If $\lambda_m := \inf_{x\in I_1} (\psi_{m,0} + \psi_{m,m})(x) \neq 0$ then the sequence $(L^k_m)_{k\ge1}$ verifies

$$\lim_{k\to\infty} (L^k_m f)(x) = f(a_1) + \frac{f(b_1) - f(a_1)}{b_1 - a_1}(x - a_1), f \in C(I_1), \text{ uniformly on } I_1.$$

3) Taking the advantage of the familiar Bohman-Korovkin arguments, see Theorem 4, here is a necessary and sufficient condition for the iterates of the sequence $(L_{m,n})$ to converge to the identity operator. Considering $(k_p)_{p\geq 1}$ an increasing sequence of positive integer numbers tending to infinity, we get: $\lim_{p\to\infty} \|L^{k_p}_{m,n}f - f\|_\infty = 0$ for each $f \in C(D)$ if and only if the same limit relation holds for $f = e_{2,0}$ and $f = e_{0,2}$.

In the light of Definition 1, Theorem 1 and relation (1) as well, choosing $X = C(D)$, we obtain

Corollary 10. *Under the hypothesis of Theorem 9, the operator $L_{m,n}$ defined by (9) is WPO for every $(m, n) \in \mathbb{N} \times \mathbb{N}$ and one has $L^\infty_{m,n}f = p^*_{[f;\mathcal{V}_D]}$, where $p^*_{[f;\mathcal{V}_D]}$ is given at (16) and (14).*

As regards WPO we can state the following general result.

Theorem 11. *Let S be an open bounded subset of $\mathbb{R}^2$ and E be a subset of $\overline{S}$. If $T : C(\overline{S}) \to C(\overline{S})$ is a linear operator satisfying the conditions:*
(i) *$(Th)(x, y) = h(x, y)$ for every $h \in C(\overline{S})$ and $(x, y) \in E$,*
(ii) *there exists $0 < \alpha < 1$ such that $\|Th\|_\infty \leq \alpha\|h\|_\infty$ for every $h \in C(\overline{S})$ having the property $h|_E = 0$,*

then T is WPO.

Proof. Setting $\Gamma := C(E)$, for each γ belonging to Γ we consider the closed set $X_\gamma := \{f \in C(\overline{S}) : f|_E = \gamma\}$. Clearly, $(X_\gamma)_{\gamma\in\Gamma}$ is a partition of $C(\overline{S})$, and each X_γ is an invariant subset of T. Moreover, if f and g belong to X_γ then $(f - g)|_E = 0$ and consequently $\|Tf - Tg\|_\infty \leq \alpha\|f - g\|_\infty$. Thus, $T|_{X_\gamma}$ is α-contraction for every $\gamma \in \Gamma$ and the conclusion follows from Theorem 1.

4. Application

In order to present concrete examples, we are looking for operators having the exactness degree 1. In this section we selected two classical examples of operators which verify all requirements formulated in section 3. For both of them we choose the domain $I_1 \times I_2$ and the net (4) as follows

$$I_1 = I_2 = [0, 1],\ \ x_{m,i} = i/m (0 \leq i \leq m),\ \ y_{n,j} = j/n\ (0 \leq j \leq n).$$

4.1. Bernstein operator of (m, n)-order

In this case, for $q = 1$ and $q = 2$ we take $\psi_{q,p,k}$, $0 \leq k \leq p$, the *fundamental Bernstein polynomials $b_{p,k}$ of p-degree*, this meaning $b_{p,k}(t) = \binom{p}{k} t^{p-k}(1 - t)^k$, $t \in [0, 1]$. The operator $L_{m,n}$ becomes $B_{m,n}$, the Bernstein operator of (m, n) order given by

$$(B_{m,n}f)(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{m,i}(x)b_{n,j}(y)f(i/m, j/n). \tag{19}$$

It is well known that $B_{m,n}$ verifies the properties (P_1), (P_2), (P_3) indicated at (6), (7), (8) respectively. Moreover, the functions introduced by (13) are defined for $(q,p)=(1,m)$ respectively $(q,p)=(2,n)$ as follows

$$_q\widetilde{\Psi}_p(t)=\frac{t(1-t)}{p},\ \ t\in[0,1],$$

consequently Theorem 4 works and $(B_{m,n})$ is an approximation process on $C(D)$. Choosing in (12) $(\delta_1,\delta_2)=(1/\sqrt{m},1/\sqrt{n})$, we reobtain a classical inequality due to Ipatov [3]. As a matter of fact, we recall that the order of approximation of bivariate functions by this type of Bernstein operators was proved in [2] as being

$$\|B_{m,n}f-f\|_\infty\le(2k-1)\omega_f(1/\sqrt{m},1/\sqrt{n}),$$

where $k=(4306+837\sqrt{6})/5832\approx 1.0898873\ldots$ is Sikkema's constant which characterizes the univariate case, see [6]. The coefficients $\lambda_{m,n}\equiv\lambda^B_{m,n}$ defined by (17) associated to these operators have the values

$$\lambda^B_{m,n}=\inf_{\substack{0\le x\le 1\\ 0\le y\le 1}}(x^m+(1-x)^m)(y^n+(1-y)^n)=\frac{1}{2^{m+n-2}},\ \ (m,n)\in\mathbb{N}\times\mathbb{N},$$

and following Theorem 9 we obtain $\lim\limits_{k\to\infty}B^k_{m,n}f=p^*_f$, uniformly on $[0,1]\times[0,1]$, where p^*_f is given by (18).

4.2. Stancu operators of (m,n)-order

This time, for $q=1$ and $q=2$ we choose $\psi_{q,p,k}$, $0\le k\le p$, the *fundamental Stancu polynomials* $w^{\langle\alpha_q\rangle}_{p,k}$ *of p-degree*, see [7], where

$$w^{\langle\alpha_q\rangle}_{p,k}(t)=\binom{p}{k}t^{[k,-\alpha_q]}(1-t)^{[p-k,-\alpha_q]}/1^{[p,-\alpha_q]},\ \ t\in[0,1].$$

Here $t^{[s,-\alpha_q]}$ stands for the generalized factorial power with the step $-\alpha_q$, $t^{[0,-\alpha_q]}:=1$ and $t^{[s,-\alpha_q]}:=t(t+\alpha_q)\ldots(t+(s-1)\alpha_q)$, $s\in\mathbb{N}$. Also, α_1,α_2 are non-negative real parameters depending on the natural numbers m respectively n.

Now, the look of $L_{m,n}$ operators is given as follows

$$(S^{\langle\alpha_1,\alpha_2\rangle}_{m,n}f)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}w^{\langle\alpha_1\rangle}_{m,i}(x)w^{\langle\alpha_2\rangle}_{n,j}(y)f(i/m,j/n).$$

Again, the properties (P_1), (P_2), (P_3) are fulfilled and one has

$$_q\widetilde{\Psi}_p(t)=\frac{t(1-t)}{\alpha_q+1}\left(\alpha_q+\frac{1}{p}\right),\ \ t\in[0,1],$$

for $q=1$ and $p=m$ respectively $q=2$ and $p=n$. Under the assumption that the couple $(\alpha_1,\alpha_2)=(\alpha_1(m),\alpha_2(n))\to(0,0)$ as $(m,n)\to(\infty,\infty)$, $(S^{\langle\alpha_1,\alpha_2\rangle}_{m,n})$ is an approximation process on the space $C(D)$.

We notice, in the particular case $\alpha_1 = \alpha_2 = 0$, the operator $S_{m,n}^{\langle 0,0\rangle}$ becomes $B_{m,n}$ defined by (19).

On the other hand, returning to (17) we deduce

$$\begin{aligned} &u_{m,n}(x,y) \\ &= ((1-x)^{[m,-\alpha_1]} + x^{[m,-\alpha_1]})((1-y)^{[n,-\alpha_2]} + y^{[n,-\alpha_2]})/1_{m,n}^{\alpha_1,\alpha_2} \\ &\geq ((1-x)^m + x^m)((1-y)^n + y^n)/1_{m,n}^{\alpha_1,\alpha_2}, \ \ 1_{m,n}^{\alpha_1,\alpha_2} := 1^{[m,-\alpha_1]}1^{[n,-\alpha_2]} \end{aligned}$$

and the coefficients $\lambda_{m,n}$ associated to Stancu operators (named $\lambda_{m,n}^S$) verify $\lambda_{m,n}^S \geq \lambda_{m,n}^B/1_{m,n}^{\alpha_1,\alpha_2}$. Thus, applying Theorem 9, we get $\lim\limits_{k\to\infty} {}^k S_{m,n}^{\langle\alpha_1,\alpha_2\rangle} f = p_f^*$, uniformly on $[0,1]\times[0,1]$, where p_f^* is again given by (18).

Finally we remark that, actually, both examples are tensor product type operators applied on functions over $D = [0,1]\times[0,1]$. More details about tensor products of Radon measures and positive operators can be found, e.g., in [1, section 1.2].

References

[1] F. Altomare and M. Campiti, *Korovkin-type Approximation Theory and its Applications*, Vol. 17, Walter de Gruyter Studies in Mathematics, Berlin-New York, 1994.

[2] I. Badea, *The modulus of oscillation for functions of two variables and some applications in the approximation by Bernstein operator*, (in Romanian), An. Univ. Craiova, Series V. **2** (1974), 43–54.

[3] A. F. Ipatov, *Estimate of the error and order of approximation of functions of two variables by S. N. Bernstein's polynomials*, (in Russian), Uč. Zap. Petrozavodsk Univ., **4**(4) (1955), 31–48.

[4] I. A. Rus,, *Weakly Picard mappings*, Commentationes Math. Univ Carolinae, **34**(4) (1993), 769–773.

[5] ———, *Generalized Contractions and Applications*, University Press, Cluj-Napoca, 2001.

[6] P. C. Sikkema, *Der Wert einiger Konstanten in der theorie der approximation mit Bernstein-Polynomen*, Numer. Math. **3** (1961), 107–116.

[7] D. D. Stancu, *Approximation of functions by a new class of linear polynomial operators*, Rev. Roum. Math. Pures et Appl., **13**(8) (1968), 1173–1194.