REVISTA DE LA
UNIÓN MATEMATICA ARGENTINA
Volumen 44, Número 2, 2003, Páginas 21-30

OPERATORS GENERATED BY A QUASI-SCALING TYPE FUNCTION

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ABSTRACT.
In this paper we introduce linear approximation operators by using a scaling type function and a general bi-dimensional net. By using the first modulus of smoothness we establish Jackson type inequalities. For a new class of general integral type operators, sufficient conditions are given both for shift invariance and for preservation of the global smoothness. Under additional assumptions, our sequence becomes an approximation process. Also, some examples are provided and the particular case of Hölder continuous functions is considered.

1 INTRODUCTION

Among numerous applications of the fascinating wavelet world, we shall consider here the construction of wavelet type linear positive operators.
In recent years, certain families of functions $\{w_{a,b} : (a,b) \in \mathbb{R}^* \times \mathbb{R}\}$ generated by dilations and translations of a single function w, i.e. given by $w_{a,b} = |a|^{1/2} w(a \cdot -b)$, have been shown to be useful in both theoretical and applied mathematics. These versatile tools are in connection with the so-called "wavelets" or "ondelettes" and

**Mathematics Subject Classification.* 41A25, 41A36
Keywords and phrases. linear positive operator, modulus of smoothness, Jackson-type inequalities, approximation process

they have led to various theories depending on the choice of w and the parameters a and b. There are many ways of restricting (a,b) to a discrete subset of $\mathbb{R}^* \times \mathbb{R}$. For example, in the Franklin-Strömberg theory this general couple is replaced by the net $(2^k, j)$, where both the dilation index k and the translation index j belong to $\mathbb{Z}$. Dilation by larger j compresses the function on the x-axis. Altering k has the effect of sliding the function along the x-axis. For more details about a historical perspective of wavelets [6; *Chapter 2*] can be consulted.

In order to approximate certain classes of functions by wavelet type operators, George Anastassiou and his collaborators followed the above mentioned trend and their quantitative approximation methods applied in wavelet's field are gathered in the recent monograph [3; *Chapter 6*].

We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In our research we use a bi-dimensional net $(a_k, \delta j)$, $(k,j) \in \mathbb{Z} \times \mathbb{Z}$, where

$$0 < a_{-p}^{-1} = a_p < a_{p+1} \text{ for every } p \in \mathbb{N}_0, \text{ and } \delta > 0 \text{ is fixed.} \tag{1}$$

Manipulating the sequence $(a_k)_{k\in\mathbb{N}_0}$ and the ratio δ we will be able to transform the net in accordance with the problem data. We refer here to those one-dimensional signals f for which we are in position to obtain information in some certain points of the real line. This way, the net is more flexible than the previous one.

The contents of this article is organized as follows. In Section 2 we recall some notations and define the notion of *quasi-scaling type function*. Two general classes (L_k), (S_k), $k \in \mathbb{Z}$, of linear operators are constructed in Section 3. Both of them are defined by using the same scaling type function φ. While L_k is a discrete operator, S_k is an integral type operator introduced through a convolution – like iteration – of another positive linear operator with the φ function. Further on, in Section 4, we investigate S_k, $k \in \mathbb{Z}$, operators giving sufficient conditions for shift invariance, for preservation of global smoothness and for convergence to the unit operator multiplied by a constant. By using the first modulus of smoothness of the approximated function, under additional conditions of the net given at (1), we prove that (S_k) becomes an approximation process.

At the same time we provide concrete examples and, in particular, the established Jackson inequality is proven to be sharp.

2 NOTATION AND PRELIMINARIES

Let $L_{1,loc}(\mathbb{R})$ be the vector space of the real-valued functions defined on $\mathbb{R}$ and integrable on any compact interval of the real line.

DEFINITION 1. A function $\varphi : \mathbb{R} \to [0, \infty)$ satisfying the following conditions:

i) φ is a bounded function belonging to $L_{1,loc}(\mathbb{R})$, (2)

ii) a positive constant s exists such that $\text{supp}(\varphi) \subset [-s, s]$, (3)

iii) a positive constant γ exists with the property

$$\sum_{j=-\infty}^{\infty} \varphi(x + j\delta) = \gamma, \ x \in \mathbb{R}, \tag{4}$$

is called a quasi-scaling type function.

LEMMA 1. *If φ is a quasi-scaling type function then one has*

$$\int_{\mathbb{R}} \varphi(x)dx = \int_{\mathbb{R}} \varphi(x + j\delta)dx = \gamma\delta, \ j \in \mathbb{Z}. \tag{5}$$

Proof. We can write successively

$$\begin{aligned} \int_{\mathbb{R}} \varphi(x)dx &= \sum_{j=-\infty}^{\infty} \int_{j\delta}^{(j+1)\delta} \varphi(x)dx = \sum_{j=-\infty}^{\infty} \int_0^{\delta} \varphi(t + j\delta)dt \\ &= \int_0^{\delta} \sum_{j=-\infty}^{\infty} \varphi(t + j\delta)dt \overset{(4)}{=} \gamma \int_0^{\delta} dt = \gamma\delta. \end{aligned}$$

The second identity is a direct consequence of the previous one. □

Throughout the paper we will denote by $\mathcal{D}_\alpha$, $\alpha \in \mathbb{R}^*$, and by $\mathcal{T}_\beta$, $\beta \in \mathbb{R}$, the *dilation operator* respectively the *translation operator*. We recall that $\mathcal{D}_\alpha f(x) = \sqrt{|\alpha|} f(\alpha x)$ and $\mathcal{T}_\beta f(x) = f(x + \beta)$ for every $f \in \mathbb{R}^{\mathbb{R}}$ and $x \in \mathbb{R}$. Using the inner product on $L_2(\mathbb{R})$, the following results can be verified

$$(f, \mathcal{D}_\alpha g) = (\mathcal{D}_{1/\alpha} f, g), \ (f, \mathcal{T}_\beta g) = (\mathcal{T}_{-\beta} f, g), \ f \in L_2(\mathbb{R}), \ g \in L_2(\mathbb{R}),$$

and, consequently, these operators preserve the L_2-norm, that is, $\|f\|_2 = \|\mathcal{D}_\alpha f\|_2 = \|\mathcal{T}_\beta f\|_2$.

As usual, the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|_2$, are defined respectively by

$$(h_1, h_2) = \int_{\mathbb{R}} h_1(t)h_2(t)dt, \ \|h\|_2 = \sqrt{(h, h)},$$

for every h_1, h_2, h belonging to $L_2(\mathbb{R})$.

On the other hand we make the following informal definition.

DEFINITION 2. Let X, Y be linear spaces of real valued functions defined on $\mathbb{R}$. If the operator $U : X \to Y$ verifies $U(\mathcal{T}_\beta f) = \mathcal{T}_\beta(Uf)$, for all $\beta \in \mathbb{R}$, then U is called a shift invariant operator.

As usual, we denote by $C(\mathbb{R})$ ($B(\mathbb{R})$, respectively) the space of all continuous (bounded) real valued functions on $\mathbb{R}$; $C_+(\mathbb{R}_+^*)$ stands for the set of all continuous functions $p : (0,\infty) \to (0,\infty)$. Also, we need the first modulus of smoothness of a bounded real function $g \in \mathbb{R}^I$, $I \subset \mathbb{R}$, denoted by $\omega_1(g;\cdot)$ and defined by $\omega_1(g;\theta) = \sup\limits_{\substack{x,y\in I \\ |x-y|\le\theta}} |g(x) - g(y)|$, for every $\theta \ge 0$. The properties of this special function can be found, e.g., in the monograph [2; §*5.1*].

Finally, we recall the notion of Lipschitz continuous function of order μ, $\mu \in (0,1]$, also known as Hölder continuous functions. For a fixed constant $A \ge 0$ we consider

$$\mathrm{Lip}_\mu A(I) = \mathrm{Lip}_\mu A$$

$$:= \{g \in \mathbb{R}^I : \ |g(x) - g(y)| \le A|x-y|^\mu, \text{ for every } (x,y) \in I \times I\}.$$

3 TWO CLASSES OF LINEAR OPERATORS

Let φ be a quasi-scaling type function. In this section our aim is to present various means to construct linear operators by using φ function. In this respect, at the beginning, we would like to mention a class $(L_k)_{k\in\mathbb{Z}}$ of discrete operators recently introduced in [1]. For every $f \in L_{1,loc}(\mathbb{R})$ and $k \in \mathbb{Z}$ we consider the operator

$$(L_k f)(x) := \sum_{j=-\infty}^{\infty} (f, \varphi_{k,j})\varphi_{k,j}(x), \ x \in \mathbb{R}, \tag{6}$$

where the functions $\varphi_{k,j}$, $(k,j) \in \mathbb{Z}\times\mathbb{Z}$, are given by

$$\varphi_{k,j}(x) := \sqrt{a_k}\,\varphi(a_k x + \delta j), \ x \in \mathbb{R}, \tag{7}$$

a_k being the same as in (1).

Because of the function φ has bounded support, for any real x the summation in (6) involves only finite terms and consequently $(L_k f)(x)$ is well-defined on $\mathbb{R}$. Examining (6) it is clear that L_k is a positive linear operator. In the particular case $a_k = 2^k$ and $\delta = 1$, operator L_k becomes operator A_k studied in [5]. Now, in relation (4) we have $\gamma = 1$.

As regards L_k operators, the main result will be read as follows.

THEOREM 1. ([1]) *Let L_k, $k \in \mathbb{Z}$, be defined by (6). For every function $f \in C(\mathbb{R})$ the following inequality*

$$|(L_k f)(x) - \gamma^2 \delta f(x)| \le \gamma^2 \delta \omega_1(f; 2sa_{-k}), \ k \in \mathbb{Z}, \ x \in \mathbb{R},$$

holds true, where γ and s are given at (4) respectively (3).
Consequently, if $(a_k)_{k\ge 0}$ tends to infinity as $k \to \infty$, then $\lim_{k\to\infty} L_k f = \gamma^2 \delta f$, uniformly on any compact interval of the real axis, for every continuous signal f.
We recall that, from the mathematical point of view, a signal is a function of time. If the variable belongs to an interval I then the signal $f = f(t)$, t in I, is called *analogical signal* and, usually, this type of signal is a continuous function of time t.
Returning to our claimed aim, with the help of the same function φ, we introduce another positive linear operator of integral type.
At first, we consider a sequence $(l_k)_{k\in\mathbb{Z}}$ of positive linear operators that maps $C(\mathbb{R})$ into itself. We fix only the operator l_0 and then we define the other operators l_k, $k \in \mathbb{Z}^*$, by the identity

$$(l_k f)(x) = (l_0 f)(a_{-k} x), \text{ for any } f \in C(\mathbb{R}) \text{ and } x \in \mathbb{R}. \tag{8}$$

Since $a_k a_{-k} = 1$, by (1), relation (8) can be rewritten as follows: $l_k f = \sqrt{a_k} l_0(\mathcal{D}_{a_k} f)$, $f \in C(\mathbb{R})$.
By using the above operators we introduce a sequence $(S_k)_{k\in\mathbb{Z}}$ of operators acting on $C(\mathbb{R})$ and defined by

$$(S_k f)(x) = \int_{\mathbb{R}} (l_k f)(u)\varphi(a_k x - u)du, \ x \in \mathbb{R}. \tag{9}$$

This construction represents a generalization of a result due to G. Anastassiou and H. Gonska [4], our approach being motivated by the quoted paper.
LEMMA 2. *If S_k, $k \in \mathbb{Z}$, are defined by (9) then, for every $f \in C(\mathbb{R})$, the following identities hold true*

$$S_k f = \mathcal{D}_{a_k} S_0 \mathcal{D}_{a_{-k}} f = \mathcal{D}_{a_k}(l_0 \mathcal{D}_{a_{-k}} f * \varphi), \ k \in \mathbb{Z},$$

where $*$ indicates the convolution product.
Proof. Since $a_0 = 1$, we have

$$(S_0 f)(x) = (l_0 f) * \varphi(x). \tag{10}$$

Relations (9) and (8) together with the above identities allow us to write

$$\begin{aligned}(S_k f)(x) &= \sqrt{a_k} \int_{\mathbb{R}} l_0(\mathcal{D}_{a_{-k}} f)(u)\varphi(a_k x - u)du = \sqrt{a_k}(S_0 \mathcal{D}_{a_{-k}} f)(a_k x)\\ &= \mathcal{D}_{a_k} S_0 \mathcal{D}_{a_{-k}} f(x) = \mathcal{D}_{a_k}(l_0 \mathcal{D}_{a_{-k}} f * \varphi). \ \square\end{aligned}$$

Clearly, (10) shows us that if $l_0 f$ is a monotone function and φ is continuous, then $S_0 f$ is also monotone.
Further on, we are going to study connections between l_k and S_k.

4 PROPERTIES OF S_K OPERATORS

At first stage we indicate a sufficient condition regarding l_k such that S_k to become a shift invariant operator.
THEOREM 2. *Let S_k be defined by (9). If one has*

$$l_0(\mathcal{D}_{a_{-k}}\mathcal{T}_\alpha f) = \mathcal{T}_{a_k\alpha}(l_0\mathcal{D}_{a_{-k}} f), \quad \alpha \in \mathbb{R}, \tag{11}$$

for every $f \in C(\mathbb{R})$, then S_k is a shift invariant operator.
Proof. The hypothesis (11) means

$$l_0(f(a_{-k}\cdot + \alpha); t) = l_0(f(a_{-k}\cdot); t + \alpha a_k), \quad \alpha \in \mathbb{R},$$

for every $t \in \mathbb{R}$. Following Definition 2 we compute

$$\begin{aligned} S_k(\mathcal{T}_\alpha f; x) &= \sqrt{a_k} S_0(\mathcal{D}_{a_{-k}}(\mathcal{T}_\alpha f); a_k x) \\ &= \sqrt{a_k}\int_{\mathbb{R}} l_0(\mathcal{D}_{a_{-k}}(\mathcal{T}_\alpha f); t)\varphi(a_k x - t)dt. \end{aligned}$$

On the other hand

$$\begin{aligned} \mathcal{T}_\alpha(S_k f; x) &= (S_k f)(x+\alpha) \\ &= \sqrt{a_k}\int_{\mathbb{R}} l_0(\mathcal{D}_{a_{-k}} f; u)\varphi(a_k x + (a_k\alpha - u))du \\ &= \sqrt{a_k}\int_{\mathbb{R}} l_0(\mathcal{D}_{a_{-k}} f; t + a_k\alpha)\varphi(a_k x - t)dt. \end{aligned}$$

Examining the previous relations and taking into account (11), the proof of our theorem is complete. □
In what follows we investigate the property of global smoothness preservation of the operators S_k.
THEOREM 3. *Let S_k be defined by (9). If the central operator l_0 verifies*

$$|(l_0 f)(x-u) - (l_0 f)(y-u)| \le \omega_1(f; |x-y|), \; x \in \mathbb{R}, \; y \in \mathbb{R}, \tag{12}$$

for every $f \in C(\mathbb{R})$ then the following inequality holds true

$$\omega_1(S_k f; \theta) \le \gamma\delta\omega_1(f; \theta), \; \theta > 0. \tag{13}$$

Proof. For a given $\theta > 0$ let us take $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $|x - y| \leq \theta$. Since φ is a non-negative function, by using (10), (5) and (12) we get

$$\begin{aligned} |(S_0 f)(x) - (S_0 f)(y)| &\leq \int_{\mathbb{R}} \varphi(u) |(l_0 f)(x-u) - (l_0 f)(y-u)| du \\ &\leq \left(\int_{\mathbb{R}} \varphi(u) du \right) \omega_1(f; |x-y|) \\ &= \gamma \delta \omega_1(f; |x-y|). \end{aligned}$$

Lemma 1 and the above result imply

$$\begin{aligned} |(S_k f)(x) - (S_k f)(y)| &= |(S_0 \mathcal{D}_{a_{-k}} f)(a_k x) - (S_0 \mathcal{D}_{a_{-k}} f)(a_k y)| \\ &\leq \gamma \delta \omega_1(\mathcal{D}_{a_{-k}} f; a_k |x-y|) = \gamma \delta \omega_1(f; |x-y|) \\ &\leq \gamma \delta \omega_1(f; \theta). \end{aligned}$$

Applying $\sup\limits_{\substack{x,y \in \mathbb{R} \\ |x-y| \leq \theta}}$ to this inequality, the result follows. □

Next, we present a result concerning the rate of convergence of S_k operators. At this moment, we add a new requirement regarding the central operator l_0. More precisely, we consider that for a fixed $t > 0$, a function $p \in C_+(\mathbb{R}_+^*)$ exists with the property

$$\sup_{\substack{u,v \in \mathbb{R} \\ |u-v| \leq t}} |(l_0 f)(u) - f(v)| \leq \omega_1(f; p(t)), \text{ for every } f \in C(\mathbb{R}). \tag{14}$$

THEOREM 4. *Let S_k be defined by (9) such that (14) is fulfilled. For every function $f \in C(\mathbb{R})$ the following inequality*

$$|(S_k f)(x) - \widetilde{c} f(x)| \leq \widetilde{c} \omega_1(f; a_{-k} p(s)), \ x \in \mathbb{R}, \tag{15}$$

holds true, where $\widetilde{c} := \gamma\delta$ and the constants γ, s are given at (4) respectively (3).

Proof. For the sake of simplicity we put $g := \mathcal{D}_{a_{-k}} f$ and this implies $f = \mathcal{D}_{a_k} g$. Starting from Lemma 2 we can write

$$|(S_k f)(x) - \widetilde{c} f(x)| \overset{(5)}{=} |(S_0 \mathcal{D}_{a_{-k}} f)(a_k x) - f(x) \int_{\mathbb{R}} \varphi(t) dt|$$

$$\overset{(9)}{=} | \int_{\mathbb{R}} l_0(\mathcal{D}_{a_{-k}} f; u) \varphi(a_k x - u) du - f(x) \int_{\mathbb{R}} \varphi(a_k x - u) du|$$

$$= | \int_{\mathbb{R}} ((l_0 g)(u) - (\mathcal{D}_{a_k} g)(x)) \varphi(a_k x - u) du|$$

$$\overset{(3)}{=} \left| \int_{a_k x - s}^{a_k x + s} ((l_0 g)(u) - g(a_k x))\varphi(a_k x - u)du \right|$$

$$\leq \gamma\delta \sup_{|u - a_k x| \leq s} |(l_0 g)(u) - g(a_k x)| \overset{(14)}{\leq} \gamma\delta\omega_1(g; p(s)).$$

Taking into account the definition of ω_1 we get $\omega_1(g;p(s)) = \omega_1(f; a_{-k}p(s))$ and this completes the proof. □

Under these circumstances, it turns out that the sequence $(\widetilde{c}^{-1}S_k)_{k\geq 0}$ has the approximation property converging to the identity operator of the space $C(\mathbb{R})$. More exactly, we ennunciate the following

COROLLARY. *Let the operators S_k, $k \in \mathbb{Z}$, be defined by (9) so that (14) holds true. If $\lim_{k\to\infty} a_k = \infty$ then*

$$\lim_{k\to\infty} L_k f = \widetilde{c} f,$$

uniformly on any compact interval of the real axis, for every $f \in C(\mathbb{R})$.

If the signal f is also a bounded one, in other words it belongs to the Banach space $C_B(\mathbb{R}) := C(\mathbb{R}) \cap B(\mathbb{R})$ endowed with the sup-norm $\|\cdot\|_{C_B(\mathbb{R})}$, then (15) leads us to the following upper-limit of the rate of convergence

$$\|\widetilde{c}^{-1}S_k f - f\|_{C_B(\mathbb{R})} \leq \omega_1(f; a_{-k}p(s)), \ k \in \mathbb{Z}.$$

For illustrating the results of this section we provide an example.

EXAMPLE. Let us choose $l_k = \sqrt{a_k}\mathcal{D}_{a_{-k}}$, $k \in \mathbb{Z}$. Condition (8) is fulfilled and we deduce $l_0 = I$ (identity operator), $S_0 f = f * \varphi$ and

$$(S_k f)(x) = \int_{\mathbb{R}} f(a_{-k}u)\varphi(a_k x - u)du, \ k \in \mathbb{Z}, \ f \in C(\mathbb{R}). \qquad (16)$$

It is easy to see that the shift invariant condition holds true, the common value of (11) being $f(a_{-k} \cdot + \alpha)$. This way S_k is a shift invariant operator. Since (12) becomes $|f(x-u) - f(y-u)| \leq \omega_1(f; |x-y|)$ for every $(x,y) \in \mathbb{R} \times \mathbb{R}$, (13) takes place. Also, (14) is verified for $p = id$ ($id(x) = x$, $x \in \mathbb{R}$) and consequently, the rate of convergence is given by (15). Moreover, by a straightforward calculation from (16) we get

$$|S_k(id;x) - S_k(id;y)| = \left| \int_{\text{supp}(\varphi)} (x-y)\varphi(t)dt \right|$$

$$= \gamma\delta|x-y| = \gamma\delta\omega_1(id; |x-y|).$$

This example ($l_0 = I$) helps us to conclude that the inequality (13) is sharp; if $f = id$, then in (13) the equality is attained. A less trivial choice would be the linear operator l_0 defined by $(l_0 f)(x) = \int_x^{x+1} f(t)dt$, which also satisfies the crucial hypothesis (11). Considerations upon the monotonicity property of S_k operators are contained in the following.

REMARK. For a given function $f \in C(\mathbb{R})$, assume that the mapping

$$(x,u) \mapsto \frac{\partial(l_0 f)}{\partial x}(x-u),\ (x,u) \in \mathbb{R} \times \text{supp}(\varphi), \tag{17}$$

exists and it is continuous in x and u on the whole domain.

Also, assume that φ is continuous. By using the Leibniz rule, relation (9) implies

$$\begin{aligned} \frac{d(S_k f)}{dx}(x) &= \int_{\mathbb{R}} a_k \frac{\partial(l_k f)}{\partial x}(a_k x - t)\varphi(t)dt \\ &= a_k \int_{\text{supp}(\varphi)} \frac{\partial l_0(\mathcal{D}_{a_{-k}} f)}{\partial x}(a_k x - t)\varphi(t)dt. \end{aligned}$$

Consequently, if the function defined by (17) is positive (negative) then $S_k f$, $k \in \mathbb{Z}$, is an increasing (decreasing) function. Obviously, if l_0 maps $C^1(\mathbb{R})$ into $C^1(\mathbb{R})$ then all operators S_k, $k \in \mathbb{Z}$, have the same property. Here, $C^1(\mathbb{R})$ denotes the subspace of all functions $g \in C(\mathbb{R})$ such that the first derivative Dg exists and belongs to $C(\mathbb{R})$.

Returning to our previous example, if $f \in C^1(\mathbb{R})$ and Df doesn't change its sign, then S_k defined by (16) is monotone. At the end of the paper we study the particular case in which these operators leave invariant the class of Lipschitz functions. In what follows we consider $(l_k)_{k\in\mathbb{Z}}$ a general sequence of linear positive operators, in other words we drop condition (8).

THEOREM 5. *For every integer k, let S_k be defined by (9) where l_k is a linear positive operator mapping $L_{1,loc}(\mathbb{R})$ into itself.*

If the function $l_k f$ belongs to $\text{Lip}_{A_k}\mu_k$ then $S_k f$ belongs to $\text{Lip}_{A'_k}\mu_k$, where $A'_k = \gamma\delta a_k^{\mu_k} A_k$.

Proof. By using (9) and knowing that $l_k f \in \text{Lip}_{A_k}\mu_k$, for every (x,y) in $\mathbb{R}\times\mathbb{R}$ we have

$$\begin{aligned} |(S_k f)(x) - (S_k f)(y)| &= |\int_{\mathbb{R}} ((l_k f)(a_k x - t) - (l_k f)(a_k y - t))\varphi(t)dt| \\ &\le \int_{\mathbb{R}} A_k |a_k(x-y)|^{\mu_k}\varphi(t)dt = \gamma\delta a_k^{\mu_k} A_k |x-y|^{\mu_k} \end{aligned}$$

and we arrive at the desired result. □

Consequently, if the knot a_k and the order μ_k are on the following link $a_k \le (\gamma\delta)^{-1/\mu_k}$, then S_k operator leaves invariant the $\text{Lip}_{A_k}\mu_k$ class.

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Recibido: 2 de junio de 2003
Aceptado: 7 de octubre de 2003