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Location of Nonnegative Solutions for Differential Equations on Finite and Semi–Infinite Intervals

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Abstract. A variety of new existence criteria are presented for boundary value problems for second order differential equations. Our results rely on upper and lower type inequalities for the appropriate Green's function.

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1. Introduction

Over the last twenty years or so, Krasnoselskii's fixed point theorem in a cone has played a major role in establishing the existence of nonnegative solutions to second and higher order boundary value problems, see [2, 3, 4, 8] and the references therein. The extension to integral equations and inclusions can be found in [1, 5]. In this paper we show how the results for integral equations can be used to establish new existence criteria for second order boundary value problems (indeed these results could also be established directly, with a lot more effort, if one uses the inequalities presented in this paper for the appropriate Green's function). To illustrate the method involved, we discuss in particular the boundary value problems

$$\begin{cases} y'' - \tau\, y + q\, f(y) = 0, \ \ 0 < t < 1 \\ y(0) = y(1) = 0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} y'' + \tau\, y' + q\, f(y) = 0, \ \ 0 < t < \infty \\ y(0) = 0, \ \lim_{t\to\infty} y(t) = 0. \end{cases} \tag{1.2}$$

In the literature (1.1), with $\tau = 0$, has received a lot of attention whereas only a handful of results [7] are available for (1.2). This paper uses the results in [5] together with upper and lower type inequalities for the appropriate Green's function to obtain new results for (1.1) and (1.2). Indeed it is of interest to note here that all the results in the literature to

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date (which make use of Krasnoselskii's fixed point theorem in a cone) on boundary value problems for second and higher order equations can be deduced from the results in [5].

2. Differential Equations on Finite Intervals

In this section we use a result of Meehan and O'Regan [5] on integral equations to obtain many new results for boundary value problems on a finite interval. Indeed the results in this section improve all known results in the literature (see [1–4] and the references therein). To motivate the general theory we begin by examining

$$\begin{cases} y'' - m^2 y + q\, f(y) = 0, & 0 < t < 1 \\ y(0) = y(1) = 0 \end{cases} \tag{2.1}$$

where

$$m > 0,\ q \in L^1[0,1] \ \text{ with } \ q \geq 0 \ \text{ a.e. on } \ [0,1]. \tag{2.2}$$

Before we state and prove a general existence result for (2.1) we recall the following existence result [5] for the integral equation

$$y(t) = \int_0^1 k(t,s)\, f(y(s))\, ds \quad \text{for} \quad t \in [0,1]. \tag{2.3}$$

THEOREM 2.1. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ and suppose the following conditions are satisfied:

$$0 \leq k_t(\,.\,) = k(t,\,.\,) \in L^1[0,1] \text{ for each } t \in [0,1], \tag{2.4}$$

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0,1] \text{ to } L^1[0,1], \tag{2.5}$$

$$\begin{cases} \exists\ 0 < M < 1,\ \kappa \in L^1[0,1] \text{ and } [a,b] \subseteq [0,1],\ a < b, \\ \text{such that } k(t,s) \geq M\kappa(s) \geq 0,\ t \in [a,b], \text{ a.e. } s \in [0,1], \end{cases} \tag{2.6}$$

$$k(t,s) \leq \kappa(s),\ t \in [0,1], \text{ a.e. } s \in [0,1], \tag{2.7}$$

$$f : \mathbf{R} \to \mathbf{R} \text{ is continuous and nondecreasing with } f(u) > 0 \text{ for } u > 0, \tag{2.8}$$

$$\exists\ \alpha > 0 \text{ with } \frac{\alpha}{f(\alpha) \sup_{t\in[0,1]} \int_0^1 k(t,s)\, ds} \geq 1 \tag{2.9}$$

and

$$\exists \beta > 0,\ \beta \neq \alpha \text{ and } t^* \in [0,1] \text{ with } \frac{\beta}{f(M\beta) \int_a^b k(t^*,s)\, ds} \leq 1. \tag{2.10}$$

Then (2.3) has at least one nonnegative solution $y \in C[0,1]$ with either

$$\textbf{(A)}\ \ 0 < \alpha \leq |y|_0 \leq \beta \ \text{ and } \ y(t) \geq M\alpha,\ t \in [a,b] \text{ if } \alpha < \beta$$

or

$$\textbf{(B)}\ \ 0 < \beta \leq |y|_0 \leq \alpha \ \text{ and } \ y(t) \geq M\beta,\ t \in [a,b] \text{ if } \beta < \alpha$$

holding; here $|y|_0 = \sup_{t\in[0,1]} |y(t)|$.

REMARK 2.1. It is possible to remove the nondecreasing assumption in (2.8) provided (2.9) and (2.10) are appropriately adjusted.

REMARK 2.2. In certain cases it is possible to apply Theorem 2.1 repeatedly to an integral equation to yield multiple nonnegative solutions.

We will apply Theorem 2.1 now to (2.1). Notice we can rewrite (2.1) as

$$y(t) = \int_0^1 G(t,s)\,q(s)\,f(y(s))\,ds,\ \ t \in [0,1]$$

where

$$G(t,s) = \begin{cases} \dfrac{\sinh(m\,s)\,\sinh(m(1-t))}{m\,\sinh\,m}, & s \le t \\ \dfrac{\sinh(m\,t)\,\sinh(m(1-s))}{m\,\sinh\,m}, & t < s. \end{cases}$$

Now let

$$k(t,s) = G(t,s)\,q(s), \tag{2.11}$$

and notice (2.4) and (2.5) hold. Let

$$\kappa(s) = G(s,s)\,q(s), \tag{2.12}$$

and (2.7) is immediate since $\sinh\,m\,x$ is increasing for $0 \le x \le 1$ and $\sinh\,m(1-x)$ is decreasing for $0 \le x \le 1$. Next take $a,\,b \in (0,1)$ with $a < b$. Let

$$M = \min\left\{\frac{\sinh\,m(1-b)}{\sinh\,m},\,\frac{\sinh\,m\,a}{\sinh\,m}\right\}, \tag{2.13}$$

and notice if $s < t$ we have

$$\frac{k(t,s)}{\kappa(s)} = \frac{\sinh\,m(1-t)}{\sinh\,m(1-s)} \ge \frac{\sinh\,m(1-t)}{\sinh\,m} \ge \frac{\sinh\,m(1-b)}{\sinh\,m}\ \ \text{for}\ \ t \in [a,b],$$

whereas if $s > t$ we have

$$\frac{k(t,s)}{\kappa(s)} = \frac{\sinh\,m\,t}{\sinh\,m\,s} \ge \frac{\sinh\,m\,t}{\sinh\,m} \ge \frac{\sinh\,m\,a}{\sinh\,m}\ \ \text{for}\ \ t \in [a,b].$$

As a result (2.6) holds. Combine this with Theorem 2.1 and we obtain the following result.

THEOREM 2.2. Assume (2.2) and (2.8) hold. Fix $a,\,b \in (0,1)$ with $a < b$ and suppose the following two conditions are satisfied:

$$\exists\ \alpha > 0\ \ \text{with}\ \ \frac{\alpha}{f(\alpha)} \ge \sup_{t\in[0,1]} \int_0^1 G(t,s)\,q(s)\,ds \tag{2.14}$$

and

$$\exists\beta > 0, \beta \ne \alpha \text{ and } t^\star \in [0,1] \text{ with } \frac{\beta}{f(M\beta)\,\int_a^b G(t^\star,s)\,q(s)\,ds} \le 1; \tag{2.15}$$

here M is as in (2.13). Then (2.1) has at least one nonnegative solution $y \in W^{2,1}[0,1]$ with either

$$\textbf{(A)}\ \ 0 < \alpha \le |y|_0 \le \beta\ \ \text{and}\ \ y(t) \ge M\alpha,\ \ t \in [a,b]\ \text{if}\ \alpha < \beta$$

or

$$\textbf{(B)}\ \ 0 < \beta \le |y|_0 \le \alpha\ \ \text{and}\ \ y(t) \ge M\beta,\ \ t \in [a,b]\ \text{if}\ \beta < \alpha$$

holding.

The argument presented for (2.1) immediately yields a new existence result for the Dirichlet boundary value problem

$$\begin{cases} y'' - r\,y + q\,f(y) = 0,\ \ 0 < t < 1 \\ y(0) = y(1) = 0, \end{cases} \tag{2.16}$$

where

$$\tau \in C(0,1) \cap L^1[0,1] \text{ with } \tau > 0 \text{ on } (0,1) \tag{2.17}$$

and

$$q \in L^1[0,1] \text{ with } q \geq 0 \text{ a.e. on } [0,1] \tag{2.18}$$

hold. Let ϕ be the unique solution [6] to

$$\begin{cases} y'' - \tau\, y = 0, & 0 < t < 1 \\ y(0) = 0, & y'(0) = 1; \end{cases} \tag{2.19}$$

notice ϕ is nondecreasing on $[0,1]$. Also let ψ be the unique solution to

$$\begin{cases} y'' - \tau\, y = 0, & 0 < t < 1 \\ y(1) = 0, & y'(1) = -1; \end{cases} \tag{2.20}$$

notice ψ is nonincreasing on $[0,1]$. We can rewrite (2.16) as

$$y(t) = \int_0^1 g(t,s)\, q(s)\, f(y(s))\, ds, \quad t \in [0,1]$$

where

$$g(t,s) = \begin{cases} \phi(s)\, \psi(t)/w_0, & s \leq t \\ \phi(t)\, \psi(s)/w_0, & t < s, \end{cases}$$

with $w_0 = \phi'(s)\, \psi(s) - \phi(s)\, \psi'(s) > 0$. Let

$$k(t,s) = g(t,s)\, q(s) \quad \text{and} \quad \kappa(s) = g(s,s)\, q(s), \tag{2.21}$$

and notice (2.4), (2.5) and (2.7) hold. Next take $a,\ b \in (0,1)$ with $a < b$. Then (2.6) holds with

$$M = \min\left\{\frac{\psi(b)}{\psi(0)},\ \frac{\phi(a)}{\phi(1)}\right\}. \tag{2.22}$$

THEOREM 2.3. Assume (2.8), (2.17) and (2.18) hold. Fix $a,\ b \in (0,1)$ with $a < b$ and suppose the following two conditions are satisfied:

$$\exists\, \alpha > 0 \text{ with } \frac{\alpha}{f(\alpha)} \geq \sup_{t \in [0,1]} \int_0^1 g(t,s)\, q(s)\, ds \tag{2.23}$$

and

$$\exists \beta > 0, \beta \neq \alpha \text{ and } t^\star \in [0,1] \text{ with } \frac{\beta}{f(M\beta) \int_a^b g(t^\star, s)\, q(s)\, ds} \leq 1; \tag{2.24}$$

here M is as in (2.22). Then (2.16) has at least one nonnegative solution $y \in W^{2,1}[0,1]$ with either

(A) $0 < \alpha \leq |y|_0 \leq \beta$ and $y(t) \geq M\alpha,\ t \in [a,b]$ if $\alpha < \beta$

or

(B) $0 < \beta \leq |y|_0 \leq \alpha$ and $y(t) \geq M\beta,\ t \in [a,b]$ if $\beta < \alpha$

holding.

REMARK 2.3. It is possible to replace $q \in L^1[0,1]$ in (2.18) by the less restrictive condition $\int_0^1 g(s,s)\, q(s)\, ds < \infty$.

REMARK 2.4. An analogue result could be presented for the Sturm Liouville problem

$$\begin{cases} y'' - \tau\, y + q\, f(y) = 0, \;\; 0 < t < 1 \\ \alpha\, y(0) \; - \; \beta\, y'(0) = 0, \;\; \alpha \geq 0, \beta \geq 0 \\ \gamma\, y(1) \; + \; \delta\, y'(1) = 0, \;\; \gamma \geq 0, \delta \geq 0 \end{cases}$$

with $\alpha\,\delta + \alpha\,\gamma + \beta\,\gamma > 0$.

In fact it is possible to use the ideas in this section to discuss the differential inclusion

$$\begin{cases} y'' - \tau\, y \in -f(t,y), \;\; 0 < t < 1 \\ y(0) = y(1) = 0; \end{cases} \tag{2.25}$$

here $f : [0,1]\times\mathbf{R} \to K(\mathbf{R})$ ($K(\mathbf{R})$ denotes the family of nonempty, compact, convex subsets of $\mathbf{R}$). Our results rely on the following existence result [1] for the integral inclusion

$$y(t) \in \int_0^1 k(t,s)\, f(s,y(s))\, ds, \quad t \in [0,1]. \tag{2.26}$$

THEOREM 2.4. Let $1 \leq p < \infty$ and q, $1 < q \leq \infty$, the conjugate to p, $k : [0,1]\times[0,1] \to \mathbf{R}$ and $f : [0,1]\times\mathbf{R} \to K(\mathbf{R})$. In addition suppose the following conditions are satisfied:

$$\text{for each } \; t \in [0,1], \;\; \text{the map } \; s \mapsto k(t,s) \; \text{ is measurable,} \tag{2.27}$$

$$\sup_{t\in[0,1]} \left(\int_0^1 |k(t,s)|^q\, ds \right)^{1/q} < \infty, \tag{2.28}$$

$$\int_0^1 |k(t',s) - k(t,s)|^q\, ds \to 0 \;\; \text{as} \;\; t \to t', \;\; \text{for each} \;\; t' \in [0,1], \tag{2.29}$$

$$\text{for each } \; t \in [0,1], \; k(t,s) \geq 0 \; \text{ for a.e. } \; s \in [0,1], \tag{2.30}$$

$$\begin{cases} \text{for each measurable } \; u : [0,1] \to \mathbf{R}, \;\; \text{the map } \; t \mapsto f(t,u(t)) \\ \text{has measurable single valued selections,} \end{cases} \tag{2.31}$$

$$\text{for a.e. } \; t \in [0,1], \;\; \text{the map } \; u \mapsto f(t,u) \; \text{ is upper semicontinuous,} \tag{2.32}$$

$$\begin{cases} \text{for each } \; r > 0, \; \exists\, h_r \in L^p[0,1] \; \text{ with } \; |f(t,y)| \leq h_r(t) \\ \text{for a.e. } \; t \in [0,1] \; \text{ and every } \; y \in \mathbf{R} \; \text{ with } \; |y| \leq r, \end{cases} \tag{2.33}$$

$$\text{for a.e. } \; t \in [0,1] \; \text{ and all } \; y \in (0,\infty), \; u > 0 \; \text{ for all } \; u \in f(t,y), \tag{2.34}$$

$$\begin{cases} \exists\, \kappa \in L^q[0,1] \; \text{ with } \; \kappa : [0,1] \to (0,\infty) \; \text{ and} \\ \text{with } \; k(t,s) \leq \kappa(s) \; \text{ for } \; t \in [0,1], \;\; \text{a.e. } \; s \in [0,1], \end{cases} \tag{2.35}$$

$$\begin{cases} \exists\, a, \, b, \; 0 < a < b < 1 \; \text{ and } \; M, \; 0 < M < 1, \\ \text{with } \; k(t,s) \geq M\, \kappa(s) \; \text{ for } \; t \in [a,b], \;\; \text{a.e. } \; s \in [0,1], \end{cases} \tag{2.36}$$

$$\begin{cases} \exists\, h \in L^p[0,1] \; \text{ with } \; h : [0,1] \to (0,\infty), \;\; \text{and } \; w \geq 0 \; \text{ continuous} \\ \text{and nondecreasing on } \; (0,\infty) \; \text{ with } \; w(y) > 0 \; \text{ for } \; y > 0, \; \text{and} \\ \text{with } \; |f(t,y)| \leq h(t)\, w(y) \; \text{ for a.e. } \; t \in [0,1] \; \text{ and all } \; y \in (0,\infty), \end{cases} \tag{2.37}$$

$$\begin{cases} \exists\, \tau \in L^p[a,b] \; \text{ with } \; \tau > 0 \; \text{ a.e. on } \; [a,b] \; \text{ and with for a.e.} \\ t \in [a,b] \; \text{ and } \; y \in (0,\infty), \; u \geq \tau(t)\, w(y) \; \text{ for all } \; u \in f(t,y), \end{cases} \tag{2.38}$$

$$\exists\, \alpha > 0 \;\; \text{with} \;\; \frac{\alpha}{w(\alpha)\, \sup_{t\in[0,1]} \int_0^1 k(t,s)\, h(s)\, ds} \geq 1 \tag{2.39}$$

and

$$\exists\beta>0,\beta\neq\alpha \text{ and } t^\star\in[0,1] \text{ with } \frac{\beta}{w(M\beta)\int_a^b k(t^\star,s)\,\tau(s)\,ds}\leq 1. \qquad (2.40)$$

Then (2.26) has at least one nonnegative solution $y\in C[0,1]$ with either

$$\textbf{(A)}\ \ 0<\alpha\leq|y|_0\leq\beta \ \text{ and } \ y(t)\geq M\alpha,\ \ t\in[a,b] \text{ if } \alpha<\beta$$

or

$$\textbf{(B)}\ \ 0<\beta\leq|y|_0\leq\alpha \ \text{ and } \ y(t)\geq M\beta,\ \ t\in[a,b] \text{ if } \beta<\alpha$$

holding.

Let ϕ (respectively ψ) be the solution of (2.19) (respectively (2.20)) and let

$$g(t,s)=\begin{cases}\phi(s)\,\psi(t)/w_0, & s\leq t\\ \phi(t)\,\psi(s)/w_0, & t<s,\end{cases}$$

with $w_0=\phi'(s)\,\psi(s)-\phi(s)\,\psi'(s)>0$. We can rewrite (2.25) as

$$y(t)\ \in\ \int_0^1 g(t,s)\,f(s,y(s))\,ds,\ \ t\in[0,1].$$

Let

$$k(t,s)=g(t,s) \ \text{ and } \ \kappa(s)=g(s,s),$$

and notice (2.27) – (2.30) and (2.35) – (2.36) are satisfied with M given in (2.22).

THEOREM 2.5. Let $1\leq p<\infty$, $f:[0,1]\times\mathbf{R}\to K(\mathbf{R})$ and fix $a,\,b\in(0,1)$ with $a<b$. Suppose (2.17), (2.31) – (2.34), (2.37), (2.38) hold and in addition assume the following conditions are satisfied:

$$\exists\ \alpha>0 \ \text{ with } \ \frac{\alpha}{w(\alpha)\ \sup_{t\in[0,1]}\int_0^1 g(t,s)\,h(s)\,ds}\geq 1 \qquad (2.41)$$

and

$$\exists\beta>0,\beta\neq\alpha \text{ and } t^\star\in[0,1] \text{ with } \frac{\beta}{w(M\beta)\int_a^b g(t^\star,s)\,\tau(s)\,ds}\leq 1; \qquad (2.42)$$

here M is as in (2.22). Then (2.25) has at least one nonnegative solution $y\in W^{2,p}[0,1]$ with either

$$\textbf{(A)}\ \ 0<\alpha\leq|y|_0\leq\beta \ \text{ and } \ y(t)\geq M\alpha,\ \ t\in[a,b] \text{ if } \alpha<\beta$$

or

$$\textbf{(B)}\ \ 0<\beta\leq|y|_0\leq\alpha \ \text{ and } \ y(t)\geq M\beta,\ \ t\in[a,b] \text{ if } \beta<\alpha$$

holding.

3. Differential Equations on Semi–infinite Intervals

In this section we discuss the boundary value problem

$$\begin{cases} y''+m^2\,y'+q\,f(y)=0, \ \ 0<t<\infty\\ y(0)=0, \ \lim_{t\to\infty}\,y(t)=0.\end{cases} \qquad (3.1)$$

Before we state and prove a general existence result for (3.1), we recall the following existence principle [5] for the integral equation

$$y(t)=\int_0^\infty k(t,s)\,f(y(s))\,ds,\ \ t\in[0,\infty). \qquad (3.2)$$

Recall $BC[0,\infty)$ denotes the space of bounded, continuous functions defined on $[0,\infty)$ with norm given by $|y|_\infty = \sup_{t\in[0,\infty)} |y(t)|$. The space $C_l[0,\infty)$ is a subset of $BC[0,\infty)$ which consists of all $y \in BC[0,\infty)$ such that $\lim_{t\to\infty} y(t)$ exists.

THEOREM 3.1. Let $k : [0,\infty)\times[0,\infty) \to \mathbf{R}$ and suppose the following conditions are satisfied:

$$0 \le k_t(s) = k(t,s) \in L^1[0,\infty) \text{ for each } t \in [0,\infty), \tag{3.3}$$

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0,\infty) \text{ to } L^1[0,\infty), \tag{3.4}$$

$$\begin{cases} \exists\, 0 < M < 1,\ \kappa \in L^1[0,\infty) \text{ and } [a,b] \subseteq [0,\infty),\ a < b, \\ \text{such that } k(t,s) \ge M\,\kappa(s) \ge 0,\ t \in [a,b], \text{ a.e. } s \in [0,\infty), \end{cases} \tag{3.5}$$

$$k(t,s) \le \kappa(s),\ t \in [0,\infty), \text{ a.e. } s \in [0,\infty), \tag{3.6}$$

$$\exists\, \tilde{k} \in L^1[0,\infty) \text{ with } k_t \to \tilde{k} \text{ in } L^1[0,\infty) \text{ as } t \to \infty, \tag{3.7}$$

$$f : \mathbf{R} \to \mathbf{R} \text{ is continuous and nondecreasing with } f(u) > 0 \text{ for } u > 0, \tag{3.8}$$

$$\exists\, \alpha > 0 \text{ with } \frac{\alpha}{f(\alpha)\, \sup_{t\in[0,\infty)} \int_0^\infty k(t,s)\,ds} \ge 1 \tag{3.9}$$

and

$$\exists \beta > 0, \beta \ne \alpha \text{ and } t^\star \in [0,\infty) \text{ with } \frac{\beta}{f(M\beta)\int_a^b k(t^\star,s)\,ds} \le 1. \tag{3.10}$$

Then (3.2) has at least one nonnegative solution $y \in C_l[0,\infty)$ with either

$$\textbf{(A)}\ 0 < \alpha \le |y|_\infty \le \beta \text{ and } y(t) \ge M\alpha,\ t \in [a,b] \text{ if } \alpha < \beta$$

or

$$\textbf{(B)}\ 0 < \beta \le |y|_\infty \le \alpha \text{ and } y(t) \ge M\beta,\ t \in [a,b] \text{ if } \beta < \alpha$$

holding.

We can apply Theorem 3.1 now to (3.1). Suppose

$$m \ne 0,\ q \in L^1[0,\infty),\ q \ge 0 \text{ a.e. on } [0,\infty) \text{ with } \lim_{t\to\infty} e^{-m^2 t}\int_0^t e^{m^2 s} q(s)\,ds = 0 \tag{3.11}$$

is satisfied. Notice we can rewrite (3.1) as

$$y(t) = \int_0^\infty G(t,s)\,q(s)\,f(y(s))\,ds,\ t \in [0,\infty)$$

where

$$G(t,s) = \begin{cases} \dfrac{e^{-m^2 t}}{m^2}\left[e^{m^2 s} - 1\right], & s \le t \\ \dfrac{1}{m^2}\left[1 - e^{-m^2 t}\right], & t < s. \end{cases}$$

Let

$$k(t,s) = G(t,s)\,q(s) \quad\text{and}\quad \kappa(s) = G(s,s)\,q(s). \tag{3.12}$$

Notice (3.3), (3.4) and (3.6) hold since

$$\frac{k(t,s)}{\kappa(s)} = \begin{cases} e^{m^2(s-t)} \le 1, & s \le t \\ \dfrac{1 - e^{-m^2 t}}{1 - e^{-m^2 s}} \le \dfrac{1 - e^{-m^2 s}}{1 - e^{-m^2 s}} = 1, & t < s. \end{cases}$$

Next take $a,\, b \in (0,\infty)$ with $a < b$. Let

$$M = \min\left\{1 - e^{-m^2 a},\, e^{-m^2 b}\right\}. \tag{3.13}$$

Notice if $s < t$ we have

$$\frac{k(t,s)}{\kappa(s)} = e^{m^2(s-t)} \geq e^{-m^2 t} \geq e^{-m^2 b} \quad \text{for } t \in [a,b],$$

whereas if $s > t$ we have

$$\frac{k(t,s)}{\kappa(s)} = \frac{1 - e^{-m^2 t}}{1 - e^{-m^2 s}} \geq 1 - e^{-m^2 t} \geq 1 - e^{-m^2 a} \quad \text{for } t \in [a,b].$$

As a result (3.5) holds. Also notice (3.7) holds with $\tilde{k} = 0$ since (3.11) implies

$$\begin{aligned}
\int_0^\infty |k_t(s)|\, ds &= \frac{e^{-m^2 t}}{m^2} \int_0^t \left[e^{m^2 s} - 1\right] q(s)\, ds + \frac{1}{m^2}\left[1 - e^{-m^2 t}\right] \int_t^\infty q(s)\, ds \\
&= \frac{1}{m^2} \int_t^\infty q(s)\, ds - \frac{e^{-m^2 t}}{m^2} \int_0^\infty q(s)\, ds + \frac{e^{-m^2 t}}{m^2} \int_0^t e^{m^2 s} q(s)\, ds \\
&\to 0 \quad \text{as } t \to \infty.
\end{aligned}$$

THEOREM 3.2. Assume (3.8) and (3.11) hold. Fix $a,\, b \in (0,\infty)$ with $a < b$ and suppose the following two conditions are satisfied:

$$\exists\, \alpha > 0 \text{ with } \frac{\alpha}{f(\alpha)} \geq \sup_{t\in[0,\infty)} \int_0^\infty G(t,s)\, q(s)\, ds \tag{3.14}$$

and

$$\exists \beta > 0, \beta \neq \alpha \text{ and } t^\star \in [0,\infty) \text{ with } \frac{\beta}{f(M\beta) \int_a^b G(t^\star,s)\, q(s)\, ds} \leq 1; \tag{3.15}$$

here M is as in (3.13). Then (3.1) has at least one nonnegative solution $y \in C_l[0,\infty)$ with either

(A) $0 < \alpha \leq |y|_\infty \leq \beta$ and $y(t) \geq M\alpha,\ t \in [a,b]$ if $\alpha < \beta$

or

(B) $0 < \beta \leq |y|_\infty \leq \alpha$ and $y(t) \geq M\beta,\ t \in [a,b]$ if $\beta < \alpha$

holding.

REMARK 3.1. A similar argument can be used (the details are left to the reader) to establish an analogue of Theorem 3.2 for the boundary value problem

$$\begin{cases} y'' + r\,y' + q\,f(y) = 0, & 0 < t < \infty \\ y(0) = 0, & \lim_{t\to\infty} y(t) = 0. \end{cases}$$

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