Applications of Fibonacci Numbers

Volume 9

Proceedings of The Tenth International Research Conference on Fibonacci Numbers and Their Applications

edited by

Frederic T. Howard

Wake Forest University,
Winston-Salem, North Carolina, U.S.A.

KLUWER ACADEMIC PUBLISHERS
DORDRECHT / BOSTON / LONDON

A GENERALIZATION OF DURRMEYER-TYPE POLYNOMIALS AND THEIR APPROXIMATION PROPERTIES

Octavian Agratini

1. INTRODUCTION

The Bernstein polynomial approximation process of discrete type defined for every function f belonging to the space $C([0,1])$ by $(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(k/n)$, where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \ x \in [0,1], \tag{1}$$

has been the object of many investigations serving as a guide for theorems that can be proved for a large class of positive linear approximation processes on a bounded interval.

Simultaneously, the Bernstein polynomial basis $\mathcal{B}_n = (p_{n,k})_{k=\overline{0,n}}$ is a treasure of nice properties.

In order to obtain an approximation process in spaces of integrable functions, J.L. Durrmeyer [5] defined the following integral modification of the Bernstein polynomial:

$$(D_n f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \ \ f \in L_1([0,1]), \ x \in [0,1], \tag{2}$$

which can be used for restoring f if its moments $\int_0^1 f(t) t^k dt$ are given. These polynomials were extensively studied by Marie Madeleine Derriennic [4]. The sequence $(D_n f)_{n \geq 1}$ appears

This paper is in final form and no version of it will be submitted for publication elsewhere.

F.T. Howard (ed.), Applications of Fibonacci Numbers, Volume 9: Proceedings of the Tenth International Research Conference on Fibonacci Numbers and their Applications, 9–18.

more complicated and maybe more difficult to compute but it possesses some desirable properties, of which the most notable are commutativity, self-adjointness and simple expansions by Legendre polynomials. It was also shown that $D_n f$, $n \in \mathbb{N}$, are positive contractions in $L_p([0,1])$, $p \geq 1$, spaces. It is the above mentioned properties that make $D_n f$ simpler than the Bernstein polynomial approximation. Therefore, we are able to prove for (2) approximation results that we are not able to prove for Bernstein polynomials from which $D_n f$ originate.

In a recent paper Michele Campiti and Giorgio Metafune [3] replaced in the polynomials $B_n f$ the binomial coefficents by general ones satisfying similar recursive properties, more precisely they replaced the sequences of constant value 1 at the sides of Pascal's triangle with arbitrary ones and defined the coefficients of their polynomials using the same rule of binomial coefficients. The new sequence does not converge to the identity operator but to an operator multiplied by an analytic function, say φ, depending on the sequences of the sides of Pascal's triangle. The authors studied the uniform convergence of these operators together with some quantitative estimates and regularity properties.

Motivated by all the above researches we propose a new general class of polynomials. The next section is devoted to construct this class. In Section 3 we investigate the convergence of the operators giving general estimates in terms of the modulus of smoothness. In the last section we focus our attention on establishing concrete examples of φ function by manipulating the numerical sequences mentioned above. Also some further ideas are presented.

2. CONSTRUCTION OF THE POLYNOMIALS $M_n f$

At first step we consider two sequences of real positive numbers $a = (a_n)_{n\geq 1}$, $b = (b_n)_{n\geq 1}$. For every $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$ we define the polynomials

$$q_{n,k}(x) = c_{n,k} x^k (1-x)^{n-k}, \quad x \in [0,1], \tag{3}$$

where the coefficients satisfy the following recursive formulas

$$c_{n+1,k} = c_{n,k} + c_{n,k-1}, \quad k = 1, \ldots, n, \quad c_{n,0} = a_n, \quad c_{n,n} = b_n. \tag{4}$$

We shall consider polynomials having the form

$$(M_n f)(x) = (n+1) \sum_{k=0}^{n} q_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad x \in [0,1], \tag{5}$$

where $f \in L_1([0,1])$.

Actually, the polynomials defined by (3) belong to the space

$$\mathcal{P}_n^+ := \left\{ s \in \mathcal{P}_n : s(x) = \sum_{i=0}^{n} \alpha_i x^i (1-x)^{n-i}, \ \alpha_i \geq 0, \ i = 0, \ldots, n \right\},$$

where $\mathcal{P}_n$ represents the set of all algebraic polynomials of degree less than or equal to n. Best of our knowledge, such polynomials were firstly studied by Jurkat and Lorentz [6] who were concerned with density and degree of approximation questions.

As regards the relations (4), if $a_j = b_j = 1$ for every $j = 1, \dots, n$, we have $c_{n,k} = \binom{n}{k}$ for every $k = 0, \dots, n$. Hence, in this case $M_n f$ becomes $D_n f$ defined by (2).

It is clear that the polynomial $M_n f$ is determined uniquely by the two sequences a and b. We can point out this fact by using a more precise notation named $M_n^{(a,b)} f$. Throughout the paper, we will use one or another of the notations as required by the context.

Remarks: (i) For every $n \in \mathbb{N}$, the operator M_n is linear positive and maps the space $L_1([0,1])$ into $\mathcal{P}_n$.

(ii) If the sequences $a^{(j)} = (a_n^{(j)})_{n\geq 1}$, $b^{(j)} = (b_n^{(j)})_{n\geq 1}$, $j \in \{1,2\}$, satisfy the following conditions $a_n^{(1)} \leq a_n^{(2)}, b_n^{(1)} \leq b_n^{(2)}$ for every $n \geq 1$, then it is easy to check that

$$M_n^{(a^{(1)},b^{(1)})} f \leq M_n^{(a^{(2)},b^{(2)})} f, \quad n \in \mathbb{N},$$

for every positive function $f \in L_1([0,1])$. In particular, if $\overline{M}$ is an upper bound for the sequences a and b, then

$$M_n^{(a,b)} f \leq M_n^{(\overline{M},\overline{M})} f = \overline{M} D_n f. \tag{6}$$

(iii) By using (5) and (4) we observe that $M_n^{(a,b)}$ depends linearly on the given sequences a and b.

Further on, we are going to investigate the sequence $(M_n f)_{n\geq 1}$.

3. PROPERTIES OF THE POLYNOMIALS $M_n f$

We will emphasize the convergence of our sequence and we will also give estimates of the rate of convergence preceded by the presentation of the following property.

Theorem 1: *Let the operator M_n be defined by (5). If f and g belong to $L_1([0,1])$ then the following identity*

$$\int_0^1 (M_n f)(x) g(x) dx = \int_0^1 f(t)(M_n g)(t) dt$$

holds. Particularly, M_n is a self-adjoint operator on the space $L_2([0,1])$.

Proof: We can write successively

$$\int_0^1 (M_n f)(x) g(x) dx = (n+1) \sum_{k=0}^{n} \int_0^1 q_{n,k}(x) g(x) dx \int_0^1 p_{n,k}(t) f(t) dt$$

$$= \int_0^1 f(t) \left\{ (n+1) \sum_{k=0}^{n} c_{n,k} t^k (1-t)^{n-k} \int_0^1 \binom{n}{k} x^k (1-x)^{n-k} g(x) dx \right\} dt$$

$$= \int_0^1 f(t)(M_n g)(t)dt.$$

If f and g belong to the complex Hilbert space $L_2([0,1])$ then the above relation implies

$$\langle M_n f, g\rangle_{L_2([0,1])} = \langle f, M_n g\rangle_{L_2([0,1])},$$

where $\langle\cdot,\cdot\rangle_{L_2([0,1])}$ stands for the inner product. We recall: $\langle f,g\rangle_{L_2([0,1])} = \int_0^1 f(x)\overline{g(x)}dx$, where $f, g \in L_2([0,1])$. This completes the proof. □

In what follows e_j stands for the j-th monomial, $e_j(t) = t^j$, $t \in [0,1]$, $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We need the following result.

Lemma 1: *If M_n is defined by (5) then the following identity*

$$(M_n e_0)(x) = \sum_{m=1}^{n-1}(a_m x(1-x)^m + b_m x^m(1-x)) + a_n(1-x)^n + b_n x^n, \tag{7}$$

$x \in [0,1]$, *holds true.*

Proof: By using the Beta function $B(\cdot,\cdot)$ we obtain

$$\int_0^1 p_{n,k}(t)dt = \binom{n}{k} B(k+1, n-k+1) = \frac{1}{n+1}, \quad k = 0,1,\ldots,n,$$

and consequently $(M_n e_0)(x) = \sum_{k=0}^n c_{n,k} x^k (1-x)^{n-k}$. Based on Remark (iii) we shall find the operators $A_{m,n}$, $B_{m,n}$, $m = \overline{1,n}$ which are associated to the sequences a, b and verify the identity

$$M_n \equiv M_n^{\langle a,b\rangle} = \sum_{m=1}^n a_m A_{m,n} + \sum_{m=1}^n b_m B_{m,n}. \tag{8}$$

Following the same technique as in [3], firstly we choose $b = 0$ and $a = \delta_m$, where $\delta_m = (\delta_{m,n})_{n\geq 1}$, $\delta_{m,n}$ being the symbol of Kronecker. We obtain $A_{m,n}e_0 = M_n^{\langle \delta_m, 0\rangle} e_0$. Taking into account the relations (4) we get

$$(A_{m,n}e_0)(x) = \sum_{k=1}^{n-m} \binom{n-m-1}{k-1} x^k (1-x)^{n-k} = x(1-x)^m, \text{ if } m = \overline{1, n-1},$$

and $(A_{n,n}e_0)(x) = (1-x)^n$.

Secondly we choose $a = 0$, $b = \delta_m$ and this leads to the identity $B_{m,n}e_0 = M_n^{(0,\delta_m)}e_0$. The same relations (4) imply

$$(B_{m,n}e_0)(x) = \sum_{k=m}^{n-1} \binom{n-m-1}{k-m} x^k(1-x)^{n-k} = (1-x)x^m, \text{ if } m = \overline{1, n-1},$$

and $(B_{n,n}e_0)(x) = x^n$.

Substituting the above expressions of the function $A_{m,n}e_0, B_{m,n}e_1$, $m = \overline{1,n}$, in the identity (8) we obtain the claimed result. □

Remark: From (7) we deduce $(M_ne_0)(0) = a_n$ and $(M_ne_0)(1) = b_n$. This means that the convergence of $(M_n)_{n\geq 1}$ implies the convergence of the sequences a and b. In what follows we assume that these sequences converge and set

$$\lim_{n\to\infty} a_n = l_a, \quad \lim_{n\to\infty} b_n = l_b. \tag{9}$$

Because of the above assumption we can define the functions σ, τ, φ belonging to $\mathbb{R}^{[0,1]}$ as follows

$$\sigma(x) = \begin{cases} l_a, & x = 0, \\ \sum_{m=1}^{\infty} a_m x(1-x)^m, & 0 < x \leq 1 \end{cases} \qquad \tau(x) = \begin{cases} \sum_{m=1}^{\infty} b_m x^m(1-x), & 0 \leq x < 1, \\ l_b, & x = 1, \end{cases}$$

and

$$\varphi = \sigma + \tau. \tag{10}$$

These definitions were suggested by the formula (7). Also, we mention that the boundedness of a and b guarantee that the power series which appear in the definition of σ and τ have radii of convergence greater than or equal to 1.

Moreover, $|\sigma(x)| \leq (1-x)\sup_{m\geq 1}|a_m|$ and $|\tau(x)| \leq x \sup_{m\geq 1}|b_m|$ for every $x \in [0,1]$.

In order to estimate the degree of convergence we involve ω_f, the first modulus of smoothness corresponding to a bounded function $f : [0,1] \to \mathbb{R}$,

$$\omega_f(\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1],\ |x-y| \leq \delta\}, \quad 0 \leq \delta \leq 1.$$

Among its remarkable properties we recall that for every $\delta > 0$

$$|f(t) - f(x)| \leq (1 + \delta^{-2}(t-x)^2)\omega_f(\delta), \quad (t,x) \in [0,1] \times [0,1], \tag{11}$$

see e.g. [2; Chapter 5, §. 1].

On the other hand we need the following identity due to Derriennic [4; page 327]:

$$\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t)(t-x)^2 dt = \frac{2nx(1-x) - 6x(1-x) + 2}{(n+1)(n+2)(n+3)}. \tag{12}$$

Theorem 2: *Let M_n be defined by (5). For every $x \in [0,1]$ one has*

$$|(M_nf)(x) - f(x)(M_ne_0)(x)| \le \mu(n)(1+\lambda_n(x))\omega_f(1/\sqrt{n+2}),$$

where

$$\mu(n) := \max_{m\le n}\{a_m, b_m\} \text{ and } \lambda_n(x) := 2((n-3)x(1-x)+1)/(n+3). \tag{13}$$

Proof: By using Remark (ii), for every natural n we get

$$|(M_nh)(x)| \le (n+1)\sum_{k=0}^{n} c_{n,k}x^k(1-x)^{n-k}\int_0^1 p_{n,k}(t)|h(t)|dt \le \mu(n)(D_n|h|)(x).$$

Further on, choosing $h = f - f(x)e_0$ and knowing that M_n is linear, the relations (11) and (12) allow us to write

$$|(M_nf)(x) - f(x)(M_ne_0)(x)| \le \mu(n)(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)|f(t)-f(x)|dt$$

$$\le \mu(n)(n+1)\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{1}{n+1} + \frac{1}{\delta^2}\int_0^1 p_{n,k}(t)(t-x)^2dt\right)\omega_f(\delta)$$

$$= \mu(n)\left(1 + \frac{2}{\delta^2}\frac{(n-3)x(1-x)+1}{(n+2)(n+3)}\right)\omega_f(\delta).$$

By taking $\delta = 1/\sqrt{n+2}$ the conclusion of our theorem follows. □

Theorem 3: *Let M_n be defined by (5) such that the sequences a and b converge. For every $x \in [0,1]$ one has*

$$|(M_nf)(x) - \varphi(x)f(x)| \le \mu(n)(1+\lambda_n(x))\omega_f(1/\sqrt{n+2}) + ((1-x)^n + x^n)\nu(n)|f(x)|,$$

where $\varphi, \mu(n), \lambda_n(x)$ are defined by (10) respectively (13) and $\nu(n)$ is given by

$$\nu(n) := \sup_{j\ge n}\max\{|a_j - a_n|, |b_j - b_n|\}. \tag{14}$$

Proof: We can write

$$|(M_nf)(x) - \varphi(x)f(x)| \le |(M_nf)(x) - f(x)(M_ne_0)(x)| + |f(x)\,||\,(M_ne_0)(x) - \varphi(x)|.$$

For the first quantity we apply Theorem 2. As regards the second quantity we use both Lemma 1 and the expression of the function φ, see (10). For every $0 < x < 1$ we have

$$|(M_n e_0)(x) - \varphi(x)|$$

$$= \left| a_n(1-x)^n + b_n x^n - \sum_{m=n}^{\infty} a_m x(1-x)^m - \sum_{m=n}^{\infty} b_n x^m (1-x) \right|$$

$$= \left| (1-x)^n \sum_{k=0}^{\infty} (a_n - a_{n+k}) x (1-x)^k + x^n \sum_{k=0}^{\infty} (b_n - b_{n+k}) x^k (1-x) \right|$$

$$\leq ((1-x)^n + x^n)\nu(n).$$

If $x = 0$ or $x = 1$ it is easy to see that the previous inequality holds true.

Combining the above statements we obtain the desired result. □

By a straightforward calculation we obtain

$$\max_{x \in [0,1]} ((1-x)^n + x^n) = 1 \text{ and } \int_0^1 \lambda_n(x) dx = \frac{1}{3}$$

for every positive integer n. Also, for every $x \in [0,1]$, we deduce $\lambda_1(x) \leq 1/2$, $\lambda_2(x) \leq 2/5$ and for $n \geq 3$, $\lambda_n(x) \leq \lambda_n(1/2) = (n-1)/(n+3)$ consequently we can state $\lambda_n(x) < 1$. Taking into account these facts, Theorem 3 leads to the following

<u>Corollary:</u> *Let M_n be defined by (5) such that the sequences a and b converge.*

(i) If $f \in C([0,1])$ then $\| M_n f - \varphi f \| \leq 2\mu(n)\omega_f(1/\sqrt{n+2}) + \nu(n) \| f \|$, where $\| \cdot \|$ is the sup-norm defined by $\| h \| = \sup_{t\in[0,1]} |h(t)|$.

(ii) If $f \in L_1([0,1])$ then $\| M_n f - \varphi f \|_1 \leq \frac{4}{3}\mu(n)\omega_f(1/\sqrt{n+2}) + \nu(n) \| f \|_1$, where $\| \cdot \|_1$ is the usual norm of this space defined by $\| h \|_1 = \int_0^1 |h(t)| dt$.

Using the Remark of this section and Theorem 3 we can completely describe the convergence of $(M_n)_{n\geq 1}$.

<u>Theorem 4:</u> *Let M_n be defined by (5) and X be a normed linear space, where $X = C([0,1])$ or $X = L_1([0,1])$. The sequence $(M_n)_{n\geq 1}$ converges on X if and only if the sequences a and b converge.*

In this case, if φ denotes the function defined by (10), we have

$$\lim_{n\to\infty} M_n f = \varphi f$$

in the norm of the space X, for every $f \in X$.

In addition we point out the behaviour of $M_n f$ when f belongs to any Lebesgue space $L_p([0,1])$, $1 \le p \le \infty$, endowed with the usual norm $\|\cdot\|_{L_p([0,1])}$.

Theorem 5: *Let M_n be defined by (5) such that the sequences a and b admit an upper bound $\overline{M}$ less or equal to 1. Then $M_n f$ is a contraction in $L_p([0,1])$ for every $f \in L_p([0,1])$, where $1 \le p \le \infty$.*

Proof: At first we recall that these classes of functions are nested as follows: $C([0,1]) \subset L_\infty([0,1]) \subset L_p([0,1]) \subset L_1([0,1])$, for any $p \in (1,\infty)$.

The proof is simple and runs taking into account that it is sufficient to prove the result for $p=\infty$ and $p=1$ as we can use the Riesz-Thorin theorem to obtain the result for $1<p<\infty$ from these special cases.

In fact, $\| f \|_{L_\infty([0,1])} = ess\,\sup_{x\in[0,1]} |f(x)|$ and by using (6) we have

$$\| M_n f \|_{L_\infty([0,1])} \le \| f \|_{L_\infty([0,1])} M_n^{(\overline{M},\overline{M})} e_0 = \overline{M} \| f \|_{L_\infty([0,1])} D_n e_0 \le \| f \|_{L_\infty([0,1])},$$

since $D_n e_0 = e_0$. At the same time, for $p=1$ choosing in the proof of Theorem 1 $g=e_0$, we easily obtain

$$\| M_n f \|_{L_1([0,1])} \le \int_0^1 |f(t)|(M_n e_0)(t)dt \le \overline{M} \int_0^1 |f(t)|(D_n e_0)(t)dt \le \| f \|_{L_1([0,1])}.$$

This way, the announced result is proved. □

4. SPECIAL CASES

It has become clear that the function φ is strongly dependent on the sequences a and b. The aim of this Section is to point out the numerous possibilities regarding the structure of the function φ when we choose the mentioned sequences.

Firstly, we keep in mind the construction of φ as a polynomial of degree less or equal to $q(\varphi \in \mathcal{P}_q)$. We consider the sequences a and b such that beginning with the rank q they are definitively constant, which means $a_n = \overline{a}$ and $b_n = \overline{b}$ for every $n \ge q$. After some manipulations, the relation (10) implies

$$\varphi(x) = \overline{a}(1-x)^q + \overline{b}x^q + x(1-x)\sum_{m=1}^{q-1}(a_m(1-x)^{m-1} + b_m x^{m-1}), \quad x \in [0,1]. \qquad (15)$$

The relations (13) and (14) will have a new look, that is

$$\mu(n) = \mu(q) \text{ and } \nu(n) = 0, \ n = q, q+1, q+2, \ldots,$$

and consequently, for every $f \in L_1([0,1])$, one has

$$\| M_n f - \varphi f \|_1 \le \frac{4}{3}\mu(q)\omega_1(1/\sqrt{n+2}), \quad n = q, q+1, q+2, \ldots.$$

Conversely, it is not difficult to prove that every polynomial φ belonging to the space $\mathcal{P}_q$ can be written as in (15) from which we can obtain the corresponding sequences a and b. In this respect we make the first step indicating the constants $\overline{a}$ and $\overline{b}$; one has $\overline{a} = \varphi(0)$, $\overline{b} = \varphi(1)$.

Secondly, let's consider a and b non-decreasing sequences. Putting $d_n := a_n - a_{n-1}$, $d'_n := b_n - b_{n-1}$ for every $n \geq 1$ with the convention $a_0 = b_0 = 0$, one has $a_m = \sum_{k=1}^{m} d_k$, $b_m = \sum_{k=1}^{m} d'_k$ and from (10), after a few calculations, we obtain

$$\varphi(x) = \sum_{m=1}^{\infty} d_m(1-x)^m + \sum_{m=1}^{\infty} d'_m x^m, \quad x \in [0,1]. \tag{16}$$

By a suitable selection of $(d_m)_{m\geq 1}$ and $(d'_m)_{m\geq 1}$ we can create a function φ with exponential growth.

For example, choosing $d_m = 0$ and $d'_m = 1/m!$, $m \geq 1$, one obtains $\varphi(x) = e^x - 1$. Alternatively if $d'_m = L_m/m!$, where (L_m) is the well-known Lucas sequence, then one arrives at the exponential generating function of this sequence, more exactly $\varphi(x) = e^{\alpha x} + e^{\beta x} - 2$ with $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

In the same relation (16) we put $d_m = 0$ and $d'_m = \frac{(\alpha)_m}{(\beta)_m m!}$, $m \geq 1$, where α and β are positive fixed numbers. We recall $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$ for $k \geq 1$. This choice leads us to the *confluent hypergeometric* function

$$\varphi(x) = {}_1F_1(\alpha, \beta; x) - 1, \quad x \in [0,1]. \tag{17}$$

However, this is a convergent series for all values of x and by using Kummer's equation, we get

$$x\frac{d^2\varphi}{dx^2} + (\beta - x)\frac{d\varphi}{dx} - \alpha\varphi = \alpha.$$

We have free hands to give α and β various values obtaining in (17) functions with a great personality, as reflected in [1, § 13.6, p. 509].

Final remarks: We consider the sequence $(M_n f)_{n\geq 1}$ a fertile field of investigation. Practically, we keep in mind the following directions: the study of the iterates, asymptotic properties as Voronovskaja-type formula, some qualitative properties of the function φ studying the sequences a and b, results concerning the convergence of derivatives of $M_n f$ for a differentiable function f, and also linear combinations of our operators.

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AMS Classification Numbers: 41A10, 41A35, 26D15