FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. 19 (2004), 53–63

MASTROIANNI OPERATORS REVISITED

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Dedicated to Prof. G. Mastroianni for his 65th birthday

Abstract. The present paper focuses on a class of linear positive operators introduced by G. Mastroianni. An integral extension in Kantorovich sense is defined and approximation properties of these two sequences are established in different normed spaces.

1. Introduction

In [8] Mastroianni introduced and studied a sequence $(M_n)_{n\geq 1}$ of discrete linear positive operators to approximate unbounded functions on the interval $[0,+\infty) := \mathbb{R}_+$. Briefly, we recall this construction. Let $(\phi_n)_{n\geq 1}$ be a sequence of real valued functions defined on $\mathbb{R}_+$ which are infinitely differentiable on $\mathbb{R}_+$ and which satisfy the following conditions:

$$\phi_n(0) = 1 \text{ for every } n \in \mathbb{N}; \tag{1.1}$$

$$(-1)^k \phi_n^{(k)}(x) \geq 0 \text{ for every } n \in \mathbb{N},\ x \in \mathbb{R}_+ \text{ and } k \in \mathbb{N}\cup\{0\} := \mathbb{N}_0; \tag{1.2}$$

for each $(n,k) \in \mathbb{N}\times\mathbb{N}_0$ there exists a number $p(n,k) \in \mathbb{N}$ and a function $\alpha_{n,k} \in \mathbb{R}^{\mathbb{R}_+}$ such that

$$\phi_n^{(i+k)}(x) = (-1)^k \phi_{p(n,k)}^{(i)}(x)\alpha_{n,k}(x), \quad i \in \mathbb{N}_0,\ x \in \mathbb{R}_+, \tag{1.3}$$

and

$$\lim_{n\to+\infty} \frac{n}{p(n,k)} = \lim_{n\to+\infty} \frac{\alpha_{n,k}(x)}{n^k} = 1. \tag{1.4}$$

Received November 11, 2004.
2000 *Mathematics Subject Classification.* Primary 41A36.

We set $E_2(\mathbb{R}_+) := \{f \in C(\mathbb{R}_+) : \dfrac{f(x)}{1+x^2}$ is convergent as $x \to +\infty\}$. This space endowed with the norm $\|\cdot\|_*$, $\|f\|_* := \sup_{x\geq 0}(1+x^2)^{-1}|f(x)|$, is a Banach space. The operators M_n, $n \in \mathbb{N}$, map $E_2(\mathbb{R}_+)$ into $C(\mathbb{R}_+)$ and are given by the following formula

$$(M_n f)(x) := \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} x^k \phi_n^{(k)}(x) f\left(\frac{k}{n}\right).$$

In time, a thoroughgoing study of this class was developed, see for instance [9], [10], [5], and new properties of it have been pointed out. A synthesis of these results can be found in the monograph [2; §*5.3.11*]. At the same time, it is fair to notice that the above construction takes its origin in a paper of Baskakov [3]. Following this way, many authors constructed similar sequences of operators.

Particular cases. Mastroianni operators include some well-known classical linear positive operators such as Szász-Favard-Mirakyan and Baskakov operators.

1° Choosing $\phi_n(x) = e^{-nx}$, $p(n,k) = n$ and $\alpha_{n,k}(x) = n^k$ (constant functions on $\mathbb{R}_+$) we obtain the first class.

2° Choosing $\phi_n(x) = (1+x)^{-n}$, $p(n,k) = n+k$ and

$$\alpha_{n,k}(x) = n(n+1)\cdots(n+k-1)(1+x)^{-k}$$

the second mentioned class is obtained.

Our aim is to present new approximation properties of Mastroianni operators. Also, an integral generalization of M_n in Kantorovich sense is investigated and special cases are revealed.

2. The class (M_n^*)

First of all, we propose a slight modification of Mastroianni operators. Instead of the net $(k/n : k \in \mathbb{N}_0)$ we can use the following $(k/a_n : k \in \mathbb{N}_0)$, where $0 < a_1 < a_2 < \cdots < a_n < \cdots$ and $\lim_{n\to+\infty} a_n = +\infty$. This way, the net is more flexible than the previous one and, practically, the properties of M_n operators do not modified. Throughout the paper we use this new net, but we keep the same notation, this means M_n, for the operators. Obviously, condition (1.4) will be replaced by the following

$$\lim_{n\to+\infty} \frac{n}{p(n,k)} = \lim_{n\to+\infty} \frac{\alpha_{n,k}(x)}{a_n^k} = 1, \quad k \in \mathbb{N}_0.$$

For every $k \in \mathbb{N}_0$ and $x \geq 0$ we have

$$\alpha_{n,k}(x) \geq 0 \text{ and } \lim_{n\to+\infty} \frac{\phi_n^{(k)}(0)}{a_n^k} = (-1)^k. \tag{2.1}$$

The above inequality is obtained multiplying identity (1.3) by $(-1)^{i+k}$ and applying condition (1.2). The second statement in (2.1) can be proved by induction on $k \in \mathbb{N}_0$, manipulating relations (1.1), (1.3) and (1.4). If e_j stands for the j-th monomial, $e_j(t) = t^j$, $t \geq 0$, $j \in \mathbb{N}_0$, then an easy computation leads us to the following identities

$$M_n e_0 = e_0, \quad M_n e_1 = -\frac{\phi_n'(0)}{a_n} e_1, \quad M_n e_2 = \frac{\phi_n''(0)}{a_n^2} e_2 - \frac{\phi_n'(0)}{a_n^2} e_1. \tag{2.2}$$

By virtue of these relations, the series which appear in the definition of M_n are absolutely convergent. Also, according to the well-known Bohman-Korovkin theorem, relations (2.2) and (2.1) guarantee that $\lim\limits_{n\to+\infty} M_n f = f$ uniformly on compact subsets of $\mathbb{R}_+$ for every $f \in E_2(\mathbb{R}_+)$. Applying a classical result due to Shisha and Mond [11], we obtain the pointwise estimate $|(M_n f)(x) - f(x)| \leq 2\omega(f, \sqrt{\delta_n(x)})$, $x \geq 0$,

$$\delta_n(x) := M_n((e_1 - xe_0)^2, x) = \left(1 + 2\frac{\phi_n'(0)}{a_n} + \frac{\phi_n''(0)}{a_n^2}\right) x^2 - \frac{\phi_n'(0)}{a_n^2} x, \tag{2.3}$$

which holds for every $f \in C(\mathbb{R}_+)$.

Setting

$$\tau_{n,j} := \phi_n^{(j)}(0)/a_n^j \quad \text{and} \quad u_n := \tau_{n,2} + 2\tau_{n,1} + 1, \tag{2.4}$$

relation (2.3) implies

$$M_n((e_1 - xe_0)^2, x) \leq \max\{u_n, |\tau_{n,1}| a_n^{-1}\}(x^2 + x) := v_n(x^2 + x). \tag{2.5}$$

In what follows we modify the M_n operators into integral form operators by replacing $f(k/a_n)$ with an integral mean of $f(x)$ over an interval $I_{n,k} := [k/a_n, (k+1)/a_n]$, as follows

$$(M_n^* f)(x) := a_n \sum_{k=0}^{+\infty} m_{n,k}(x) \int_{k/a_n}^{(k+1)/a_n} f(t)dt, \quad x \geq 0, \ n \in \mathbb{N}, \tag{2.6}$$

where $m_{n,k}(x) := \dfrac{1}{k!}(-1)^k x^k \phi_n^{(k)}(x)$ and $f \in \mathcal{M}_{\text{loc}}(\mathbb{R}_+)$, the class of all measurable functions on $\mathbb{R}_+$ and bounded on every compact subinterval of

$\mathbb{R}_+$. Clearly, the operator M_n^* is a linear positive one and it can be written as a singular integral of the type $(M_n^*f)(x) = \int_0^{+\infty} K_n(x,t)f(t)dt$, with the kernel $K_n(x,t) := a_n \sum_{k\geq 0} m_{n,k}(x)\chi_{n,k}(t)$, where $\chi_{n,k}$ is the characteristic function of the interval $I_{n,k}$, $k \in \mathbb{N}_0$.

We mention that these kind of extensions are familiar to several discrete operators. For a quick information see [6, p. 115]. In [1] the first author developed a similar approach for Balázs-Szabados operators.

Denoting by $\Omega_{n,r}$ the r-th central moment of M_n^*, that is $\Omega_{n,r}(x) := M_n^*((e_1 - xe_0)^r, x)$, $r \in \mathbb{N}_0$, $x \in \mathbb{R}_+$, by a straightforward calculation we get

Lemma 2.1. *For every $n \in \mathbb{N}$, the operator M_n^* defined by (2.6) verifies*

$$M_n^*e_0 = e_0, \quad M_n^*e_1 = -\tau_{n,1}e_1 + \frac{1}{2a_n}, \quad M_n^*e_2 = \tau_{n,2}e_2 - \frac{2\tau_{n,1}}{a_n}e_1 + \frac{1}{3a_n^2},$$

$$\Omega_{n,1} = -(1+\tau_{n,1})e_1 + \frac{1}{2a_n}, \quad \Omega_{n,2} = u_n e_2 - \frac{1+2\tau_{n,1}}{a_n}e_1 + \frac{1}{3a_n^2}, \tag{2.7}$$

where $\tau_{n,j}$ and u_n are defined by (2.4).

3. Approximation Properties of M_n and M_n^*

In the first part of this section, coming back to $(M_n)_{n\geq 1}$, we establish some pointwise estimates of the rate of convergence of this approximation process. More precisely, we present the relationship between the local smoothness of f and the local approximation. For the sake of completeness, we recall that a function $f \in C(\mathbb{R}_+)$ is locally Lipα on E, $0 < \alpha \leq 1$, $E \subset \mathbb{R}_+$, if it satisfies the condition

$$|f(x) - f(y)| \leq c_f|x-y|^\alpha, \quad (x,y) \in \mathbb{R}_+ \times E, \tag{3.1}$$

where c_f is a constant depending only on α and f.

Theorem 3.1. *If f is locally* Lip α *on $E \subset \mathbb{R}_+$, $\alpha \in (0,1]$, then one has*

$$|(M_nf)(x) - f(x)| \leq c_f \left(v_n^{\alpha/2}(x^2+x)^{\alpha/2} + 2d^\alpha(x,E) \right), \quad x \geq 0,$$

where v_n is defined at (2.5) and $d(x,E)$ represents the distance between x and E.

Proof. It is clear that (3.1) holds true for any $x \in \mathbb{R}_+$ and $y \in \overline{E}$, the closure of the set E in $\mathbb{R}$. Let $(x, x_0) \in \mathbb{R}_+ \times \overline{E}$ such that $|x - x_0| = d(x, E) := \inf\{|x - y| : y \in E\}$. Since $|f - f(x)| \le |f - f(x_0)| + |f(x_0) - f(x)|$ and M_n is a linear positive operator reproducing the constants, we get

$$\begin{aligned}(3.2)\qquad |(M_n f)(x) - f(x)| &\le M_n(|f - f(x_0)|, x) + |f(x) - f(x_0)| \\ &\le M_n(c_f|e_1 - x_0|^\alpha, x) + c_f|x - x_0|^\alpha.\end{aligned}$$

Based on Hölder's inequality, one has $M_n h^\alpha \le M_n^{\alpha/2} h^2$ for every function $h \in \mathbb{R}_+^{\mathbb{R}_+}$. Consequently, for every $x \ge 0$ we deduce

$$(3.3)\qquad M_n(|e_1 - x|^\alpha, x) \le \delta_n^{\alpha/2}(x),$$

where $\delta_n(x)$ is given at (2.3). Since $|t - x_0| \le |t - x| + |x - x_0|$ and M_n is monotone, the elementary inequality

$$(a + b)^\alpha \le a^\alpha + b^\alpha, \quad a \ge 0,\ b \ge 0,\ 0 < \alpha \le 1,$$

and relation (3.3) imply

$$\begin{aligned}M_n(c_f|e_1 - x_0|^\alpha, x) &\le c_f(M_n(|e_1 - x|^\alpha, x) + |x - x_0|^\alpha) \\ &\le c_f(\delta_n^{\alpha/2}(x) + |x - x_0|^\alpha).\end{aligned}$$

Returning to (3.2) and taken into account (2.5), the conclusion follows. □

In particular for $E = \mathbb{R}_+$, if f satisfies $\omega(f, t) = O(t^\alpha)$ then a constant c_f independent of n and x exists, such that $|M_n f - f| \le c_f v_n^{\alpha/2}(e_2 + e_1)^{\alpha/2}$.

In order to increase the degree of exactness of M_n operators, we consider the following condition to be fulfilled

$$(3.4)\qquad a_n = -\phi_n'(0), \quad n \in \mathbb{N},$$

in other words $\tau_{n,1} = -1$, which guarantees that $M_n e_1 = e_1$. Taking into account (1.3), our requirement is equivalent with the relation $a_n = \alpha_{n,1}(0)$, $n \in \mathbb{N}$. Moreover, we get $\phi_n^{(k)}(0) = (-1)^k \alpha_{n,k}(0)$ for every $k \in \mathbb{N}_0$.

Following the line of Ditzian-Totik [6, § 1.2], we consider $\varphi \in \mathbb{R}^{\mathbb{R}_+}$ an admissible weight function. In order to give another estimate of the approximation error, we need to use the weighted K-functional of second order for $f \in C_B(\mathbb{R}_+)$ defined as follows

$$K_{2,\varphi}(f, t) := \inf_g \Big\{ \|f - g\| + t\|\varphi^2 g''\| : g' \in AC_{\text{loc}}(\mathbb{R}_+) \Big\}, \quad t > 0,$$

where $\|\cdot\|$ stands for the supremum norm and $g' \in AC_{\text{loc}}(\mathbb{R}_+)$ means that g is differentiable and g' is absolutely continuous on every compact of $\mathbb{R}_+$.

Theorem 3.2. *If (3.4) takes place and φ is an admissible weight function such that φ^2 is concave, then*

$$|(M_nf)(x) - f(x)| \leq 2K_{2,\varphi}\left(f, \frac{v_n x(x+1)}{2\varphi^2(x)}\right)$$

holds true for every $x > 0$, where v_n is defined by (2.5).

Proof. Let $x > 0$ be fixed and $g : \mathbb{R}_+ \to \mathbb{R}$ be twice differentiable such that $g' \in AC_{\text{loc}}(\mathbb{R}_+)$. Starting from Taylor's expansion

$$g(u) = g(x) + g'(x)(u-x) + \int_x^u g''(t)(u-t)dt, \quad u \geq 0,$$

and knowing that (3.4) holds true, in other words M_n reproduces linear functions, we have

$$(M_ng)(x) - g(x) = M_n\left(\int_{xe_0}^{e_1} g''(t)(e_1 - t)dt, x\right).$$

Since φ^2 is concave, for every $t = (1-\lambda)u + \lambda x$, $\lambda \in (0,1)$, we get

$$\varphi^2(t) \geq (1-\lambda)\varphi^2(u) + \lambda\varphi^2(x) \geq \lambda\varphi^2(x)$$

and consequently

$$\frac{|t-u|}{\varphi^2(t)} = \frac{\lambda|x-u|}{\varphi^2(t)} \leq \frac{|x-u|}{\varphi^2(x)}.$$

It turns out that

$$\begin{aligned}\left|\int_x^u g''(u)(u-t)dt\right| &\leq \|\varphi^2 g''\|\left|\int_x^u \frac{|t-u|}{\varphi^2(t)}dt\right| \\ &\leq \|\varphi^2 g''\|\left|\int_x^u \frac{|x-u|}{\varphi^2(x)}dt\right| = \|\varphi^2 g''\|\frac{(x-u)^2}{\varphi^2(x)}.\end{aligned}$$

Applying the linear positive operator M_n, we have

$$M_n\left(\int_{xe_0}^{e_1} g''(t)(e_1-t)dt, x\right) \leq \|\varphi^2 g''\|\frac{\delta_n(x)}{\varphi^2(x)},$$

and further

$$\begin{aligned}|(M_nf)(x) - f(x)| &\leq |M_n(f-g,x)| + |g(x) - f(x)| + |(M_ng)(x) - g(x)| \\ &\leq 2\|f-g\| + v_n\|\varphi^2 g''\|\frac{x^2+x}{\varphi^2(x)}.\end{aligned}$$

In the above we used that every operator M_n maps continuously $C_B(\mathbb{R}_+)$ into itself: for each $h \in C_B(\mathbb{R}_+)$ and $x \geq 0$ one has $|(M_n h)(x)| \leq \|h\|$. At this point, taking the infimum over all g with $g' \in AC_{\text{loc}}(\mathbb{R}_+)$, we get the desired result. □

It is known that $K_{2,\varphi}(f, t^2)$ and Ditzian-Totik modulus of smoothness of second order are equivalent, that is $K_{2,\varphi}(f, t^2) \sim \omega_{2,\varphi}(f, t)_\infty$. We recall

$$(3.5)\quad \omega_{2,\varphi}(f,t)_\infty := \sup_{0\leq h\leq t} \sup_{x\pm h\varphi(x)\geq 0} |f(x-\varphi(x)h) - 2f(x) + f(x+\varphi(x)h)|.$$

On the light of this fact, Theorem 3.2 implies: a constant C_φ independent of f and n exists, such that

$$|(M_n f)(x) - f(x)| \leq C_\varphi \omega_{2,\varphi}\left(f, \frac{\sqrt{v_n x(x+1)}}{\sqrt{2}\varphi(x)}\right)_\infty, \quad x > 0.$$

We notice that the construction of M_n operators requires an estimation of an infinite sum which, from computational point of view, restricts the operators usefulness. In this respect, in order to approximate a function f, it is useful to consider partial sums of $M_n f$ which have finite terms depending upon n. In other words, the operators are truncated fading away their "tails". For a fixed constant $\lambda > 0$, we consider the operators defined as follows

$$(M_n^{\langle\lambda\rangle} f)(x) = \sum_{k=0}^{[\lambda a_n]} m_{n,k}(x) f\left(\frac{k}{a_n}\right), \quad x \geq 0,\ n \in \mathbb{N},$$

where $[\alpha]$ indicates the largest integer not exceeding α.

Theorem 3.3. *The operators $M_n^{\langle\lambda\rangle}$, $n \in \mathbb{N}$, have the property*

$$\lim_{n\to+\infty} (M_n^{\langle\lambda\rangle} f)(x) = f(x) \ \text{ for all } \ f \in C([0,\lambda]),$$

uniformly on every compact $K_\lambda \subset [0, \lambda]$.

Proof. For every function $f \in C([0,\lambda])$ we introduce the function $f_\lambda \in C(\mathbb{R}_+)$ given by

$$f_\lambda(x) = \begin{cases} f(x), & 0 \leq x \leq \lambda, \\ f(\lambda), & x > \lambda. \end{cases}$$

For every $x \in [0, \lambda)$ we have

$$(M_n^{\langle\lambda\rangle} f)(x) = (M_n f_\lambda)(x) - f(\lambda) r_n(x), \quad \text{where } r_n(x) = \sum_{k=[\lambda a_n]+1}^{+\infty} m_{n,k}(x).$$

If $k/a_n > \lambda$ and $0 \le x < \lambda$, then $1 < (\lambda - x)^{-2}(k/a_n - x)^2$ holds true. Consequently we can write

$$\begin{aligned} r_n(x) &\le \frac{1}{(\lambda - x)^2} \sum_{\substack{k \\ |\frac{k}{a_n} - x| > \lambda - x}} m_{n,k}(x) \left(\frac{k}{a_n} - x\right)^2 \\ &\le \frac{1}{(\lambda - x)^2} \sum_{k=0}^{+\infty} m_{n,k}(x) \left(\frac{k}{a_n} - x\right)^2 \le \frac{x(x+1)}{(\lambda - x)^2} v_n, \end{aligned}$$

where v_n was defined by (2.5). Since relation (2.1) guarantees that $r_n(x) = o(1)$ $(n \to +\infty)$, uniformly on every compact subinterval $K_\lambda \subset [0, \lambda)$, the proof is finished. □

If M_n turns into Szász operator, see *Particular cases* (1°), then, choosing $a_n = n$ and $\lambda = 1$, the truncated operator $M_n^{\langle 1\rangle} \equiv S_{n,1}$ is given by the formula

$$(S_{n,1} f)(x) = e^{-nx} \sum_{k=0}^{[n]} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad f \in E_2(\mathbb{R}_+),\ x \ge 0.$$

In this particular case, Theorem 3.3 encounters a result obtained by Lehnhoff [7, Theorem 5].

Further on, we are going to present an approximation property for smooth functions of the operators defined by (2.6). In this respect we recall some definitions and preliminary results. The vector space

$$C_B^2(\mathbb{R}_+) := \left\{ f \in C_B(\mathbb{R}_+) :\ f' \text{ and } f'' \text{ exist and belong to } C_B(\mathbb{R}_+) \right\},$$

endowed with norm $\|\cdot\|_{C_B^2}$, $\|f\|_{C_B^2} := \sum_{j=0}^{2} \|f^{(j)}\|$, is a Banach space.

The K-functional for the couple $(C_B(\mathbb{R}_+), C_B^2(\mathbb{R}_+))$ is given by

$$K(f, t) := \inf_{g \in C_B^2(\mathbb{R}_+)} \left\{ \|f - g\| + t \|g\|_{C_B^2} \right\}, \quad f \in C_B(\mathbb{R}_+),\ t > 0.$$

For every $t > 0$, the following inequality

$$K(f, t) \le c\left\{\omega_2(f, \sqrt{t})_\infty + \min(1, t)\|f\|\right\} \tag{3.6}$$

holds true. The constant c is independent of f and $\omega_2(f, \cdot)_\infty$ is defined at (3.5) by choosing $\varphi \equiv 1$. Actually, we can describe the K-functional in terms of the moduli of smoothness in a more general frame of Besov and Sobolev spaces, see e.g. [4, Theorem 4.12].

Theorem 3.4. *Let M_n^*, $n \ge 1$, be defined by (2.6). For each $f \in C_B(\mathbb{R}_+)$ and $x \ge 0$ one has*

$$|(M_n^* f)(x) - f(x)| \le C\left\{\omega_2(f, \sqrt{\lambda_{n,x}}\,) + \min(1, \lambda_{n,x})\|f\|\right\},$$

where C is a constant independent of f and n, $\lambda_{n,x} := \frac{1}{2} w_n \max\{1, x^2\}$ and

$$w_n := |1 + \tau_{n,1}| + \frac{1}{2}|u_n| + \frac{1}{2a_n}\left(1 + 2|1 + 2\tau_{n,1}| + \frac{1}{3a_n}\right) = o(1), \tag{3.7}$$

when $n \to +\infty$, and the quantities u_n and $\tau_{n,1}$ being defined by (2.4).

Proof. For a given function $g \in C_B^2(\mathbb{R}_+)$ and $x \ge 0$, we get

$$g(t) - g(x) = (t - x)g'(x) + \frac{1}{2}(t - x)^2 g''(\xi),$$

where $\xi = \xi(t, x)$ is a point of the interval determined by t and x, $t \in \mathbb{R}_+$.

Consequently, by applying M_n^*, we obtain

$$(M_n^* g)(x) - g(x) = g'(x)\Omega_{n,1}(x) + \frac{1}{2} g''(\xi)\Omega_{n,2}(x).$$

By using (2.7) and (3.7) we can write successively

$$\begin{aligned} &|(M_n^* g)(x) - g(x)| \\ &\le \|g'\|\left(\frac{1}{2a_n} + |\tau_{n,1} + 1|x\right) + \frac{1}{2}\|g''\|\left(|u_n|x^2 + \frac{|1 + 2\tau_{n,1}|}{a_n}x + \frac{1}{3a_n^2}\right) \\ &\le w_n \|g\|_{C_B^2} \max\{1, x^2\}. \end{aligned}$$

Since $a_n^{-1} = o(1)$ $(n \to +\infty)$, based on (2.4) and (2.1), clearly $w_n = o(1)$ $(n \to +\infty)$.

Further on, for every $f \in C_B(\mathbb{R}_+)$ and $g \in C_B^2(\mathbb{R}_+)$ we have

$$\begin{aligned}&|(M_n^*f)(x) - f(x)| \\ &\quad \le |(M_n^*f)(x) - (M_n^*g)(x)| + |(M_n^*g)(x) - g(x)| + |g(x) - f(x)| \\ &\quad \le 2\|f - g\| + w_n\|g\|_{C_B^2} \max\{1, x^2\}.\end{aligned}$$

Passing to the infimum over all functions $g \in C_B^2(\mathbb{R}_+)$, we get

$$|(M_n^*f)(x) - f(x)| \le 2K\left(f, \frac{w_n}{2}\max\{1, x^2\}\right).$$

By using (3.6) the proof of the theorem is complete. □

Remark 3.1. Under the additional assumption specified at (3.4), the new look of w_n is the following

$$w_n = \frac{1}{2}(|\tau_{n,2} - 1| + 3a_n^{-1} + a_n^{-2}/3).$$

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