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Linear Operators Generated by a Probability Density Function

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Abstract. In this paper we deal with linear positive operators of both discrete and integral type. For the former we obtain the limit of iterates and for the latter we investigate the order of approximation in various spaces of functions proving that the sequence becomes an approximation process.

§1. Introduction

The trend of using probability methods in Approximation Theory has become of common usage. Following this approach, the aim of the paper split into five sections, is to introduce and investigate a general sequence of linear positive operators of integral type. The starting point is a given approximation process of discrete type, which we manipulate with the help of the density function of a certain random variable; our construction is described in Section 2. Further on, in Sections 3 and 4 we establish both pointwise and global estimates of the rate of convergence of our operators in the framework of various function spaces. For this purpose we use the modulus of smoothness, a Lipschitz-type maximal function, the Peetre functional K_2 and the Hardy-Littlewood maximal function. We estimate approximation order in L_p-spaces for smooth functions.

In order to illustrate the general class of discrete operators used in our construction, in the last section we appeal to a polynomial sequence introduced by E. W. Cheney and A. Sharma. We use this opportunity to present a new property regarding the limit of iterates of this sequence.

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§2. The Class (Λ_n)

Firstly, we present the general notation and definitions we shall use in the paper.

Let J be a real interval and $\dot{J}$ its interior. Since an affine substitution maps (a,b), $-\infty \le a < b \le \infty$, onto $(0,1)$, $\mathbb{R}_+^* = (0,\infty)$ or $\mathbb{R}$, it is enough to consider these intervals as being $\dot{J}$. Let $I_n \subset \mathbb{Z}$ be a set of indices consistent with J, this meaning $\{k/n : k \in I_n\} \subset J$. Throughout the paper we will denote by $C(J)$ the space of all real-valued continuous functions on J; $B(J)$ represents the Banach space of all real-valued bounded functions on J endowed with the sup-norm $\|\cdot\|$ defined by $\|f\| := \sup_{x\in J} |f(x)|$, $f \in B(J)$. Furthermore, we set $C_B(J) := C(J) \cap B(J)$, which is endowed with the same norm $\|\cdot\|$.
Also used in the sequel are the Lebesgue spaces $(L_p(J), \|\cdot\|_p)$, $1 \le p \le \infty$, where $\|f\|_p := \left(\int_J |f(x)|^p dt\right)^{1/p}$ for $1 \le p < \infty$, and $\|f\|_\infty := \operatorname{ess\,sup}_{x\in J} |f(x)|$ for $p = \infty$. Also, e_j stands for the j-th monomial, $e_j(t) = t^j$, $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let L_n, $n \in \mathbb{N}$, be linear operators having the form

$$(L_n f)(x) = \sum_{k\in I_n} a_{n,k}(x) f(k/n), \quad x \in J, \tag{1}$$

where $a_{n,k} \in C(J)$, $a_{n,k} \ge 0$, for every $(n,k) \in \mathbb{N} \times I_n$ and $f \in \mathcal{F} \subset \mathbb{R}^J$ such that (1) is well defined. In order for (L_n) to become an approximation process we require the following conditions to be fulfilled:

$$L_n e_0 = e_0, \quad L_n e_1 = e_1, \quad L_n e_2 = e_2 + w/u_n, \tag{2}$$

where $w \in C(J)$, $w(x) > 0$ for every $x \in \dot{J}$ and the sequence $(u_n)_{n\ge 1}$ satisfies $u_n = \mathcal{O}(n^\lambda)$ $(n \to \infty)$ with $0 < \lambda < 2$.

Actually, the above requirements imply that L_n, $n \in \mathbb{N}$, have the degree of exactness 1 and, according to the well-known Bohman-Korovkin theorem, one has $\|L_n f - f\| \to 0$ on each compact $K \subset J$, for all $f \in C(J) \cap \mathcal{F}$. More complete details in this direction can be found, for instance, in [2]. Moreover, $a_{n,k} \ge 0$ and $\sum_{k\in I_n} a_{n,k} = e_0$ guarantee that each $a_{n,k}$ belongs to $C_B(J)$.

Next, let X be a real random variable on a probability space $(\Omega, \mathcal{F}, P)$. Denoting by ψ its probability density function, we assume that $\psi \in L_2(\mathbb{R})$ and $\operatorname{supp}(\psi) \subset [-\mu, \mu] \cap J$, $\mu > 0$. A bounded compactly supported $\psi \in L^2(\mathbb{R})$ is automatically in $L_1(\mathbb{R})$. Also, one has $\psi \ge 0$ and

$$\|\psi\|_1 = \int_{\mathbb{R}} \psi(t)dt = 1. \tag{3}$$

We set $E(X) := e$, $Var(X) := \sigma^2$, the expectation and the variance of X respectively.

Starting from X we generate the random variables $X_{n,k}$ defined by

$$X_{n,k} = n^{-1}(X + k - e), \quad (n,k) \in \mathbb{N} \times I_n. \tag{4}$$

Consequently $P_{X_{n,k}}$, the distribution function of $X_{n,k}$, satisfies $dP_{X_{n,k}} = n\psi(n \cdot -k + e)$ and one has $E(X_{n,k}) = k/n$, representing exactly the mesh of the L_n operators.

Letting $\mathcal{D} := \{f \in \mathbb{R}^J \mid E(|f \circ X_{n,k}|) < \infty$ for every $(n,k) \in \mathbb{N} \times I_n\}$ and taking into account L_n defined at (1), we introduce the operators $\Lambda_n : \mathcal{D} \to C(\mathbb{R})$, $n \in \mathbb{N}$, as follows:

$$\Lambda_n f = \sum_{k \in I_n} a_{n,k} E(f \circ X_{n,k}) = \sum_{k \in I_n} a_{n,k} \int_{\Omega} f \circ X_{n,k} dP. \tag{5}$$

It is obvious that Λ_n, $n \in \mathbb{N}$, are linear positive operators and the following relations

$$(\Lambda_n f)(x) = n \sum_{k \in I_n} a_{n,k}(x) \int_{\mathbb{R}} f(t)\psi(nt - k + e)dt$$

$$= \sum_{k \in I_n} a_{n,k}(x) \int_{\mathrm{supp}(\psi)} f((t + k - e)/n)\psi(t)dt,$$

hold true for every $f \in \mathcal{D}$ and $x \in J$.

We mention that these operators are different from Feller operators [6] and other generalizations following this line. All of them were based on independence, and as a rule, identically distributed random variables. Among the extensions of Feller type we quote a general one due to Mohammad Kazim Khan [7; *Eq. (2.1)*]. Since X is non-constant, by examining (4) we deduce that for any $(k_1, k_2) \in I_n \times I_n$, $k_1 \neq k_2$, the variables X_{n,k_1}, X_{n,k_2} are not independent. All variables $X_{n,k}$, $(n,k) \in \mathbb{N} \times I_n$, represent scaled versions of the same variable X, they being obtained from it by contractions ($1/n$, $n \in \mathbb{N}$) and by translations ($(k-e)/n$, $k \in I_n$). As regards the domain $\mathcal{D}$ it is easy to see that it includes the space $L_{loc}(J)$ consisting of all real-valued functions that are locally Lebesgue integrable, i.e., integrable on every compact subset of the interval J.

As we will see in the sequel, the operators Λ_n defined at (5) have the advantage that they can be used for L_p-approximation. While these operators are of integral type, we will prove that they keep the degree of exactness 1.

§3. Estimates for Continuous Functions

At first we present some technical results gathered in the following

Lemma 1. *Let Λ_n, $n \in \mathbb{N}$, be defined by (5). Then*

(i) *the degree of exactness of Λ_n is 1;*

(ii) $\Lambda_n e_2 = e_2 + \sigma^2/n^2 + w/u_n$;

(iii) *if $f \in C_B(J)$ then $\|\Lambda_n f\| \le \|f\|$.*

Proof: For the first statement it is enough to show $\Lambda_n e_0 = e_0$ and $\Lambda_n e_1 = e_1$. These are consequences of the identities (2) satisfied by L_n.
The term $L_n e_2$ from (2) and definition (5) imply the second statement. Since

$$|(\Lambda_n f)(x)| \le \sum_{k\in I_n} a_{n,k}(x)\Big(\int_{\mathrm{supp}(\psi)} \psi(t)dt\Big)\|f\| = \|f\|, \quad f \in C_B(J),$$

we deduce that $\Lambda_n f$ is non-expansive and the last statement is proved. □

Denoting by $\Omega_s T$, $s \in \mathbb{N}_0$, the s-th order central moment of the operator T, that is $\Omega_s T(x) := T((e_1 - xe_0)^s, x)$, Lemma 1 implies

$$\Omega_0\Lambda_n = 1, \quad \Omega_1\Lambda_n = 0, \quad \Omega_2\Lambda_n = \sigma^2/n^2 + w/u_n. \tag{6}$$

We give a general estimate of the rate of convergence in terms of the modulus of smoothness ω_h associated to the function $h \in C(J)$.

Theorem 1. *Let Λ_n, $n \in \mathbb{N}$, be defined by (5). For every $f \in C(J)$,*

(i) $\lim_{n\to\infty} \Lambda_n f = f$ *uniformly on any compact $K \subset J$;*

(ii) $|(\Lambda_n f)(x) - f(x)| \le (1 + c_n(\sigma,\lambda,x))\omega_f(n^{-\lambda/2})$, $x \in J$;

(iii) *if f is differentiable on J and $f' \in C(J)$ then*

$|(\Lambda_n f)(x) - f(x)| \le (\sigma/n + \sqrt{w(x)/u_n})(1 + c_n(\sigma,\lambda,x))\omega_{f'}(n^{-\lambda/2})$, $x \in J$.
Here $c_n(\sigma,\lambda,x) := n^{\lambda/2}(\sigma/n + \sqrt{w(x)/u_n})$.

Proof: Our first assertion follows directly from Lemma 1 - (i) and (ii) - and the theorem of Bohman-Korovkin.

Since $\Lambda_n e_j = e_j$, $j \in \{0,1\}$, a general estimate for linear positive operators (see, e.g. [2; *Theorem 5.1.2*]) allows us to write

$$|(\Lambda_n f)(x) - f(x)| \le \left(1 + \frac{1}{\delta}(\Omega_2\Lambda_n)^{1/2}(x)\right)\omega_f(\delta),$$

respectively

$$|(\Lambda_n f)(x) - f(x)| \le (\Omega_2\Lambda_n)^{1/2}(x)\left(1 + \frac{1}{\delta}(\Omega_2\Lambda_n)^{1/2}(x)\right)\omega_{f'}(\delta),$$

for every $\delta > 0$ and $x \in J$. Because $u_n = \mathcal{O}(n^{\lambda})$ $(n \to \infty)$, with $0 < \lambda < 2$, in the above we choose $\delta = n^{-\lambda/2}$ and by using the inequality $\sqrt{\alpha+\beta} \le \sqrt{\alpha} + \sqrt{\beta}$ the proof of the last two statements is finished. □

As regards the quantities $c_n(\sigma, \lambda, x)$, $(n, x) \in \mathbb{N} \times J$, we observe that $\lim_{n\to\infty} c_n(\sigma, \lambda, x) = \widetilde{c}\sqrt{w(x)}$, where $\widetilde{c}$ is a constant which does not depend on σ.

Further on, we present the relationship between the local smoothness of f and the local approximation. To do this, we recall that a function $f \in C(J)$ is locally *Lip*α on E $(0 < \alpha \le 1,\ E \subset J)$ if it satisfies the condition

$$|f(x) - f(y)| \le M_f |x - y|^{\alpha}, \quad (x, y) \in J \times E, \tag{7}$$

where M_f is a constant depending only on α and f.

It is clear that (7) holds for any $x \in J$ and $y \in \overline{E}$, the closure of the set E in $\mathbb{R}$. Let $(x, x_0) \in J \times \overline{E}$ such that $|x - x_0| = d(x, E) := \inf\{|x - y| : y \in E\}$, the distance between x and E. Since $|f - f(x)| \le |f - f(x_0)| + |f(x_0) - f(x)|$ and Λ_n is a positive linear operator, we get

$$\begin{aligned} |(\Lambda_n f)(x) - f(x)| &\le \Lambda_n(|f - f(x_0)|, x) + |f(x) - f(x_0)| \\ &\le \Lambda_n(M_f |e_1 - x_0|^{\alpha}, x) + M_f |x - x_0|^{\alpha}. \end{aligned} \tag{8}$$

Knowing that $\Lambda_n h^{\alpha} \le \Lambda_n^{\alpha/2} h^2$ for any $h \ge 0$, $h \in \mathcal{D}$, (see Hölder's inequality with parameters $r := 2/\alpha$ and $s := 2/(2-\alpha)$), we deduce

$$\Lambda_n(|e_1 - x|^{\alpha}, x) \le (\Omega_2 \Lambda_n)^{\alpha/2}(x), \quad x \in J. \tag{9}$$

On the other hand, in the inequality $(a+b)^{\alpha} \le a^{\alpha} + b^{\alpha}$, $a \ge 0$, $b \ge 0$, $0 < \alpha \le 1$, putting $a = |t - x|$, $b = |x - x_0|$ and using (6), relation (9) implies

$$\begin{aligned} \Lambda_n(M_f |e_1 - x_0|^{\alpha}, x) &\le M_f(\Lambda_n(|e_1 - x|^{\alpha}, x) + |x - x_0|^{\alpha}) \\ &\le M_f((\Omega_2 \Lambda_n)^{\alpha/2}(x) + |x - x_0|^{\alpha}) \\ &\le M_f((\sigma/n)^{\alpha} + (w(x)/u_n)^{\alpha/2} + |x - x_0|^{\alpha}). \end{aligned}$$

Returning to (8) we can state

Theorem 2. *Let Λ_n, $n \in \mathbb{N}$, be defined by (5), $0 < \alpha \le 1$, and E be any subset of J. If f is locally Lipα on E then*

$$|(\Lambda_n f)(x) - f(x)| \le M_f \left(\left(\frac{\sigma}{n}\right)^{\alpha} + \left(\frac{w(x)}{u_n}\right)^{\alpha/2} + 2d^{\alpha}(x, E) \right), \quad x \in J.$$

Examining this result we deduce: in particular for $E = J$, if f satisfies $\omega_f(t) = \mathcal{O}(t^{\alpha})$ then there exists a constant M_f, independent of n and x, such that $|\Lambda_n f - f| \le M_f((\sigma/n)^{\alpha} + (w/u_n)^{\alpha/2})$.

The local behaviour of a function can be measured by the Lipschitz-type maximal function of order α introduced by B. Lenze [8] as

$$f_\alpha^\sim(x) := \sup_{\substack{t\neq x\\ t\in J}} \frac{|f(x)-f(t)|}{|x-t|^\alpha}, \quad x\in J,\ \alpha\in(0,1],$$

for every bounded $f \in L_{loc}(J)$. The finiteness of $f_\alpha^\sim$ gives a local control for the smoothness of f. Boundedness of $f_\alpha^\sim$ is, roughly speaking, equivalent to $f \in Lip\alpha$ on J.

Theorem 3. *Let Λ_n, $n \in \mathbb{N}$, be defined by (5), $\alpha \in (0,1]$ and $f \in \mathcal{D}$ be bounded. Then*

$$|(\Lambda_n f)(x) - f(x)| \le (2\mu)^{\alpha/2}\left(\frac{w(x)}{u_n} + \frac{\mu^2+3e^2}{3n^2}\right)^{\alpha/2} f_\alpha^\sim(x), \quad x\in J.$$

Proof: Since $\left|f(x) - f\left(\frac{t+k-e}{n}\right)\right| \le f_\alpha^\sim(x)\left|x - \frac{t+k-e}{n}\right|^\alpha$, with the help of Hölder's integral inequality ($r := 2/\alpha$, $s := 2/(2-\alpha)$), we can write

$$|(\Lambda_n f)(x) - f(x)| \le f_\alpha^\sim(x) \sum_{k\in I_n} a_{n,k}(x) \int_{\text{supp}(\psi)} \left|x - \frac{t+k-e}{n}\right|^\alpha \psi(t)dt \le$$

$$\sum_{k\in I_n} a_{n,k}(x)\Big(\int_{\text{supp}(\psi)} \Big(x - \frac{t+k-e}{n}\Big)^2 dt\Big)^{\frac{\alpha}{2}}\Big(\int_{\text{supp}(\psi)} \psi^{2/(2-\alpha)}(t)dt\Big)^{\frac{2-\alpha}{2}} f_\alpha^\sim(x). \tag{10}$$

Denoting the first integral by $c_{n,k}(x)$ and knowing that $\text{supp}(\psi) \subset [-\mu,\mu]$, we can proceed to write

$$\sum_{k\in I_n} a_{n,k}c_{n,k}^{\alpha/2} = \sum_{k\in I_n} a_{n,k}^{1-\alpha/2}(a_{n,k}c_{n,k})^{\alpha/2}$$

$$\le \Big(\sum_{k\in I_n} (a_{n,k}^{1-\alpha/2})^s\Big)^{1/s}\Big(\sum_{k\in I_n} (a_{n,k}c_{n,k})^{\alpha r/2}\Big)^{1/r} = \Big(\sum_{k\in I_n} a_{n,k}c_{n,k}\Big)^{\alpha/2}$$

$$\le \Big(\sum_{k\in I_n} a_{n,k}\int_{-\mu}^{\mu} (\cdot-(t+k-e)/n)^2 dt\Big)^{\alpha/2} = \left(2\mu\Omega_2 L_n + \frac{2\mu^3}{3n^2} + \frac{2\mu e^2}{n^2}\right)^{\alpha/2}.$$

Relation (2) guarantees $\Omega_2 L_n = w/u_n$. Returning to (10) and taking into account (3), our conclusion follows. □

The last quantitative estimate of this section will be given by using the Peetre functional K_2 defined as follows

$$K_2(f,t) := \inf\{\|f-g\| + t\|g''\| :\ g \in C^2(J)\cap C_B(J)\}, \quad t>0.$$

Theorem 4. *Let Λ_n, $n \in \mathbb{N}$, be defined by (5). If $f \in C_B(J)$ then*

$$|(\Lambda_n f)(x) - f(x)| \le 2K_2(f, 2^{-1}(\sigma/n + \sqrt{w(x)/u_n})), \quad x \in J.$$

Proof: Let us fix g in $C^2(J) \cap C_B(J)$ and x in J. Taylor's formula implies

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

At the first step, by using the linearity of Λ_n, the values $\Lambda_n e_j$, $j \in \{0,1\}$, the inequality $\left| \int_x^{(t+k-e)/n} |(t+k-e)/n - u| du \right| \le ((t+k-e)/n - x)^2$ and relation (3) we get successively

$$|(\Lambda_n g)(x) - g(x)| = \left| \Lambda_n \Big(\int_x^{\bullet} (\cdot - u) g''(u) du, x \Big) \right|$$

$$= \Big| \sum_{k \in I_n} a_{n,k}(x) \int_{\mathrm{supp}(\psi)} \Big(\int_x^{(t+k-e)/n} ((t+k-e)/n - u) g''(u) du \Big) \psi(t) dt \Big|$$

$$\le \sum_{k \in I_n} a_{n,k}(x) \int_{\mathrm{supp}(\psi)} \Big| \int_x^{(t+k-e)/n} |(t+k-e)/n - u| g''(u) du \Big| \psi(t) dt \quad (11)$$

$$\le \|g''\| \sum_{k \in I_n} a_{n,k}(x) \Big\{ \int_{\mathrm{supp}(\psi)} ((t+k-e)/n - x)^2 \psi(t) dt \Big\}^2 \|\psi\|_1^{1/2}$$

$$= \|g''\| \sum_{k \in I_n} a_{n,k}^{1/2}(x) \Big\{ a_{n,k}(x) \int_{\mathrm{supp}(\psi)} ((t+k-e)/n - x)^2 \psi(t) dt \Big\}^{1/2}$$

$$\le \|g''\| \Big(\sum_{k \in I_n} a_{n,k}(x) \Big)^{1/2} \Big(\sum_{k \in I_n} a_{n,k}(x) \int_{\mathrm{supp}(\psi)} ((t+k-e)/n - x)^2 \psi(t) dt \Big)^{1/2}$$

$$= \|g''\| (\Omega_2 \Lambda_n)^{1/2}(x) \le \|g''\| (\sigma/n + \sqrt{w(x)/u_n}). \quad (12)$$

At the second step, by using (12) for an arbitrary $f \in C_B(J)$ we have

$$\begin{aligned} |(\Lambda_n f)(x) - f(x)| &= |\Lambda_n(f - g, x) + g(x) - f(x) + (\Lambda_n g)(x) - g(x)| \\ &\le \|\Lambda_n(f-g)\| + \|g - f\| + \|g''\| (\sigma/n + \sqrt{w(x)/u_n}). \end{aligned}$$

We use Lemma 1 and taking the infimum over $g \in C^2(J) \cap C_B(J)$ we obtain the claimed result. □

It is known [5] that Peetre functional K_2 is equivalent to the regular modulus of smoothness ω_2, in other words there exist some constants $\beta > 0$ and $t_0 > 0$ such that

$$\beta^{-1}\omega_2(f,t) \le K_2(f,t^2) \le \beta\omega_2(f,t), \quad f \in C_B(J),\ 0 < t \le t_0,$$

where $\omega_2(f,t) := \sup_{0<h\le t} \|\Delta_h^2 f\|$. Here $\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h)$ for $x \pm h \in J$ and vanishes otherwise.

In the light of this equivalence, Theorem 4 implies the following. If $\omega_2(f,t) = \mathcal{O}(t^\alpha)$, $0 < \alpha < 2$, then a certain constant $\widetilde{\beta}$ exists such that

$$|(\Lambda_n f)(x) - f(x)| \le \widetilde{\beta}(\sigma/(2n) + \sqrt{w(x)/(4u_n)})^{\alpha/2}.$$

§4. Estimates in L_p Spaces

We will focus on the case $J = [0,\infty)$ as it exhibits the problems caused by a finite endpoint and by the unboundedness of the interval. In the sequel AC_{loc}^+ denotes the space of all real-valued functions that are absolutely continuous in every closed bounded and positive interval. Regarding the function w that appeared in (2), we suppose additional hypotheses to be fulfilled. More precisely, we impose

$$w(x) = x^\tau \overline{w}(x), \quad x \in J = [0,\infty),\ 0 < \tau \le 1, \tag{13}$$

where $\overline{w} \in C(J)$ satisfies $0 < a := \inf_{x\in J} \overline{w}(x)$ and $\sup_{x\in J} \overline{w}(x) := b < \infty$.
We need the Hardy-Littlewood maximal function $\mathcal{M}g$ of $g \in L_{loc}(J)$,

$$(\mathcal{M}g)(x) := \sup_{t\ne x} \left| \frac{1}{t-x} \int_x^t |g(u)|du \right|, \quad x \in J. \tag{14}$$

Let $1 < p \le \infty$ and suppose g belongs to $L_p(J)$. Then $\mathcal{M}g$ belongs to $L_p(J)$ and a classical result due to Hardy-Littlewood says that

$$\|\mathcal{M}g\|_p \le \gamma_p \|g\|_p \tag{15}$$

where γ_p is a constant depending only on p. Obviously $\gamma_\infty = 1$. For more details, [3; *Chapter 3*, §3] can be consulted.

Lemma 2. *Let Λ_n, $n \in \mathbb{N}$, be given by (5) such that (13) is fulfilled.*

(i) *If f' exists and f, f' belong to AC_{loc}^+ then, for $0 \le x \le n^{-\lambda}$,*

$$|(\Lambda_n f)(x) - f(x)| \le c_1(n)(\mathcal{M}f')(x),$$

where $c_1(n) := \sigma/n + (\max_{x\in[0,1]} \sqrt{w(x)})/\sqrt{u_n}$.

(ii) *If f'' exists and f, wf'' belong to AC^+_{loc} then, for $x > n^{-\lambda}$,*

$$|(\Lambda_n f)(x) - f(x)| \le c_2(n)(\mathcal{M}wf'')(x),$$

where $c_2(n) := a^{-1}(\sigma^2/n^{2-\tau\lambda} + b/u_n)$.

Proof: (i) Since $f((t+k-e)/n) - f(x) = \int_x^{(t+k-e)/n} f'(u)du$ and using both (14) and (9) (with $\alpha = 1$) we obtain

$$|(\Lambda_n f)(x) - f(x)| \le \sum_{k\in I_n} a_{n,k}(x) \int_{\text{supp}(\psi)} \left|\frac{t+k-e}{n} - x\right| \psi(t)dt(\mathcal{M}f')(x)$$
$$= \Lambda_n(|e_1 - x|, x)(\mathcal{M}f')(x) \le \sqrt{\frac{\sigma^2}{n^2} + \frac{w(x)}{u_n}}(\mathcal{M}f')(x).$$

Since $0 \le x \le n^{-\lambda} \le 1$, the first conclusion follows.

(ii) It is easy to prove that $x^\tau|v-u| \le u^\tau|x-v|$ for any $\tau \in (0,1]$, where u lies between x and v, $x \ge 0$, $v \ge 0$. Using this inequality and relation (13), we have

$$\frac{|v-u|}{w(u)} \le \frac{|x-v|}{x^\tau \overline{w}(u)} \le \frac{|x-v|}{ax^\tau}.$$

Choosing, in the above, $v := (t+k-e)/n$ we can write

$$|f''(u)||(t+k-e)/n - u| \le \frac{|w(u)f''(u)|}{|x-(t+k-e)/n|} \frac{(x-(t+k-e)/n)^2}{ax^\tau}.$$

Now we rewrite (11) for f and taking into account (14) and (6) we get

$$|(\Lambda_n f)(x) - f(x)| \le (\mathcal{M}wf'')(x) \sum_{k\in I_n} a_{n,k} \int_{\text{supp}(\psi)} \frac{(x-(t+k-e)/n)^2}{ax^\tau}\psi(t)dt$$
$$= \frac{(\Omega_2\Lambda_n)(x)}{ax^\tau}(\mathcal{M}wf'')(x) = \frac{1}{a}\left(\frac{\sigma^2}{x^\tau n^2} + \frac{\overline{w}(x)}{u_n}\right)(\mathcal{M}wf'')(x).$$

Since $x > n^{-\lambda}$, the second conclusion follows. □

For smooth functions in L_p-spaces, the following property holds.

Theorem 5. *Let $1 < p \le \infty$. Let Λ_n, $n \in \mathbb{N}$, be defined by (5) such that (13) is fulfilled. If f', f'' exist and f, f', wf'' belong to $AC_{loc}^{+} \cap L_p(J)$, then the following inequality*

$$\|\Lambda_n f - f\|_p \le \widetilde{c}_p(n)(\|f'\|_p + \|wf''\|_p), \tag{16}$$

holds, where $\widetilde{c}_p(n)$ is a constant depending on n and p with the property $\lim_{n\to\infty} \widetilde{c}_p(n) = 0$.

Proof: Combining both cases of Lemma 2, for any $x \ge 0$ we get

$$|(\Lambda_n f)(x) - f(x)| \le c_3(n)((\mathcal{M}f')(x) + (\mathcal{M}wf'')(x)),$$

where $c_3(n)$ can be chosen to be $c_1(n)+c_2(n)$. Examining these constants, clearly we have

$$\lim_{n\to\infty} c_3(n) = 0. \tag{17}$$

For $1 < p \le \infty$, the preceding inequality and Minkowski's inequality imply

$$\|\Lambda_n f - f\|_p \le c_3(n)(\|\mathcal{M}f'\|_p + \|\mathcal{M}wf''\|_p).$$

By virtue of (15) we find out $\|\mathcal{M}f'\|_p \le \gamma_p'\|f'\|_p$, $\|\mathcal{M}wf''\|_p \le \gamma_p''\|wf''\|_p$ and we arrive at (16) with $\widetilde{c}_p(n) = c_3(n)\max\{\gamma_p', \gamma_p''\}$. Relation (17) completes the proof of our assertion. □

In view of the proofs of this section, we conclude that the case J bounded, more exactly $J = [0,1]$, implies that the number $\|w\|_{C(J)}$ exists. Instead of (16), the first part of Lemma 2 leads us to the following relation

$$\|\Lambda_n f - f\|_p \le c_p(n)\|f'\|_p, \text{ if } f' \text{ exists and } f, f' \text{ belong to } \mathcal{D} \cap L_p(J).$$

§5. Cheney-Sharma Operators Revisited

Among numerous examples of discrete operators satisfying conditions (1) we decided upon the following classical sequence which was enriched in time with new properties. Based on a combinatorial identity of Abel-Jensen, E. W. Cheney and A. Sharma [4] have investigated the operators

$$(Q_n f)(x) = \sum_{k=0}^{n} q_{n,k}(x;\beta) f\left(\frac{k}{n}\right), \quad f \in C([0,1]),\ x \in [0,1],\ n \in \mathbb{N},$$

where

$$q_{n,k}(x;\beta) = (1+n\beta)^{1-n}\binom{n}{k} x(x+k\beta)^{k-1}(1-x)[1-x+(n-k)\beta]^{n-1-k},$$

and β is a non-negative parameter.

The authors proved that $(Q_n)_n$ is an approximation process preserving the constant functions. In [10] was shown that Q_n reproduces the linear functions, thus Q_n has the degree of exactness 1 and (1) is fulfilled. Setting $Q_n^1 = Q_n$, $Q_n^{m+1} = Q_n^m \circ Q_n$, $n \in \mathbb{N}$, we bring to light a new property, proving that these iterates satisfy the following limiting relation

$$\lim_{m\to\infty} (Q_n^m f)(x) = f(0) + (f(1) - f(0))x, \quad f \in C([0,1]), \tag{18}$$

uniformly on $[0,1]$, for any $\beta \geq 0$.

Taking in view the approach presented in [1], the proof runs as follows.

Defining $S_{a,b} = \{f \in C([0,1]) : \ f(0) = a, \ f(1) = b\}$, $(a,b) \in \mathbb{R} \times \mathbb{R}$, every $S_{a,b}$ is a closed subset of $C([0,1])$ and the system $(S_{a,b})_{a,b}$ makes up a partition of this space. It is easy to see that each $Q_n f$ interpolates the function f in 0 and 1. Consequently, for all $(a,b) \in \mathbb{R} \times \mathbb{R}$, $S_{a,b}$ is an invariant set of Q_n. On the other hand, $Q_n|_{S_{a,b}} : S_{a,b} \to S_{a,b}$ is a contraction for every $(a,b) \in \mathbb{R} \times \mathbb{R}$ and $n \in \mathbb{N}$. Indeed, if f_1 and f_2 belong to $S_{a,b}$ then we get

$$\begin{aligned} |(Q_n f_1)(x) - (Q_n f_2)(x)| &\leq (1 - q_{n,0}(x;\beta) - q_{n,n}(x;\beta)) \sup_{x\in[0,1]} |f_1(x) - f_2(x)| \\ &\leq (1 - 2^{1-n}(1+n\beta)^{1-n}) \|f_1 - f_2\|. \end{aligned}$$

At this moment we introduce $p_{a,b}$, $p_{a,b}(x) = a + (b-a)x$, $x \in [0,1]$. One has $p_{a,b} \in S_{a,b}$. Since Q_n reproduces the affine functions, $p_{a,b}$ is a fixed point of Q_n. For any $f \in C([0,1])$ one has $f \in S_{f(0),f(1)}$. By using the characterization of weakly Picard operators due to I. A. Rus [9] and the contraction principle, we arrived at (18).

In the terminology of [9] this means that the Cheney-Sharma operator is a weakly Picard operator.

References

1. Agratini, O., On some Bernstein type operators: iterates and generalizations, East Journal on Approximations, 9(2003), 4, 415-426.
2. Altomare, F. and M. Campiti, *Korovkin-type Approximation Theory and its Applications*, de Gruyter Series Studies in Mathematics, Vol. 17, Walter de Gruyter & Co., Berlin, New York, 1994.
3. Bennett, C. and R. Sharpley, *Interpolation of Operators*, Vol. 129, Series: Pure and Applied Mathematics, Academic Press, Inc., 1988.
4. Cheney, E. W. and A. Sharma, On a generalization of Bernstein polynomials, Riv. Mat. Univ. Parma (2), 5(1964), 77–84.

5. Ditzian, Z. and V. Totik, *Moduli of Smoothness*, Springer Series in Computational Mathematics, Vol. 9, Springer-Verlag, New York Inc., 1987.
6. Feller, W., *An Introduction to Probability Theory and its Applications II*, Wiley, New York, 1966.
7. Khan, M. K., On the rate of convergence of Bernstein power series for functions of bounded variation, J. Approx. Theory, **57**(1989), 90–103.
8. Lenze, B., Bernstein-Baskakov-Kantorovič operators and Lipschitz-type maximal functions, in *Colloquia Mathematica Societatis János Bolyai 58, Approximation Theory*, Kecskemét, Hungary, J. Szabados and K. Tandori (eds.), North-Holland Publishing Company, Amsterdam-Oxford-New York, 1990, 469–496.
9. Rus, I. A., Weakly Picard mappings, Commentationes Math. Univ. Carolinae, **34**(1993), 4, 769–773.
10. Stancu, D. D. and C. Cismaşiu, On an approximating linear positive operators of Cheney-Sharma, Revue d'Analyse Numérique et de Théorie de l'Approximation, **26**(1997), Nos. 1-2, 221–227.

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