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Compression–expansion fixed point theorem in two norms and applications

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Abstract

In this paper we present a two-norms version of Krasnoselskii's fixed point theorem in cones. The abstract result is then applied to prove the existence of positive L^p solutions of Hammerstein integral equations with better integrability properties on the kernels.

Keywords: Positive solution; Integral equation; Fixed point theorem in cones

1. Introduction

Krasnoselskii's fixed point theorem in cones is one of the most useful principles in proving existence, localization and multiplicity results for various nonlinear problems (see [1–3,5–7]). For example, in [5], Krasnoselskii's theorem is used to establish existence results for positive L^p solutions of Hammerstein integral equations. The aim of this paper is to express the compression condition and the expansion condition in Krasnoselskii's theorem with respect to different norms. This allows us to refine the conditions from [5] for

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Hammerstein integral equations with better integrability properties on the kernels and, in consequence, to obtain extensions of the results in [5].

Following Granas and Dugundji [4, p. 120], we recall some notions and results which will be used throughout the next sections.

Let $(E, |.|)$ be a normed linear space and $C \subset E$ a convex set. By a *pair* (X, A) in C, we mean an arbitrary subset X of C and a nonempty $A \subset X$ closed in X. We denote by $\mathcal{K}_A(X, C)$ the set of all compact maps $F : X \to C$ with $F(x) \neq x$ for all $x \in A$. A map $F \in \mathcal{K}_A(X, C)$ is called *essential* provided every $G \in \mathcal{K}_A(X, C)$ such that $F|_A = G|_A$ has a fixed point. A map that is not essential is called *inessential*. Two maps $F, G \in \mathcal{K}_A(X, C)$ are called *homotopic*, written $F \sim G$ in $\mathcal{K}_A(X, C)$, provided that there is a compact map $H : X \times [0, 1] \to C$ such that $H(., 0) = F$, $H(., 1) = G$ and $H(x, \lambda) \neq x$ for all $x \in A$ and $\lambda \in [0, 1]$.

Theorem 1.1 (Topological transversality). *Let (X, A) be a pair in a convex $C \subset E$, and let F, G be maps in $\mathcal{K}_A(X, C)$ such that $F \sim G$ in $\mathcal{K}_A(X, C)$. Then F is essential if and only if G is essential.*

As an example of essential and inessential maps, we have

Theorem 1.2. *Let U be an open subset of a convex set $C \subset E$, $U \neq C$, let $(\bar{U}, \partial U)$ be the pair consisting of the closure of U in C and the boundary of U in C, and let $x_0 \in C$. Then the constant map $F(x) = x_0$ $(x \in \bar{U})$ is essential in $\mathcal{K}_{\partial U}(\bar{U}, C)$ if $x_0 \in U$ and inessential in $\mathcal{K}_{\partial U}(\bar{U}, C)$ if $x_0 \in C \setminus \bar{U}$.*

We conclude the introduction by stating Krasnoselskii's theorem, in the form given in Granas and Dugundji [4, p. 325].

Theorem 1.3 (Krasnoselskii). *Let $(E, |.|)$ be a normed linear space, $C \subset E$ a proper wedge and $N : C \to C$ a completely continuous map. Assume that for some numbers ρ and R with $0 < \rho < R$, one of the following conditions is satisfied*:

(a) $|N(x)| \leqslant |x|$ *for* $|x| = \rho$ *and* $|N(x)| \geqslant |x|$ *for* $|x| = R$,
(b) $|N(x)| \geqslant |x|$ *for* $|x| = \rho$ *and* $|N(x)| \leqslant |x|$ *for* $|x| = R$.

Then N has a fixed point x with $\rho \leqslant |x| \leqslant R$.

In this paper, a continuous map $N : X \to Y$, where X, Y are topological spaces, is said to be *compact* if $N(X)$ is contained in a compact subset of Y. If X is a metric space, then N is said to be *completely continuous* if the image of each bounded set in X is contained in a compact subset of Y.

2. Krasnoselskii's theorem in two norms

Throughout this section $(E, |.|)$ will be a normed linear space and $\|.\|$ will be another norm on E. Also $C \subset E$ will be a nonempty convex (not necessarily closed) set with $0 \notin C$ and $\lambda C \subset C$ for all $\lambda > 0$. We shall assume that there exist constants $c_1, c_2 > 0$ such that

$$c_1|x| \leqslant \|x\| \leqslant c_2|x| \quad \text{for all } x \in C. \tag{2.1}$$

Hence the norms $|.|$ and $\|.\|$ are topologically equivalent on C (but not necessarily on E).

For two numbers ρ, R with $0 < c_2\rho < R$ we shall denote

$$B_\rho = \{x \in C\colon\ |x| \leqslant \rho\}, \qquad S_\rho = \{x \in C\colon\ |x| = \rho\},$$
$$D_R = \{x \in C\colon\ \|x\| \leqslant R\}, \qquad \Sigma_R = \{x \in C\colon\ \|x\| = R\},$$
$$C_{\rho,R} = \{x \in C\colon\ \rho \leqslant |x|,\ \|x\| \leqslant R\}.$$

Obviously, since $c_2\rho < R$, one has $B_\rho \subset D_R$. Also $C_{\rho,R} = (D_R \setminus B_\rho) \cup S_\rho$ and (B_ρ, S_ρ), (D_R, Σ_R) are pairs in C.

Theorem 2.1. *Assume $0 < c_2\rho < R$, the map $N : D_R \to C$ is compact and $N(x) \neq x$ for all $x \in S_\rho \cup \Sigma_R$. In addition, assume that the following conditions are satisfied*:

(i) *$N|_{B_\rho}$ is essential in $\mathcal{K}_{S_\rho}(B_\rho, C)$,*
(ii) *N is inessential in $\mathcal{K}_{\Sigma_R}(D_R, C)$.*

Then N has at least two fixed points $x_1, x_2 \in C$ with $|x_1| < \rho < |x_2|$ and $\|x_2\| < R$.

Proof. We have from (i) that N has a fixed point $x_1 \in B_\rho$. Since $N(x) \neq x$ in S_ρ, one has $|x_1| < \rho$. Thus it remains to show that N has fixed points in $C_{\rho,R}$. Assume the contrary, i.e., $N(x) \neq x$ in $C_{\rho,R}$. Now N being inessential in $\mathcal{K}_{\Sigma_R}(D_R, C)$, there exists a map $G \in \mathcal{K}_{\Sigma_R}(D_R, C)$ with $G|_{\Sigma_R} = N|_{\Sigma_R}$ and $G(x) \neq x$ for all $x \in D_R$.

We shall define a map $H : B_\rho \to C$ with the following properties: $H \in \mathcal{K}_{S_\rho}(B_\rho, C)$, $H|_{S_\rho} = N|_{S_\rho}$ and $H(x) \neq x$ for all $x \in B_\rho$. This shows that N is inessential in $\mathcal{K}_{S_\rho}(B_\rho, C)$ contrary to (i) and finishes the proof. Recall $0 \notin C$. Let $H : B_\rho \to C$ be given by

$$H(x) = \begin{cases} a^2 G\left(\frac{1}{a^2}x\right), & 0 < |x| \leqslant a\rho, \\ \frac{|x|^2}{\rho^2} N\left(\frac{\rho^2}{|x|^2}x\right), & a\rho \leqslant |x| \leqslant \rho. \end{cases}$$

Here $a = a(x) := \frac{\rho\|x\|}{R|x|}$. From (2.1) we have $\frac{\rho c_1}{R} \leqslant a(x) \leqslant \frac{\rho c_2}{R}$ for every $x \in B_\rho$. We note that if $0 < |x| \leqslant a\rho$, then

$$\left\| \frac{1}{a^2}x \right\| = \frac{1}{a^2}\|x\| = \frac{1}{a^2}\frac{aR|x|}{\rho} = \frac{R|x|}{a\rho} \leqslant R.$$

Hence $\frac{1}{a^2}x \in D_R$. Also, for $a\rho \leqslant |x| \leqslant \rho$, we have

$$\left| \frac{\rho^2}{|x|^2}x \right| = \frac{\rho^2}{|x|} \geqslant \frac{\rho^2}{\rho} = \rho$$

and

$$\left\| \frac{\rho^2}{|x|^2} x \right\| = \frac{\rho^2}{|x|^2} \|x\| = \frac{\rho^2 a R |x|}{|x|^2 \rho} = \frac{\rho a R}{|x|} \leqslant \frac{\rho a R}{a\rho} = R.$$

Thus $\frac{\rho^2}{|x|^2} x \in C_{\rho,R}$. Therefore H is well defined and has no fixed points. Also if $|x| = a\rho$, then

$$a^2 = \frac{|x|^2}{\rho^2} \quad \text{and} \quad \left\| \frac{1}{a^2} x \right\| = R.$$

This together with $G|_{\Sigma_R} = N|_{\Sigma_R}$ guarantees that H is continuous. Now the compactness of N and G implies that H is a compact map. Therefore $H \in \mathcal{K}_{S_\rho}(B_\rho, C)$. □

Theorem 2.2. *Assume* $0 < c_2\rho < R$, *the map* $N : D_R \to C$ *is compact and that the following conditions are satisfied*:

(a) $|N(x)| < |x|$ *for all* $x \in S_\rho$,
(b) *there exists* $e \in C$ *with* $x \neq N(x) + \delta e$ *for any* $\delta > 0$ *and* $x \in \Sigma_R$,
(c) *there exists* $\delta_0 > \frac{R}{\|e\|}$ *with* $x \neq \lambda N(x) + \delta_0 e$ *for all* $\lambda \in (0,1)$ *and* $x \in \Sigma_R$.

Then N *has at least two fixed points* $x_1, x_2 \in C$ *with* $|x_1| < \rho \leqslant |x_2|$ *and* $\|x_2\| \leqslant R$.

Proof. Assume $N(x) \neq x$ for all $x \in \Sigma_R$ (otherwise we are finished). We shall prove that conditions (i) and (ii) in Theorem 2.1 hold. Let $x_0 \in C$ with $|x_0| = \rho$ be fixed. We claim that (a) implies the existence of a $\eta \in (0,1)$ such that

$$x \neq (1-\lambda)\eta x_0 + \lambda N(x) \quad \text{for all } x \in S_\rho,\ \lambda \in (0,1). \tag{2.2}$$

Otherwise, for each $n \in \mathbf{N}$, $n \geqslant 1$, there are $\lambda_n \in (0,1)$ and $x_n \in S_\rho$ with

$$x_n = (1-\lambda_n)\frac{1}{n} x_0 + \lambda_n N(x_n). \tag{2.3}$$

Since N is compact, we may assume (passing eventually to a subsequence) that $N(x_n)$ is convergent. Also we may suppose $\lambda_n \to \bar{\lambda}$ for some $\bar{\lambda} \in [0,1]$. Then (2.3) implies $x_n \to \bar{x}$. Clearly $\bar{x} \in S_\rho$ and $\bar{x} = \bar{\lambda} N(\bar{x})$. We have $\bar{\lambda} < 1$ by the assumption $N(x) \neq x$ on S_ρ. Then (a) implies $|\bar{x}| = \bar{\lambda}|N(\bar{x})| \leqslant \bar{\lambda}|\bar{x}|$ and so $1 \leqslant \bar{\lambda}$, a contradiction. Hence (2.2) is true for some $\eta \in (0,1)$. This shows that the constant map ηx_0 and N are homotopic in $\mathcal{K}_{S_\rho}(B_\rho, C)$. Since $0 < |\eta x_0| < \rho$, from Theorem 1.2 we have that ηx_0 is essential in $\mathcal{K}_{S_\rho}(B_\rho, C)$. Now Theorem 1.1 guarantees that N is essential in $\mathcal{K}_{S_\rho}(B_\rho, C)$. Thus (i) holds.

Consider the homotopy $H : D_R \times [0,1] \to C$ defined by $H(x,\lambda) = N(x) + \lambda\delta_0 e$. Notice H is a compact map, $H(.,0) = N$, $H(.,1) = N + \delta_0 e$ and, from (b), $H(x,\lambda) \neq x$ for all $x \in \Sigma_R, \lambda \in (0,1]$. Hence $N \sim N + \delta_0 e$ in $\mathcal{K}_{\Sigma_R}(D_R, C)$. Also, from (c) we immediately see that $N + \delta_0 e \sim \delta_0 e$ in $\mathcal{K}_{\Sigma_R}(D_R, C)$. Consequently, $N \sim \delta_0 e$ in $\mathcal{K}_{\Sigma_R}(D_R, C)$. Since $\|\delta_0 e\| > R$, we have from Theorem 1.2, that $\delta_0 e$ is inessential in $\mathcal{K}_{\Sigma_R}(D_R, C)$. Thus N is inessential in $\mathcal{K}_{\Sigma_R}(D_R, C)$ too. Therefore (ii) holds. □

Now we are ready to state the new version of Krasnoselskii's theorem, in terms of two norms.

Theorem 2.3. *Assume $0 < c_2\rho < R$, $\|.\|$ is increasing with respect to C, that is $\|x+y\| > \|x\|$ for all $x, y \in C$, and the map $N : D_R \to C$ is compact. In addition, assume that the following conditions are satisfied*:

(h1) $|N(x)| < |x|$ *for all* $x \in C$ *with* $|x| = \rho$,
(h2) $\|N(x)\| \geqslant \|x\|$ *for all* $x \in C$ *with* $\|x\| = R$.

Then N has at least two fixed points $x_1, x_2 \in C$ with $|x_1| < \rho \leqslant |x_2|$ and $\|x_2\| \leqslant R$.

Proof. We shall prove that conditions (b) and (c) in Theorem 2.2 are satisfied with any $e \in C$ and $\delta_0 > \frac{R}{\|e\|}$. Indeed, if (b) does not hold for a given $e \in C$, then there are $x \in \Sigma_R$ and $\delta > 0$ with $x = N(x) + \delta e$. Then we obtain

$$R = \|x\| = \left\|N(x) + \delta e\right\| > \left\|N(x)\right\| \geqslant \|x\|,$$

a contradiction.

If (c) does not hold for a fixed $e \in C$ and a given $\delta_0 > \frac{R}{\|e\|}$, then there are $x \in \Sigma_R$ and $\lambda \in (0, 1)$ such that $x = \lambda N(x) + \delta_0 e$. Then we obtain

$$R = \|x\| = \left\|\lambda N(x) + \delta_0 e\right\| > \|\delta_0 e\| = \delta_0 \|e\|,$$

a contradiction.

Thus all the assumptions of Theorem 2.2 hold. □

Obviously, the following dual proposition is also true.

Theorem 2.4. *Assume $0 < \frac{1}{c_1}\rho < R$, $|.|$ is increasing with respect to C, and the map $N : \{x \in C\colon |x| \leqslant R\} \to C$ is compact. In addition, assume that the following conditions are satisfied*:

(h1) $\|N(x)\| < \|x\|$ *for all* $x \in C$ *with* $\|x\| = \rho$,
(h2) $|N(x)| \geqslant |x|$ *for all* $x \in C$ *with* $|x| = R$.

Then N has at least two fixed points $x_1, x_2 \in C$ with $\|x_1\| < \rho \leqslant \|x_2\|$ and $|x_2| \leqslant R$.

3. Application

Consider the nonlinear integral equation

$$u(t) = \int_0^1 k(t,s) f\bigl(s, u(s)\bigr)\, ds \quad \text{for a.e. } t \in [0,1]. \tag{3.1}$$

We seek positive solutions $u \in L^p[0,1]$ where $1 \leqslant p < \infty$. Here by a positive solution we mean $u(t) > 0$ for a.e. $t \in [0,1]$.

We shall assume that the following conditions are satisfied:

(A) $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function (i.e., the map $t \mapsto f(t,y)$ is measurable for all $y \in \mathbf{R}$ and the map $y \mapsto f(t,y)$ is continuous for a.e. $t \in [0,1]$) and there exists $p_2 \in [1,\infty)$; $a_0, a_1 \in L^{p_2}([0,1]; \mathbf{R}_+)$ with $a_0(t) > 0$ on a set of positive measure, and $a_2 > 0$ such that

$$\begin{aligned} &\left|f(t,y)\right| \leqslant a_1(t) + a_2|y|^{\frac{p}{p_2}} \quad \text{for all } y \in \mathbf{R},\ \text{a.e. } t \in [0,1],\\ &f(t,y) \geqslant a_0(t) \quad \text{for all } y \in \mathbf{R}_+,\ \text{a.e. } t \in [0,1], \end{aligned} \tag{3.2}$$

and

$$f(t,y) \text{ is nondecreasing in } y \text{ on } (0,\infty) \text{ for a.e. } t \in [0,1].$$

(B) $k : [0,1] \times [0,1] \to \mathbf{R}$ is measurable, there exists $0 < M \leqslant 1$, $k_1 \in L^p[0,1]$, $q \in [1,p]$ with $q \geqslant \frac{p}{p_2}$ and $k_2 \in L^{\frac{p_2 q}{p_2 q - p}}[0,1]$ such that $0 < k_1(t), k_2(t)$ for a.e. $t \in [0,1]$ and

$$Mk_1(t)k_2(s) \leqslant k(t,s) \leqslant k_1(t)k_2(s)$$

for a.e. $t \in [0,1]$, a.e. $s \in [0,1]$.

Notice since $\frac{p}{p_2} \leqslant q \leqslant p$, one has $\bar{p}_1 := \frac{p_2 q}{p_2 q - p} \geqslant p_1$, where $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then $L^{\frac{p_2 q}{p_2 q - p}}[0,1] \subset L^{p_1}[0,1]$ and consequently $k_2 \in L^{p_1}[0,1]$, which is assumed in [5]. Thus, to be a "better integrability property" (with respect to the $\frac{p}{p_2}$-growth of f) of kernel k means that $k_2 \in L^{\bar{p}_1}[0,1]$ for some $\bar{p}_1 \geqslant p_1$.

We shall use $|u|_\gamma$ to denote the norm on $L^\gamma[0,1]$ with

$$|u|_\gamma := \left(\int_0^1 \left|u(t)\right|^\gamma dt\right)^{\frac{1}{\gamma}} \quad \text{if } 1 \leqslant \gamma < \infty$$

and

$$|u|_\infty := \operatorname{ess}\sup_{t\in[0,1]} \left|u(t)\right| \quad \text{if } \gamma = \infty.$$

Let $\bar{p}_2$ be such that $\frac{1}{\bar{p}_1} + \frac{1}{\bar{p}_2} = 1$ and denote

$$\begin{aligned} &\psi(t) := |a_1|_{\bar{p}_2} + a_2 t^{\frac{p}{p_2}} \quad (t \geqslant 0),\\ &b_\gamma(t) := M\frac{k_1(t)}{|k_1|_\gamma} \quad \text{a.e. } t \in [0,1]\ (1 \leqslant \gamma \leqslant p). \end{aligned}$$

Theorem 3.1. *Assume* (A) *and* (B). *In addition assume that there exists ρ, R with $0 < \rho < R$ and $r \in [1,q]$ such that*:

$$|k_1|_q |k_2|_{\bar{p}_1} \psi(\rho) < \rho, \tag{3.3}$$

$$M|k_1|_r \int_0^1 k_2(s) f\big(s, b_r(s)R\big)\, ds \geqslant R. \tag{3.4}$$

Then (3.1) *has at least two positive solutions u_1, u_2 with $|u_1|_q < \rho \leqslant |u_2|_q$ and $|u_2|_r \leqslant R$.*

Proof. Let $E := L^p[0,1]$ and $N = KF$, where

$$F : L^p[0,1] \to L^{p_2}[0,1], \qquad F(u)(t) = f\big(t, u(t)\big),$$
$$K : L^{p_2}[0,1] \to L^p[0,1], \qquad K(v)(t) = \int_0^1 k(t,s)v(s)\,ds.$$

Also, let

$$C = \big\{u \in L^p[0,1]\colon\ u \neq 0,\ u(t) \geqslant b_\gamma(t)|u|_\gamma \text{ a.e. } t \in [0,1] \text{ for all } \gamma \in [1,p]\big\}.$$

Notice that for each $v \in L^{p_2}([0,1];\mathbf{R}_+)$, $v \neq 0$, we have $u := K(v) \in C$. Indeed,

$$u(t) = \int_0^1 k(t,s)v(s)\,ds \geqslant Mk_1(t)\int_0^1 k_2(s)v(s)\,ds \tag{3.5}$$

and for $\gamma \in [1,p]$,

$$|u|_\gamma = \left(\int_0^1 \left(\int_0^1 k(t,s)v(s)\,ds\right)^\gamma dt\right)^{\frac{1}{\gamma}} \leqslant |k_1|_\gamma \int_0^1 k_2(s)v(s)\,ds. \tag{3.6}$$

Now (3.5) and (3.6) imply

$$u(t) \geqslant b_\gamma(t)|u|_\gamma \quad \text{for a.e. } t \in [0,1]. \tag{3.7}$$

As a result, $N(C) \subset C$. Also note that the set $C \cup \{0\}$ is closed in $L^p[0,1]$.

We shall apply Theorem 2.3 with $|.| := |.|_q$ and $\|.\| := |.|_r$. Clearly $|u|_r \leqslant |u|_q$ for all $u \in L^p[0,1]$ since $r \leqslant q$, and $|.|_r$ is increasing with respect to C. Also for $u \in C$, from $u(t) \geqslant b_q(t)|u|_q$ we deduce that $|u|_r \geqslant |b_q|_r|u|_q$. Hence (2.1) holds with $c_1 = |b_q|_r$ and $c_2 = 1$. Moreover, for every $\gamma \in [1,p]$ and $u \in C$, we have $|b_p|_\gamma |u|_p \leqslant |u|_\gamma \leqslant |u|_p$, which shows that the L^γ-norms with $1 \leqslant \gamma \leqslant p$ are topologically equivalent on C.

According to [5] the map F from $L^p[0,1]$ to $L^{p_2}[0,1]$ is well defined, continuous and bounded, while K from $L^{p_2}[0,1]$ to $L^p[0,1]$ is completely continuous. This implies that the map N is completely continuous from $L^p[0,1]$ to $L^p[0,1]$. Consequently $N(D_R)$ is contained in a compact subset C_0 of the closed set $C \cup \{0\}$. From (3.2) we have

$$\big|N(u)\big|_p = \big|KF(u)\big|_p \geqslant \big|K(a_0)\big|_p > 0 \quad \text{for all } u \in D_R.$$

It follows that we may assume $C_0 \subset C$. Therefore N is a compact map from D_R to C.

Let $u \in S_\rho$. Using Hölder's inequality, the growth property of f and (3.3), we obtain

$$\begin{aligned}\big|N(u)\big|_q &= \left(\int_0^1 \left(\int_0^1 k(t,s)f\big(s,u(s)\big)\,ds\right)^q dt\right)^{\frac{1}{q}} \leqslant |k_1|_q \int_0^1 k_2(s)f\big(s,u(s)\big)\,ds \\ &\leqslant |k_1|_q|k_2|_{\bar p_1}\big|F(u)\big|_{\bar p_2} \leqslant |k_1|_q|k_2|_{\bar p_1}\big(|a_1|_{\bar p_2} + a_2|u|_q^{p/p_2}\big) \\ &= |k_1|_q|k_2|_{\bar p_1}\psi(\rho) < \rho = |u|_q.\end{aligned}$$

Hence condition (h1) in Theorem 2.3 is satisfied.

Now, if $u \in \Sigma_R$, that is $|u|_r = R$, then using (3.7) and (3.4), we obtain

$$\begin{aligned}\left|N(u)\right|_r &= \left(\int_0^1 \left(\int_0^1 k(t,s) f\big(s,u(s)\big)\,ds\right)^r dt\right)^{\frac{1}{r}} \\ &\geqslant M|k_1|_r \int_0^1 k_2(s) f\big(s,u(s)\big)\,ds \geqslant M|k_1|_r \int_0^1 k_2(s) f\big(s, b_r(s)|u|_r\big)\,ds \\ &\geqslant R = |u|_r.\end{aligned}$$

Thus (h2) also holds and the conclusion follows from Theorem 2.3. □

Remark. If in Theorem 3.1, instead of (3.2) we only require that

$$f(t,y) \geqslant 0 \quad \text{for all } y \in \mathbf{R}_+ \text{ and a.e. } t \in [0,1],$$

then we obtain that (3.1) has at least one positive solution u_2 with $\rho \leqslant |u_2|_q$ and $|u_2|_r \leqslant R$. If in addition, the null function does not solve (3.1), then there is a second positive solution u_1 to Eq. (3.1) with $|u_1|_q < \rho$.

Indeed, for every small enough $\varepsilon > 0$, all the assumptions of Theorem 3.1 are satisfied with $f^\varepsilon(t,y) = f(t,y) + \varepsilon$, $a_0^\varepsilon(t) = \varepsilon$, $a_1^\varepsilon(t) = a_1(t) + \varepsilon$ and $\psi^\varepsilon(t) = |a_1^\varepsilon|_{\bar{p}_2} + a_2 t^{\frac{p}{p_2}}$ in the place of $f(t,y)$, $a_0(t)$, $a_1(t)$ and $\psi(t)$, respectively. Theorem 3.1 guarantees the existence of two positive solutions u_1^ε and u_2^ε to the equation

$$u = KF(u) + K(\varepsilon)$$

with $|u_1^\varepsilon|_q < \rho \leqslant |u_2^\varepsilon|_q$ and $|u_2^\varepsilon|_r \leqslant R$. Now a standard limit argument leads to the desired conclusion.

Example. Let k be as in Theorem 3.1 and assume that $f(t,y) = |y|^n$ with $n = \frac{p}{p_2} > 1$ (superlinear growth). Here $\psi(t) = t^n$ and (3.3), (3.4) become

$$|k_1|_q |k_2|_{\bar{p}_1} \rho^{n-1} < 1 \tag{3.8}$$

and respectively

$$M^{n+1} R^{n-1} \frac{1}{|k_1|_r^{n-1}} \int_0^1 k_2(s) k_1^n(s)\,ds \geqslant 1. \tag{3.9}$$

When $r = q = p$, one has $\bar{p}_1 = p_1$ and (3.8), (3.9) reduce to the inequalities

$$|k_1|_p |k_2|_{p_1} \rho^{n-1} < 1, \qquad M^{n+1} R^{n-1} \frac{1}{|k_1|_p^{n-1}} \int_0^1 k_2(s) k_1^n(s)\,ds \geqslant 1,$$

which are assumed in [5, Example 2.2].

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