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ON THE RATE OF CONVERGENCE OF SOME INTEGRAL OPERATORS FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract

In the present paper we define a general class $B_{n,\alpha}$, $\alpha \geqq 1$, of Durrmeyer–Bézier type of linear positive operators. Our main aim is to estimate the rate of pointwise convergence for functions f at those points x at which the one-sided limits $f(x+)$ and $f(x-)$ exist. As regards these functions defined on an interval J certain conditions are required. We discuss two distinct cases: $\operatorname{Int}(J) = (0, \infty)$ and $\operatorname{Int}(J) = (0, 1)$.

1. Introduction

Let $(\Lambda_n)_n$ denote a sequence of linear operators acting on a real function space $\mathcal{S}$, $\mathcal{S} \subset \mathbb{R}^J$, J is an interval. For any $f \in \mathcal{S}$ the rate of convergence is determined by estimating $\left|(\Lambda_n f)(x) - f(x)\right|$ in terms of certain bounds. Let $x_0 \in \operatorname{Int}(J)$ be a discontinuity point of the first kind for f. In the last two decades it comes out a further development investigating the behaviour of Λ_n in connection with estimates concerning the deviation

$$(1)\qquad \left|(\Lambda_n f)(x_0) - \frac{1}{2}\big(f(x_0+) + f(x_0-)\big)\right|.$$

As regards Λ_n, $n \in \mathbb{N}$, in time have been used both discrete-type operators such as Bernstein, Szász, Baskakov, Meyer–König and Zeller operators and their integral analogue in Kantorovich or Durrmeyer sense.

As regards the space $\mathcal{S}$, it has been intensively considered functions of bounded variation. All discontinuities of such a function are only of first kind, consequently the study of (1) is well raised.

We recall: the *total variation* of a function f on $[a, b]$ is defined as the upper bound of the numbers $v(f, \Delta_n) := \sum_{k=0}^{n-1} \left|f(x_{k+1}) - f(x_k)\right|$, for any

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$n \in \mathbb{N}$ and all meshes Δ_n $(a = x_0 < x_1 < \cdots < x_n = b)$. Setting $\bigvee_a^b(f) := \sup_{\Delta_n} v(f, \Delta_n) \in [0, \infty]$, whenever this quantity is finite we shall say that f is of *bounded variation* on the interval $[a, b]$.

The estimate of (1) for functions of bounded variation is usually given in terms of the arithmetic means of the sequence of total variation. We point out that a pioneer work in this direction is due to R. Bojanic and M. Vuilleumier – in [5] they deepened a technique later often used in many papers.

Best of our knowledge, here are some authors who approached the above trend studying various classes of operators: Fuhua Cheng [6], Ranko Bojanic and Mohammad Kazim Khan [4], [15], Xiao-Ming Zeng and Wenzhong Chen [18], Ashok Sahai and Govind Prasad [16], Shunsheng Guo [8]. The papers of Grazyna Aniol [1], [2] deal in this respect both with some discrete operators and Kantorovich-type operators. A real contribution in this field is due to Vijay Gupta and his collaborators [9], [10], [11], [12], [13].

In this paper we are dealing with a general class of linear operators of Durrmeyer–Bézier type, investigating their rate of convergence for functions of bounded variation. The article is organized as follows. In Section 2 we construct the announced sequence of summation-integral operators, named $B_{n,\alpha}$, $\alpha \geqq 1$. In Section 3 the basic notations used throughout the paper are indicated. Next we give several preliminary results. Mainly these are estimates of the quantities in which we split the expression $\left|(B_{n,\alpha}f)(x) - (\alpha+1)^{-1}\big(f(x+) + \alpha f(x-)\big)\right|$. The last section is devoted to give an upper pointwise bound of the mentioned deviation under some additional conditions imposed to f. We consider both the cases when J is unbounded and when J is bounded. Some particular cases are also analyzed.

We point out that this class is a very general one including many classical sequences. On the other hand, instead of using subintervals with their endpoints $x \pm x/\sqrt{n}$ as in the previously quoted papers, here the considered endpoints are $x \pm x/n^{\beta}$ which offer more flexibility to our operators ($\beta > 0$ is arbitrary). We also remark that the construction of the best known operators which activate for $\operatorname{Int}(J) = (0, \infty)$ – as Szász or Baskakov type – requires an estimation of infinite sums which in a certain sense restricts usefulness of the operators from the computational point of view. In our case, for $\operatorname{Int}(J) = (0, \infty)$ we use index sets I_n which can be finite. We admit that an inconvenient feature of our research is the following: the evaluation given in Section 5 is not asymptotically optimal.

2. Construction of the operators $B_{n,\alpha}$

Let J be a given interval of the real line. Let I_n, $n \in \mathbb{N}$, be sets of indexes such that $I_n \subset I_{n+1}$ holds. We start from a sequence $(b_n)_n$ of linear positive operators of discrete type, that is, operators of the form $(b_n f)(x) = \sum\limits_{k\in I_n} u_{n,k}(x) f(x_{n,k})$, where $u_{n,k} \in \mathbb{R}_+^J$ and $x_{n,k} \in J$, $k \in I_n$. In order to generalize b_n to a summation-integral operator B_n, we follow J. L. Durrmeyer and use a non-negative family $\omega_{n,k}$, $k \in I_n$, of real functions belonging to Lebesgue space $L_p(J)$, $p = 1$ if J is bounded and $p = \infty$ if J is unbounded. We define B_n as follows:

$$(2) \qquad (B_n f)(x) = \sum_{k\in I_n} u_{n,k}(x) \int_J \omega_{n,k}(t) f(t)\,dt, \quad x \in J,\ f \in \mathcal{F},$$

where $\mathcal{F}$ contains all functions $f \in \mathbb{R}^J$ for which the right-hand side in (2) is well defined.

For example, choosing $J = [0,1]$, $I_n = \{0, 1, \ldots, n\}$,

$$u_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad \omega_{n,k}(t) = (n+1) u_{n,k}(t)$$

we obtain the original Durrmeyer operators studied by M. M. Derriennic [7], here $\mathcal{F}$ being $L_1(J)$.

So that our operators, both b_n and B_n ($n \in \mathbb{N}$), have the degree of exactness zero, we assume throughout the paper

$$(3) \qquad \sum_{k\in I_n} u_{n,k}(x) = 1,\ x \in J \quad \text{and} \quad \int_J \omega_{n,k}(t)\,dt = 1,\ k \in I_n.$$

Moreover, for each $n \in \mathbb{N}$ we assume that a function $\phi_n \in \mathbb{R}_+^J$ exists with the property

$$(4) \qquad u_{n,k}(x) \leqq \phi_n(x), \quad k \in I_n,\ x \in \operatorname{Int}(J).$$

We have in mind the variants: I_n finite thus as a model can be chosen $\{0, 1, \ldots, s_n\}$, $s_n = \#(I_n) - 1$, or I_n is infinite thus our model can be considered $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

At this moment we can define the Bézier variant of B_n operators. Let α be a real number, $\alpha \geqq 1$. We consider the operators $B_{n,\alpha}$, $n \in \mathbb{N}$, given as follows:

$$(5) \qquad (B_{n,\alpha} f)(x) = \sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x) \int_J \omega_{n,k}(t) f(t)\,dt, \quad x \in J,\ f \in \mathcal{F},$$

where

$$Q_{n,k}^{(\alpha)}(x) := S_{n,k}^{\alpha}(x) - S_{n,k+1}^{\alpha}(x), \quad S_{n,k}(x) := \sum_{\substack{j \geqq k \\ j \in I_n}} u_{n,j}(x), \tag{6}$$

for every $x \in J$ and $k \in I_n$.

If $k \leqq \inf(I_n)$, $S_{n,k} = 1$, see (3); if $k > \sup(I_n)$ we agree to take $S_{n,k} = 0$.

Clearly, the operator $B_{n,\alpha}$ is a linear positive one and it can be written as a singular integral of the type

$$(B_{n,\alpha}f)(x) = \int_J K_{n,\alpha}(x,t)f(t)\,dt, \quad x \in J,\ f \in \mathcal{F},$$

with the kernel $K_{n,\alpha}(x,t) := \sum_{k \in I_n} Q_{n,k}^{(\alpha)}(x)\omega_{n,k}(t)$, $(x,t) \in J \times J$.

We gather some direct properties of $Q_{n,k}^{(\alpha)}$, $S_{n,k}$ and $K_{n,\alpha}$ useful in the proofs inserted in Section 4.

LEMMA 1. *For all $k \in I_n$, $x \in J$ and $\alpha \geqq 1$ one has*

$$S_{n,k}(x) - S_{n,k+1}(x) = u_{n,k}(x), \quad 0 \leqq S_{n,k}(x) \leqq 1, \tag{7}$$

$$\sum_{k \in I_n} Q_{n,k}^{(\alpha)}(x) = 1, \quad \int_J K_{n,\alpha}(x,t)\,dt = 1, \tag{8}$$

$$\sum_{k \in I_n} Q_{n,k}^{(\alpha)}(x) \sum_{\substack{j \leqq k \\ j \in I_n}} u_{n,j}(x) = \sum_{j \in I_n} u_{n,j}(x) \sum_{\substack{k \geqq j \\ k \in I_n}} Q_{n,k}^{(\alpha)}(x), \tag{9}$$

(10) *there exists $\tau_{n,k,x}$ such that $Q_{n,k}^{(\alpha+1)}(x) = (\alpha+1)u_{n,k}(x)\tau_{n,k,x}^{\alpha}$,*

$$\left|S_{n,k}^{\alpha}(x) - S_{n,k+1}^{\alpha}(x)\right| \leqq \alpha u_{n,k}(x) \leqq \alpha\phi_n(x). \tag{11}$$

PROOF. (7) and (8) are implied by (6) combined with (3). The next statement is implied by the identity

$$b_0a_0 + b_1(a_0 + a_1) + b_2(a_0 + a_1 + a_2) + \cdots = (b_0 + b_1 + b_2 + \dots)a_0$$
$$+(b_1 + b_2 + \dots)a_1 + (b_2 + \dots)a_2 + \dots .$$

Obviously, if $a, b \in [0,1]$ and $\nu \geqq 1$ then $c_{a,b}$ between a, b exists such that $|b^\nu - a^\nu| = \nu|b-a|c_{a,b}^{\nu-1} \leqq \nu|b-a|$. Based on this relation, (7) and (4) imply both (10) and (11). We notice that $\tau_{n,k,x}$ lies between $S_{n,k}(x)$ and $S_{n,k+1}(x)$. □

We also deduce that $B_{n,1}$ becomes B_n defined by (2) and $B_{n,\alpha}$, $\alpha \geqq 1$, reproduces the constants, that is $(B_{n,\alpha}1)(x) = 1$, $x \in J$.

In what follows we make a crucial assumption as regards the families $(u_{n,k})_k$, $(\omega_{n,k})_k$. More precisely, we impose the following condition to be fulfilled

$$\int_{\{t\in J:\, t>x\}} \omega_{n,k}(t)\,dt = \sum_{\substack{j\leqq k\\ j\in I_n}} u_{n,j}(x), \quad x \in \mathrm{Int}\,(J), \tag{12}$$

for all $k \in I_n$, $k \neq \sup(I_n)$ if I_n is finite, and $k \in I_n$ if I_n is infinite.

At first glance it seems to be a very tough request. The following examples remove this feeling.

EXAMPLES. 1° Taking $J = [0,1]$, $I_n = \{0,1,\ldots,n\}$, $n \geqq 2$, $u_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$, $\omega_{n,k}(t) = nu_{n-1,k}(t)$ for $1 \leqq k \leqq n-1$ and $\omega_{n,n}(t) = 1$, B_n defined by (2) becomes Bernstein–Durrmeyer operator in a slight modified form; (3) takes place and (12) is fulfilled, see [18, *Eq. (19)*]. Further on we consider $J = [0,\infty)$ and $I_n = \mathbb{N}_0$.

2° Choosing $u_{n,k}(x) = e^{-nx}(nx)^k/k!$ and $\omega_{n,k}(t) = nu_{n,k}(t)$, B_n becomes modified Szász–Mirakjan operator. Condition (12) is fulfilled, see [16, *Lemma 5*].

3° Choosing

$$u_{n,k}(x) = \binom{n+k-1}{k} x^k(1+x)^{-n-k}$$

and $\omega_{n,k}(t) = (n+k)(1+t)^{-1}u_{n,k}(t)$, B_n becomes a modified Baskakov operator and condition (12) is again fulfilled, see [12, *Lemma 2.4*]. As regards these operators, further results have been obtained by H. Heilmann and M. W. Müller [14].

The first class of operators was introduced by V. Gupta in [9] and the Bézier variant of the last two classes were introduced and studied by Vijay Gupta and Ulrich Abel in [11] and Vijay Gupta in [10] respectively.

As a matter of fact we indicate for each of the above three examples a possibility to select ϕ_n function which verifies condition (4).

$$\phi_n(x) = \frac{1}{\sqrt{2enx(1-x)}}, \quad 0 < x < 1,$$

$$\phi_n(x) = \frac{2(4x^2+3x+1)}{\sqrt{nx}}, \quad \phi_n(x) = \frac{8\sqrt{9x(1+x)+1}+2}{5\sqrt{nx(x+1)}}, \ x>0,$$

see [17], [16, *Lemma 2*], [12, *Lemma 2.3*], respectively. It is fair to notice that in [9] there are improvements and corrections of some results obtained in [16].

We point out that recent results in this area improve the values of $\phi_n(x)$ for Szász and Baskakov basis functions, see [11] and [3]. These new values are given by

$$\phi_n(x) = \frac{1}{\sqrt{2enx}} \quad \text{and} \quad \phi_n(x) = \frac{C}{\sqrt{nx(x+1)}},$$

respectively. In the above expression C is a constant. For $n=1$, $C=1$; for $n \geqq 2$, the value of C depends on n and based on [3, *Théorème 3*], it is given by $\max\{(2/3)^{3/2}, (3n/2)^{3/2}(n-1)^{n-1}/(n+1/2)^{n+1/2}\}$. However our purpose is attained by any choice of ϕ_n function.

In the next section we gather all notations which will be used for ennunciation and proving our results.

3. Basic notation

Let $a<b$ be real numbers. We set $J^-(a) := J \cap (-\infty, a)$, $J(a,b) := J \cap [a,b]$, $J^+(b) := J \cap (b, \infty)$. For any point $x \in \mathrm{Int}\,(J)$ we consider the following decomposition of the interval J

$$J = J^-(x-\delta) \cup J(x-\delta, x+\delta) \cup J^+(x+\delta), \quad \delta>0, \ x \pm \delta \in \mathrm{Int}\,(J).$$

Let β be a given real number, $\beta>0$. In order to be brief we introduce the quantities

$$u_{n,x} := x - xn^{-\beta}, \quad v_{n,x} := x + xn^{-\beta}, \quad w_{n,x} := x + (1-x)n^{-\beta}, \quad n \in \mathbb{N}.$$

Next we define the functions g_x, sgn_x, δ_x as usual:

$$g_x(t) := \begin{cases} f(t) - f(x-), & t<x, \\ 0, & t=x, \\ f(t)-f(x+), & t>x, \end{cases} \qquad \mathrm{sgn}_x(t) := \begin{cases} -1, & t<x, \\ 0, & t=x, \\ 1, & t>x, \end{cases}$$

$$\delta_x(t) := \begin{cases} 1, & t=x, \\ 0, & t \neq x, \end{cases}$$

where $t \in J$. Since g_x is continuous at $t = x$, the map $t \mapsto \bigvee_a^t (g_x)$, ($a \in J$, $t \in J$ such that x is between a and t), is continuous at the same point x. With the help of g_x we introduce $\widehat{g}_x \in \mathbb{R}^J$ given as follows:

$$\widehat{g}_x(t) := \begin{cases} g_x(t), & t \leqq 2x, \\ g_x(2x), & t > 2x, \end{cases} \qquad (t \in J).$$

We set $s_f(x) := \big(f(x+) - f(x-)\big)/2$, the half-jump of f at the point x. For any integer $s \geqq 0$ we introduce the s-th order central moment of the operator $B_{n,\alpha}$, that is

$$\mu_{n,s}^{\langle\alpha\rangle}(x) := (B_{n,\alpha}\psi_{x,s})(x), \quad \psi_{x,s}(t) := (t-x)^s, \quad (t,x) \in J \times J.$$

If $\alpha = 1$ then we will simply denote these moments by $\mu_{n,s}$. We associate with the kernel $K_{n,\alpha}$ the following map:

$$\lambda_{n,\alpha}(x,t) := \int_{J^-(t)} K_{n,\alpha}(x,u)\,du, \quad x \in J,\ t \in \operatorname{Int}(J). \tag{13}$$

For a given $N \in \mathbb{N}$, $BV_N(J)$ stands for the class of all functions $f \in \mathbb{R}^J$ of bounded variation on every compact subinterval of J (denoted by $BV(J)$) and satisfying the growth condition $\big|f(t)\big| \leqq M_f\big(1 + |t|^N\big)$, $t \in J$, where M_f is a positive constant depending on f.

Next we present some technical results involving the above elements and the kernel $K_{n,\alpha}$ as well.

4. Preliminary results

LEMMA 2. *For every $(n,s) \in \mathbb{N} \times \mathbb{N}$ and $\alpha \geqq 1$ one has*

$$\mu_{n,2s}^{\langle\alpha\rangle} \leqq \alpha\mu_{n,2s}. \tag{14}$$

PROOF. By using the relation $B^\alpha - A^\alpha \leqq \alpha(B - A)$, $0 \leqq A \leqq B \leqq 1$, $\alpha \geqq 1$, and taking into account (6) we get

$$Q_{n,k}^{(\alpha)}(x) = S_{n,k}^\alpha(x) - S_{n,k+1}^\alpha(x) \leqq \alpha\big(S_{n,k}(x) - S_{n,k+1}(x)\big) = \alpha Q_{n,k}^{(1)}(x).$$

Consequently, for every $h \in \mathbb{R}_+^J$ one has $B_{n,\alpha}h \leqq \alpha B_{n,1}h$ and choosing $h = \psi_{x,2s} \geqq 0$ the conclusion follows. $\square$

LEMMA 3. *If $x \in \operatorname{Int}(J)$ then the following relations hold*

$$\text{(15)} \qquad \text{(i) } \textit{for each } y \in J,\ y < x, \quad \int_{J^-(y)} K_{n,\alpha}(x,t)\,dt \leqq \frac{\alpha\mu_{n,2}(x)}{(x-y)^2},$$

$$\text{(16)} \qquad \text{(ii) } \textit{for each } z \in J,\ z > x, \quad \int_{J^+(z)} K_{n,\alpha}(x,t)\,dt \leqq \frac{\alpha\mu_{n,2}(x)}{(z-x)^2}.$$

PROOF. Let $x \in \operatorname{Int}(J)$. If $y \in J$, $y < x$, then $1 \leqq (t-x)^2(x-y)^{-2}$, $(\forall)\ t \in J^-(y)$. If $z \in J$, $z > x$, then $1 \leqq (t-x)^2(z-x)^{-2}$, $(\forall)\ t \in J^+(z)$. These inequalities combined with (14) lead us to the desired result. $\square$

LEMMA 4. *If* $A_{n,\alpha}(x) := \int_{J^-(u_{n,x})} g_x(t)K_{n,\alpha}(x,t)\,dt$, *then one has*

$$\bigl|A_{n,\alpha}(x)\bigr| \leqq \frac{\alpha\mu_{n,2}(x)}{x^2}\Bigl(\bigvee_0^x(g_x) + \sum_{k=1}^{n-1}\bigl((k+1)^{2\beta} - k^{2\beta}\bigr)\bigvee_{u_{k,x}}^x(g_x)\Bigr), \quad n \geqq 2.$$

PROOF. Recalling (13) and integrating by parts, we have

$$\text{(17)} \quad A_{n,\alpha}(x) = g_x(t)\lambda_{n,\alpha}(x,t)\Big|_{t\in J^-(u_{n,x})} + \int_{J^-(u_{n,x})} \lambda_{n,\alpha}(x,t)\,d_t\bigl(-g_x(t)\bigr).$$

For $u \in J^-(t)$, $t < u_{n,x} < x$, applying (15) we get

$$\lambda_{n,\alpha}(x,t) \leqq \frac{\alpha\mu_{n,2}(x)}{(x-t)^2}, \quad \lambda_{n,\alpha}(x,u_{n,x}-) \leqq \frac{\alpha\mu_{n,2}(x)}{x^2}n^{2\beta}.$$

At the same time $\bigl|g_x(u_{n,x}-)\bigr| = \bigl|g_x(u_{n,x}-) - g_x(x)\bigr| \leqq \bigvee_{u_{n,x}}^x(g_x)$ and the map $t \mapsto -\bigvee_t^x(g_x)$ is a nondecreasing one, $t \in J^-(x)$. Gathering these relations, identity (17) implies

$$(18)\quad \left|A_{n,\alpha}(x)\right| \leqq \lambda_{n,\alpha}(x, u_{n,x}-) \bigvee_{u_{n,x}}^{x} (g_x)$$

$$+ \alpha\mu_{n,2}(x) \int_{J^-(u_{n,x})} (x-t)^{-2}\, d_t \left(- \bigvee_{t}^{x} (g_x) \right)$$

$$\leqq \frac{\alpha\mu_{n,2}(x)}{x^2} n^{2\beta} \bigvee_{u_{n,x}}^{x} (g_x)$$

$$+ \alpha\mu_{n,2}(x) \left\{ -(x-t)^{-2} \bigvee_{t}^{x} (g_x) \Big|_{t\in J^-(u_{n,x})} + 2 \int_{J^-(u_{n,x})} \bigvee_{t}^{x} (g_x) \frac{dt}{(x-t)^3} \right\}$$

$$= \frac{\alpha\mu_{n,2}(x)}{x^2} \bigvee_{0}^{x} (g_x) + 2\alpha\mu_{n,2}(x) \int_{J^-(u_{n,x})} \bigvee_{t}^{x} (g_x) \frac{dt}{(x-t)^3}.$$

In the last integral making the change $t = x - x/y^{\beta}$, one gets $1 \leqq y < n$ $(n \geqq 2)$ and it becomes

$$\int_{1}^{n} \bigvee_{x-xy^{-\beta}}^{x} (g_x) \frac{\beta}{x^2} y^{2\beta-1}\, dy = \frac{\beta}{x^2} \sum_{k=1}^{n-1} \int_{k}^{k+1} \bigvee_{x-xy^{-\beta}}^{x} (g_x) y^{2\beta-1}\, dy$$

$$\leqq \frac{\beta}{x^2} \sum_{k=1}^{n-1} \int_{k}^{k+1} \bigvee_{u_{k,x}}^{x} (g_x) y^{2\beta-1}\, dy = \frac{1}{2x^2} \sum_{k=1}^{n-1} \left((k+1)^{2\beta} - k^{2\beta} \right) \bigvee_{u_{k,x}}^{x} (g_x).$$

We considered that $y \in [k, k+1]$ implies $[x - xy^{-\beta}, x] \subset [u_{k,x}, x]$. Returning to (18) we obtain the claimed result. □

LEMMA 5. *If* $\mathcal{B}_{n,\alpha}(x) := \int_{J(u_{n,x}, v_{n,x})} g_x(t) K_{n,\alpha}(x,t)\, dt$ *then one has*

$$(19)\qquad \left|\mathcal{B}_{n,\alpha}(x)\right| \leqq \bigvee_{u_{n,x}}^{v_{n,x}} (g_x) \leqq \frac{1}{n} \sum_{k=1}^{n} \bigvee_{u_{k,x}}^{v_{k,x}} (g_x), \quad n \in \mathbb{N}.$$

The same relations are true if we substitute $v_{k,x}$ *by* $w_{k,x}$, $k = \overline{1,n}$.

PROOF. For $t \in J(u_{n,x}, v_{n,x})$ one has $\left|g_x(t)\right| = \left|g_x(t) - g_x(x)\right| \leqq \bigvee_{u_{n,x}}^{v_{n,x}} (g_x)$ and knowing that $0 \leqq \int_{J(u_{n,x},v_{n,x})} K_{n,\alpha}(x,t)\,dt \leqq 1$, the first inequality is proved.

For each $k = \overline{1,n}$, $J(u_{n,x}, v_{n,x}) \subset J(u_{k,x}, v_{k,x})$ takes place and consequently $\bigvee_{u_{n,x}}^{v_{n,x}} (g_x) \leqq \bigvee_{u_{k,x}}^{v_{k,x}} (g_x)$. The second inequality is based on the well-known property of the arithmetic mean: $\min_{k=\overline{1,n}} A_k \leqq \frac{1}{n} \sum_{k=1}^{n} A_k$.

The last assertion of our lemma is evident. □

LEMMA 6. *If* $C_{n,\alpha}(x) := \int_{J^+(v_{n,x})} \widehat{g}_x(t) K_{n,\alpha}(x,t)\,dt$ *then one has*

$$\left|C_{n,\alpha}(x)\right| \leqq \frac{\alpha\mu_{n,2}(x)}{x^2}\left(\bigvee_{x}^{2x}(g_x) + \sum_{k=1}^{n-1}\left((k+1)^{2\beta} - k^{2\beta}\right)\bigvee_{x}^{v_{k,x}}(g_x)\right), \quad n \geqq 2.$$

PROOF. Recalling (13) and integrating by parts, we have

$$C_{n,\alpha}(x) = \lambda_{n,\alpha}(x,t)\widehat{g}_x(t)\Big|_{t\in J^+(v_{n,x})} - \int_{J^+(v_{n,x})} \lambda_{n,\alpha}(x,t)\,d_t\big(\widehat{g}_x(t)\big).$$

Taking into account both $\lim_{t\to\infty} \lambda_{n,\alpha}(x,t) = 1$, $\lim_{t\to\infty} \widehat{g}_x(t) = g_x(2x)$ and the form of $\widehat{g}_x(t)$ for $t \in J^+(v_{n,x})$, we get

$$C_{n,\alpha}(x) = g_x(2x) - \lambda_{n,\alpha}(x, v_{n,x}+)g_x(v_{n,x}+) - \int_{v_{n,x}}^{2x} \lambda_{n,\alpha}(x,t)\,d_t\big(g_x(t)\big).$$

Since $g_x(2x) = \int_{v_{n,x}}^{2x} d_t\big(g_x(t)\big) + g_x(v_{n,x}+)$ holds, we obtain

$$C_{n,\alpha}(x) = \big(1 - \lambda_{n,\alpha}(x, v_{n,x}+)\big)\,g_x(v_{n,x}+) + \int_{v_{n,x}}^{2x} \big(1 - \lambda_{n,\alpha}(x,t)\big)\,d_t\big(g_x(t)\big).$$

On the other hand $1-\lambda_{n,\alpha}(x,z)=\int\limits_{J^+(z)} K_{n,\alpha}(x,u)\,du$, $\left|g_x(v_{n,x}+)\right| = \left|g_x(v_{n,x}+)-g_x(x)\right| \leqq \bigvee\limits_x^{v_{n,x}}(g_x)$ and $t\mapsto \bigvee\limits_x^{t}(g_x)-g_x(t)$ is a nondecreasing map, $t\in J^+(x)$. By using (16) both for $z=v_{n,x}$ and $z=t>x$ one gets

$$\left|C_{n,\alpha}(x)\right| \leqq \frac{\alpha\mu_{n,2}(x)}{(v_{n,x}-x)^2}\bigvee_x^{v_{n,x}}(g_x)+\int\limits_{v_{n,x}}^{2x}\frac{\alpha\mu_{n,2}(x)}{(t-x)^2}\,d_t\Big(\bigvee_x^{t}(g_x)\Big) \tag{20}$$

$$=\frac{\alpha\mu_{n,2}(x)}{(v_{n,x}-x)^2}\bigvee_x^{v_{n,x}}(g_x)$$

$$+\,\alpha\mu_{n,2}(x)\left\{(t-x)^{-2}\bigvee_x^{t}(g_x)\Big|_{t=v_{n,x}}^{t=2x}+2\int\limits_{v_{n,x}}^{2x}\bigvee_x^{t}(g_x)\frac{dt}{(t-x)^3}\right\}$$

$$=\frac{\alpha\mu_{n,2}(x)}{x^2}\bigvee_x^{2x}(g_x)+2\alpha\mu_{n,2}(x)\int\limits_{v_{n,x}}^{2x}\bigvee_x^{t}(g_x)\frac{dt}{(t-x)^3}.$$

In the above integral substituting $t=x+x/z^\beta$ it becomes

$$\int\limits_n^1\bigvee_x^{x+xz^{-\beta}}(g_x)\frac{(-\beta)}{x^2}z^{2\beta-1}\,dz=\frac{\beta}{x^2}\sum_{k=1}^{n-1}\int\limits_k^{k+1}\bigvee_x^{x+xz^{-\beta}}(g_x)z^{2\beta-1}\,dz$$

$$\leqq\frac{\beta}{x^2}\sum_{k=1}^{n-1}\int\limits_k^{k+1}\bigvee_x^{v_{k,x}}(g_x)z^{2\beta-1}\,dz=\frac{1}{2x^2}\sum_{k=1}^{n-1}\left((k+1)^{2\beta}-k^{2\beta}\right)\bigvee_x^{v_{k,x}}(g_x).$$

We used: $z\in[k,k+1]$ implies $[x,x+xz^{-\beta}]\subset[x,v_{k,x}]$, $k=\overline{1,n-1}$. Returning to (20) the proof is complete. □

LEMMA 7. *Let $f\in BV_N(J)$, $\operatorname{Int}(J)=(0,\infty)$.*

If $D_{n,\alpha}(x):=\int\limits_{J^+(2x)}\big(g_x(t)-g_x(2x)\big)K_{n,\alpha}(x,t)\,dt$ then one has

$$\left|D_{n,\alpha}(x)\right|\leqq\alpha M_f 2^N\left(\frac{2^{1-N}+x^N}{x^2}\mu_{n,2}(x)+\sqrt{\alpha^{-1}\mu_{n,2N}(x)}\right).$$

PROOF. Because of $t > 2x$ and $f \in BV_N(J)$ we obtain

$$\left| g_x(t) - g_x(2x) \right| = \left| f(t) - f(2x) \right| \leqq M_f\big((1+t^N) + (1+2^N x^N) \big).$$

Consequently,

$$\left| D_{n,\alpha}(x) \right| \leqq M_f \left\{ (2+2^N x^N) \int\limits_{J^+(2x)} K_{n,\alpha}(x,t)\,dt + \int\limits_{J^+(2x)} t^N K_{n,\alpha}(x,t)\,dt \right\}.$$

For the first integral we apply (16). In order to increase the second one, under the hypothesis $t > 2x$, we use Schwarz inequality.

$$\int\limits_{J^+(2x)} t^N K_{n,\alpha}(x,t)\,dt \leqq 2^N \int\limits_{J^+(2x)} (t-x)^N K_{n,\alpha}(x,t)\,dt$$

$$\leqq 2^N \left\{ \int\limits_{J^+(2x)} (t-x)^{2N} K_{n,\alpha}(x,t)\,dt \right\}^{1/2} \left\{ \int\limits_{J^+(2x)} K_{n,\alpha}(x,t)\,dt \right\}^{1/2}$$

$$\leqq 2^N \sqrt{\mu_{n,2N}^{\langle\alpha\rangle}(x)},$$

because of $J^+(2x) \subset J$ and (8). Lemma 2 finishes the proof. □

LEMMA 8. *Let* $\operatorname{Int}(J) = (0,1)$. *If* $E_{n,\alpha}(x) := \int\limits_{J^+(w_{n,x})} g_x(t) K_{n,\alpha}(x,t)\,dt$ *then one has*

$$\left| E_{n,\alpha}(x) \right| \leqq \frac{\alpha \mu_{n,2}(x)}{(1-x)^2} \left(\bigvee_x^1 (g_x) + \sum_{k=1}^{n-1} \big((k+1)^{2\beta} - k^{2\beta} \big) \bigvee_x^{w_{k,x}} (g_x) \right), \quad n \geqq 2.$$

PROOF. Taking the advantage of (13) and (16) we follow similar steps as in Lemma 6.

$$E_{n,\alpha}(x) = \int\limits_{J^+(w_{n,x})} \left(\int\limits_{J^+(t)} K_{n,\alpha}(x,u)\,du \right) d_t\big(g_x(t)\big), \quad u > t > w_{n,x} > x;$$

$$\left|E_{n,\alpha}(x)\right| \leqq \alpha\mu_{n,2}(x) \int\limits_{J^+(w_{n,x})} (t-x)^{-2}\, d_t\Bigl(\bigvee_x^t (g_x)\Bigr)$$

$$= \alpha\mu_{n,2}(x)\Bigl\{(t-x)^{-2}\bigvee_x^t(g_x)\Big|_{t\in J^+(w_{n,x})} + 2\int\limits_{J^+(w_{n,x})} (t-x)^{-3}\bigvee_x^t(g_x)\, dt\Bigr\};$$

$$\int\limits_{w_{n,x}}^{1} \bigvee_x^t (g_x)\frac{dt}{(t-x)^3} = \frac{\beta}{(1-x)^2}\int\limits_1^n \bigvee_x^{x+\frac{1-x}{z^\beta}} (g_x) z^{2\beta-1}\, dz$$

$$\leqq \frac{\beta}{(1-x)^2}\sum_{k=1}^{n-1}\int\limits_k^{k+1}\bigvee_x^{w_{k,x}} (g_x) z^{2\beta-1}\, dz$$

$$= \frac{1}{2(1-x)^2}\sum_{k=1}^{n-1}\bigl((k+1)^{2\beta}-k^{2\beta}\bigr)\bigvee_x^{w_{k,x}}(g_x).$$

In the above we replaced $t = x + (1-x)/z^\beta$ and used $\left[x, x+(1-x)/z^\beta\right] \subset [x, w_{k,x}]$ for $z \in [k, k+1]$, $k = \overline{1, n-1}$. Assembling all relations, the proof is complete. □

LEMMA 9. *Under the hypotheses* (12) *and* (4), *the operator defined by* (5) *verifies*

$$\left|(B_{n,\alpha}\operatorname{sgn}_x)(x) + \frac{\alpha-1}{\alpha+1}\right| \leqq 2\alpha\phi_n(x), \quad x \in \operatorname{Int}(J).$$

PROOF. First we consider the case when I_n is infinite. Taking in view both relation (8), our hypothesis (12) and property (9) as well, we can write

$$(B_{n,\alpha}\operatorname{sgn}_x)(x) + 1 = \int\limits_{J^+(x)} K_{n,\alpha}(x,t)\, dt - \int\limits_{J^-(x)} K_{n,\alpha}(x,t)\, dt + 1$$

$$= 2\int\limits_{J^+(x)} K_{n,\alpha}(x,t)\, dt = 2\sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x)\int\limits_{J^+(x)} \omega_{n,k}(t)\, dt$$

$$= 2\sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x)\sum_{\substack{j\leqq k\\ j\in I_n}} u_{n,j}(x)$$

$$= 2\sum_{j\in I_n} u_{n,j}(x)\sum_{\substack{k\geqq j\\ k\in I_n}} Q_{n,k}^{(\alpha)}(x) = 2\sum_{j\in I_n} u_{n,j}(x)S_{n,j}^{\alpha}(x).$$

Further on, because of $1 = \sum_{j\in I_n} Q_{n,j}^{(\alpha+1)}(x)$ and (10) we get

$$(B_{n,\alpha}\,\mathrm{sgn}_x)(x) + \frac{\alpha-1}{\alpha+1} = 2\sum_{j\in I_n}\left(u_{n,j}(x)S_{n,j}^{\alpha}(x) - \frac{1}{\alpha+1}Q_{n,j}^{(\alpha+1)}(x)\right)$$

$$= 2\sum_{j\in I_n} u_{n,j}(x)(S_{n,j}^{\alpha}(x) - \tau_{n,j,x}^{\alpha}).$$

Clearly, $\left|S_{n,j}^{\alpha}(x) - \tau_{n,j,x}^{\alpha}\right| \leqq \left|S_{n,j}^{\alpha}(x) - S_{n,j+1}^{\alpha}(x)\right|$. By using (11) and (3) we obtain the assertion of our lemma.

For the case when I_n is finite, putting $\overline{n} := \sup(I_n)$ and $I_n^* = I_n \setminus \{\overline{n}\}$, we have $Q_{n,\overline{n}}^{(\alpha)} = S_{n,\overline{n}}^{\alpha} = u_{n,\overline{n}}^{\alpha}$. Now we decompose $\sum_{k\in I_n}$ into two parts: the sum $\sum_{k\in I_n^*}$ and the term corresponding to $k = \overline{n}$. The proof running similarly as in the previous case, we obtain

$$\left|(B_{n,\alpha}\,\mathrm{sgn}_x)(x) + \frac{\alpha-1}{\alpha+1}\right| \leqq 2\sum_{j\in I_n^*} u_{n,j}(x)\left|S_{n,j}^{\alpha}(x) - S_{n,j+1}^{\alpha}(x)\right| + T_{\overline{n}}^{(\alpha)}(x),$$

where

$$T_{\overline{n}}^{(\alpha)}(x) := u_{n,\overline{n}}^{\alpha}(x)\left|\int_{J^+(x)} \omega_{n,\overline{n}}(t)\,dt - \frac{2u_{n,\overline{n}}(x)}{\alpha+1}\right|.$$

Since $T_{\overline{n}}^{(\alpha)} \leqq \left(1 + \dfrac{2}{\alpha+1}\right)u_{n,\overline{n}} \leqq 2\alpha\phi_{\overline{n}}$, we arrive at the same result. □

5. Main results

Since an affine substitution maps (a,b), $-\infty \leqq a < b \leqq \infty$, onto $(0,1)$, $(0,\infty)$ or $\mathbb{R}$, it is enough to consider these intervals as being Int (J).

For the first two situations, we are going to present the rate of pointwise convergence of $B_{n,\alpha}$ operators for functions of bounded variation. Our main results may be stated as follows.

THEOREM 1. *Let* $\operatorname{Int}(J) = (0, \infty)$. *Let* $B_{n,\alpha}$ *be defined by* (5) *such that* (4) *and* (12) *are fulfilled. For every* $\beta > 0$, $f \in BV_N(J) \cap \mathcal{F}$, $x > 0$ *and the integer* $n \geqq 2$, *the inequality*

$$\text{(21)} \qquad \left|(B_{n,\alpha}f)(x) - (\alpha+1)^{-1}\big(f(x+) + \alpha f(x-)\big)\right|$$

$$\leqq \frac{\alpha \mu_{n,2}(x)}{x^2} \Delta_n(\beta, f; x) + \bigvee_{x-x/n^\beta}^{x+x/n^\beta} (g_x)$$

$$+ 2^N M_f \sqrt{\alpha \mu_{n,2N}(x)} + 2\alpha \left| s_f(x) \right| \phi_n(x)$$

holds, where

$$\Delta_n(\beta, f; x) = \bigvee_0^{2x} (g_x) + M_f(2 + 2^N x^N) + \sum_{k=1}^{n-1} \left((k+1)^{2\beta} - k^{2\beta}\right) \bigvee_{x-x/k^\beta}^{x+x/k^\beta} (g_x).$$

PROOF. Setting $c_{f,\alpha}(x) := \big(f(x+) + \alpha f(x-)\big)/(\alpha+1)$ a convex combination of the real numbers $f(x\pm)$, and having in mind Section 3, for each $t \in J$ we can write

$$f(t) = c_{f,\alpha}(x) + g_x(t) + s_f(x)\left(\operatorname{sgn}_x(t) + \frac{\alpha-1}{\alpha+1}\right) + \big(f(x) - s_f(x)\big)\,\delta_x(t).$$

In the above we apply the linear operator $B_{n,\alpha}$. Since $B_{n,\alpha}$ reproduces the constants and $B_{n,\alpha}\delta_x$ is null, one obtains

$$\left|(B_{n,\alpha}f)(x) - c_{f,\alpha}(x)\right| \leqq \left|(B_{n,\alpha}g_x)(x)\right| + \left|s_f(x)\right| \left|(B_{n,\alpha}\operatorname{sgn}_x)(x) + \frac{\alpha-1}{\alpha+1}\right|$$

$$= \left|A_{n,\alpha}(x) + \mathcal{B}_{n,\alpha}(x) + C_{n,\alpha}(x) + D_{n,\alpha}(x)\right|$$

$$+ \left|s_f(x)\right| \left|(B_{n,\alpha}\operatorname{sgn}_x)(x) + \frac{\alpha-1}{\alpha+1}\right|,$$

where $A_{n,\alpha}(x)$, $\mathcal{B}_{n,\alpha}(x)$, $C_{n,\alpha}(x)$, $D_{n,\alpha}(x)$ have been defined in Lemma 4, Lemma 5, Lemma 6 and Lemma 7, respectively. Using the statements of these lemmas together with Lemma 9, after some arrangements we arrive at the claimed result. □

REMARKS. 1° Certainly we are interested in those sequences $(B_{n,\alpha})_n$ which form an approximation process, in other words $\lim_n B_{n,\alpha} f = f$, $f \in \mathcal{S}$, the convergence being understood with respect to a suitable topology on the involved function space $\mathcal{S}$. In this respect, for our integral linear operators it is natural $\mu_{n,2} = o(1)$ $(n \to \infty)$ to be fulfilled. Here o represents the Landau symbol. On the other hand, continuity of g_x at x implies that $\bigvee_{x-\beta}^{x+\alpha}(g_x) \to 0$ as $\alpha, \beta \to 0^+$. These facts allow us to state the following.

If $\phi_n(x) = o(1)$ $(n \to \infty)$ and $\mu_{n,2}(x)\Delta_n(\beta, f; x) = o(1)$ $(n \to \infty)$ then

$$\lim_{n\to\infty} (B_{n,\alpha} f)(x) = \frac{f(x+) + \alpha f(x-)}{1+\alpha}, \tag{22}$$

for every $f \in BV_N(J) \cap \mathcal{F}$.

2° If x is a continuity point of f then relation (21) becomes

$$\big|(B_{n,\alpha} f)(x) - f(x)\big| \leqq \frac{\alpha\mu_{n,2}(x)}{x^2}\Delta_n(\beta, f; x) + \bigvee_{x-x/n^\beta}^{x+x/n^\beta}(g_x)$$
$$+ 2^N M_f \sqrt{\alpha\mu_{n,2N}(x)}.$$

3° If $\beta \in (0, 1/2)$ then $(k+1)^{2\beta} - k^{2\beta} < 1$ and one has

$$\Delta_n(\beta, f; x) < \bigvee_0^{2x}(g_x) + M_f(2 + 2^N x^N) + \sum_{k=1}^{n-1} \bigvee_{x-x/k^\beta}^{x+x/k^\beta}(g_x).$$

Also, $\Delta_n(1/2, f; x)$ has a simple form.

THEOREM 2. *Let* $\operatorname{Int}(J) = (0,1)$. *Let* $B_{n,\alpha}$ *be defined by* (5) *such that* (4) *and* (12) *are fulfilled. For every* $\beta > 0$, $f \in BV(J) \cap \mathcal{F}$, $x \in (0,1)$ *and the integer* $n \geqq 2$, *the following inequality*

$$\big|(B_{n,\alpha} f)(x) - (\alpha+1)^{-1}\big(f(x+) + \alpha f(x-)\big)\big|$$
$$\leqq \alpha\mu_{n,2}(x)\psi(x)\nabla_n(\beta, f; x) + \bigvee_{x-x/n^\beta}^{x+(1-x)/n^\beta}(g_x) + 2\alpha\big|s_f(x)\big|\phi_n(x),$$

holds, where

$$\nabla_n(\beta, f; x) = \bigvee_0^1 (g_x) + \sum_{k=1}^{n-1} \left((k+1)^{2\beta} - k^{2\beta}\right) \bigvee_{x-x/k^\beta}^{x+(1-x)/k^\beta} (g_x)$$

and $\psi(x) = \max\left\{x^{-2}, (1-x)^{-2}\right\}$.

PROOF. We use lemmas 4, 5, 8 and 9. This time we can write $(B_{n,\alpha} g_x)(x) = A_{n,\alpha}(x) + \mathcal{B}_{n,\alpha}(x) + E_{n,\alpha}(x)$ noticing that now $\mathcal{B}_{n,\alpha}(x)$ contains the knots $w_{k,x}$. A short calculation justifies our assertion. □

REMARKS. The established inequality offers possibility to discuss the particular cases:

(i) x is a continuity point of f;
(ii) $\beta \in (0, 1/2)$ and $\beta = 1/2$.

Also (22) is true for every $f \in BV(J) \cap \mathcal{F}$.

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