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Rate of convergence of a class of Bézier type operators for functions of bounded variation

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Abstract. By using probability methods we introduce a general class of Bézier type linear operators. The aim of the present paper is to estimate the rate of pointwise convergence of this class for functions of bounded variation defined on an interval J. Two cases are analyzed: $Int(J) = (0, \infty)$ and $Int(J) = (0, 1)$. In a particular case, our operators turn into the Kantorovich-Bézier operators. Also some examples are delivered.

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1 Introduction

Recently, in [1] we introduced Λ_n, $n \in \mathbb{N}$, a general class of linear operators acting on a function real space $\mathcal{S}$, $\mathcal{S} \subset \mathbb{R}^J$, J a real interval. Using a Lipschitz-type maximal function, the Peetre functional K_2 and the Hardy-Littlewood maximal function, has been estimated approximation order in L_p-spaces for smooth functions.

The first aim of this paper is to present the Bézier variant $\Lambda_{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \geq 1$, of the above operators. Section 2 contains both this construction and some direct properties of these integral operators. Our main goal is to estimate the rate of pointwise convergence for functions f at those

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points x at which the one-sided limits $f(x+)$ and $f(x-)$ exist, $x \in Int(J)$. Actually, in the last two decades it has been intensively considered functions of bounded variation. All discontinuities of a such function are only of first kind, consequently the proposed study is well raised.

We recall: the *total variation* of a function $f \in \mathcal{S}$ on $[a,b] \subset J$ is defined as the upper bound of the numbers $v(f;\Delta_n) := \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$, for any $n \in \mathbb{N}$ and all meshes $\Delta_n(a = x_0 < x_1 < \cdots < x_n = b)$. Setting $\bigvee_a^b(f) := \sup_{\Delta_n} v(f;\Delta_n) \in [0,\infty]$, whenever this quantity is finite we shall say that f is of *bounded variation* on $[a,b]$. Throughout the paper, BV stands for the class of all functions of bounded variation on every compact subinterval of $\mathbb{R}$.

In Section 3 we gather all notations which will be used for enunciation our main result. We also prove several preliminary results. Mainly these are estimates of the quantities in which we split the expression

$$|(\Lambda_{n,\alpha}f)(x) - q^{-\alpha}f(x+) - (1-q^{-\alpha})f(x-)|, \tag{1.1}$$

where $q > 1$ plays an important role in the structure of the initial operators Λ_n.

Section 4 is devoted to give an upper pointwise bound of the mentioned deviation under some additional conditions imposed to f. We consider here both the case J unbounded and J bounded. Considerations concerning the convergence of our sequence of operators are delivered. Also particular cases are analyzed. We reobtain the Bézier variant of some Kantorovich type operators.

At the end of this section we mention that the rate of convergence of some operators of functions with bounded variation is usually given in terms of the arithmetic means of the sequence of total variation. A pioneer work in this direction is due to R. Bojanic and M. Vuilleumier [6], they deepening a technique later often used in many papers.

In time, for $q = 2$ and $\alpha = -1$ the deviation (1.1) was intensively studied by a large number of mathematicians. Best of our knowledge, we mention some of their papers in connection with functions of bounded variation. F. Cheng [9] established the rate of convergence for Bernstein operators. His results have been extended by S. S. Guo and M. K. Khan [10] the approximation of functions of bounded variation on $\mathbb{R}$ has been achieved and several classical operators (Bernstein, Szász, Baskakov, Gamma, Weierstrass) have been discussed as examples. At the same time M. K. Khan

[15] investigated Bernstein power series operators and later, in a joint paper with R. Bojanic [7], functions with derivative of bounded variation on $\mathbb{R}$ have been considered. For the same values, $q = 2$, $\alpha = -1$, we quote [17], [14], [13], in which the authors gave estimates of (1.1) for modified Szász and Baskakov operators in Durrmeyer sense, respectively. Actually, Vijay Gupta has studied this problem in several papers for different classes of discrete and integral operators. The pointwise convergence of Meyer-König and Zeller operators for bounded functions was investigated by X.-M. Zeng and J.-N. Zhao [21].

2 The class $(\Lambda_{n,\alpha})$

Let J be a given interval of the real line. Let I_n, $n \in \mathbb{N}$, be sets of indices such that $I_n \subset I_{n+1}$ holds true. We have in mind the variants: I_n finite thus as a model can be chosen $\{0, 1, 2, \ldots, s_n\}$, or I_n is infinite thus our model can be considered $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each integer $n \geq 1$ we consider a net on J namely $(kn^{-\beta})_{k \in I_n}$, where $\beta > 0$ is a fixed real number. We start from a sequence $(L_n)_n$ of linear positive operators of discrete type having the form $(L_n f)(x) = \sum_{k \in I_n} a_{n,k}(x) f(k/n^\beta)$, $x \in J$, where $a_{n,k} \in C(J)$, $a_{n,k} \geq 0$, for every $(n,k) \in \mathbb{N} \times I_n$ and f belongs to a vectorial subspace of $\mathbb{R}^J$ such that the operators are well defined. Setting e_j the j-th monomial, $e_j(t) = t^j$, $j \in \mathbb{N}_0$, for every $n \in \mathbb{N}$ we require the following conditions to be fulfilled

$$\begin{cases} L_n e_0 = e_0, \ L_n e_1 = e_1, \ L_n e_2 = e_2 + \varphi_n, \\ a_{n,k}(x) \leq \phi_n(x), \ k \in I_n, \ x \in Int(J), \end{cases} \tag{2.1}$$

where $\varphi_n \in C(J)$, $\phi_n \in \mathbb{R}_+^J$ are certain functions. These requirements imply that each operator L_n has the degree of exactness 1.

Next, let X be a non constant real random variable on a probability space $(\Omega, \mathcal{F}, P)$. Denoting by ψ its probability density function, we assume that $\psi \in L_2(\mathbb{R})$ and $\mathrm{supp}(\psi) \subset [-\mu, \mu] \cap J$, $\mu > 0$. A bounded compactly supported $\psi \in L_2(\mathbb{R})$ is automatically in $L_1(\mathbb{R})$. Also, one has $\psi \geq 0$ and

$$\|\psi\|_1 = \int_{\mathbb{R}} \psi(t)dt = 1. \tag{2.2}$$

We set $E(X) := e$, $Var(X) := \sigma^2$, the expectation and the variance of X respectively. Starting from X we generate the random variables $X_{n,k}$ defined by

$$X_{n,k} := \frac{1}{n^\beta}(X + k - e), \quad (n,k) \in \mathbb{N} \times I_n. \tag{2.3}$$

Consequently, $P_{X_{n,k}}$, the distribution function of $X_{n,k}$, satisfies $dP_{X_{n,k}} = n^\beta\psi(n^\beta\cdot -k+e)$ and one has $E(X_{n,k}) = k/n^\beta$, representing exactly the mesh of the L_n operator.

Letting $\mathcal{S} := \{f \in \mathbb{R}^{\mathbb{R}} : \ E(|f\circ X_{n,k}|) < \infty \text{ for every } (n,k) \in \mathbb{N}\times I_n\}$, we introduce the operators $\Lambda_n : \mathcal{S} \to C(J)$, $n \in \mathbb{N}$, as follows

$$\Lambda_n f = \sum_{k\in I_n} a_{n,k}E(f\circ X_{n,k}) = \sum_{k\in I_n} a_{n,k}\int_{\Omega} f\circ X_{n,k}dP, \tag{2.4}$$

this meaning $(\Lambda_n f)(x) = n^\beta \sum_{k\in I_n} a_{n,k}(x)\int_{\mathbb{R}} f(t)\psi(n^\beta t-k+e)dt$, $x \in J$.

Following Altomare and Campiti monograph [3; §5.2] this is a positive approximation process generated by a random scheme on J. Since X is non-constant, by examining (2.3) we deduce that for any $(k_1,k_2) \in I_n\times I_n$ the variables X_{n,k_1}, X_{n,k_2} are not independent. All these variables represent scaled versions of the same variable X, they being obtained from it by contractions $(n^{-\beta},\ n\in\mathbb{N})$ and by translations $((k-e)n^{-\beta},\ k\in I_n)$. Moreover, by using (2.1) a simple computation shows us that the operators Λ_n keep the degree of exactness 1 and $\Lambda_n e_2 = e_2 + \varphi_n + \sigma^2/n^{2\beta}$.

The next step is to define the Bézier variant of Λ_n operators. Let α be a real number, $\alpha \geq 1$. We consider the operators $\Lambda_{n,\alpha}$, $n \in \mathbb{N}$, given as follows

$$\begin{aligned}(\Lambda_{n,\alpha} f)(x) &= n^\beta \sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x)\int_{\mathbb{R}} f(t)\psi(n^\beta t-k+e)dt\\ &= \sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x)\int_{\mathrm{supp}(\psi)} \psi(t) f\Big(\frac{t+k-e}{n^\beta}\Big)dt,\end{aligned} \tag{2.5}$$

where

$$Q_{n,k}^{(\alpha)}(x) := S_{n,k}^\alpha(x) - S_{n,k+1}^\alpha(x), \quad S_{n,k}(x) := \sum_{\substack{j\geq k\\ j\in I_n}} a_{n,j}(x), \tag{2.6}$$

for every $x \in J$ and $k \in I_n$.

Based on (2.2) and knowing that $\sum_{k\in I_n} a_{n,k} = 1$, we deduce $S_{n,k} = 1$ for $k \leq inf(I_n)$. If $k > \sup(I_n)$, we agree to take $S_{n,k} = 0$.

In the last years the operators of Bézier-type have also been studied. By using probabilistic tools, X. M. Zeng and W. Chen [19] estimated (1.1) for

Durrmeyer-Bézier operators (instead of q^α appeared as valid $(\alpha+1)^{-1}$). X.-M. Zeng and V. Gupta [20], [11] approached the Bézier variants of Baskakov and Baskakov-Kantorovich operators, respectively. As regards the Bézier variant of Szász-Durrmeyer operators a fruitful investigation has been carried out by U. Abel and V. Gupta [12]. Also a general class of Durrmeyer-Bézier type operators was recently presented in [2]. A similar technique as in [2] will be applied in this study. However, the results aim at different classes of operators and arise from distinct hypotheses.

Remark 2.1 *The operator $\Lambda_{n,\alpha}$ is a linear positive one and it can be written as a singular integral of the type*

$$(\Lambda_{n,\alpha}f)(x) = \int_{\mathbb{R}} K_{n,\alpha}(x,t)f(t)dt, \quad x \in J,$$

with the kernel $K_{n,\alpha}(x,t) := n^\beta \sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x)\psi(n^\beta t - k + e)$, $(x,t) \in J \times \mathbb{R}$. *Clearly,* $\Lambda_{n,1}$ *becomes just* Λ_n *defined by (2.4).*

Remark 2.2 *For all $k \in I_n$, $x \in J$ and $\alpha \geq 1$, the quantities $Q_{n,k}^{(\alpha)}$, $S_{n,k}$, $K_{n,\alpha}$ verify the following direct properties useful in the sequel.*

$$S_{n,k}(x) - S_{n,k+1}(x) = a_{n,k}(x), \quad 0 \leq S_{n,k}(x) \leq 1, \tag{2.7}$$

$$\sum_{k\in I_n} Q_{n,k}^{(\alpha)}(x) = 1, \quad \int_{\mathbb{R}} K_{n,\alpha}(x,t)dt = 1, \tag{2.8}$$

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha a_{n,k}(x). \tag{2.9}$$

Relations (2.7) and (2.8) are implied by (2.6) combined with (2.2). Relation (2.8) means that $\Lambda_{n,\alpha}$, $\alpha \geq 1$, reproduces the constants, in other words $\Lambda_{n,\alpha}e_0 = e_0$. Further on, if $a, b \in [0,1]$ and $\alpha \geq 1$ then $c_{a,b}$ between a, b exists such that $|b^\alpha - a^\alpha| = \alpha|b-a|c_{a,b}^{\alpha-1} \leq \alpha|b-a|$. Based on this inequality, (2.9) is implied by (2.6) and (2.7).

In what follows, as regards the family $(a_{n,k})_k$, we make an additional assumption considering that a real number q, $q > 1$, exists such that

$$\Big|\sum_{k>n^\beta x} a_{n,k}(x) - \frac{1}{q}\Big| \leq \widetilde{a}(n,x), \quad x \in Int(J), \tag{2.10}$$

where $\widetilde{a}(n,\cdot)$ is continuous on $Int(J)$ for each $n \in \mathbb{N}$ and $\widetilde{a}(n,x) = o(1)$ $(n \to \infty)$. Here $o(\cdot)$ represents, as usual, the Landau symbol. In order to justify that this is not an unusual request we deliver the following examples.

Let $J = [0,\infty)$, $I_n = \mathbb{N}_0$ and $\beta = 1$. We consider the Banach lattice

$$E_2 := \left\{ f \in C([0,\infty)) \mid \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}$$

endowed with the norm $\|\cdot\|_*$, $\|f\|_* := \sup\limits_{x\geq 0}(1+x^2)^{-1}|f(x)|$.

Example 2.3 *Choosing* $a_{n,k}(x) = e^{-nx}(nx)^k/k!$, L_n *become Mirakjan-Favard-Szász operators,* $n \in \mathbb{N}$. *The domain* $\mathcal{S}$ *can be considered* E_2. *For these operators X. M. Zeng [18; Lemma 2] has proved that (2.10) is fulfilled with* $q = 2$ *and* $\widetilde{a}(n,x) = \sqrt{1+3x}/\sqrt{nx}$.

Example 2.4 *Choosing* $a_{n,k}(x) = (1+x)^{-n}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k$, L_n *become Baskakov operators,* $n \in \mathbb{N}$, *with* $\mathcal{S} = E_2$. *Again (2.10) holds true with* $q = 2$ *and* $\widetilde{a}(n,x) = (3x+C)/\sqrt{nx(1+x)}$ *in concordance with [20; Lemma 5]. In the above* C *indicates a constant.*

As a matter of fact, following (2.1), for these examples we specify that $\varphi_n(x) = x/n$ and $\varphi_n(x) = (x+x^2)/n$, respectively.

Note that a similar relation as (2.10) was proved for other discrete operators, like as Meyer-König and Zeller, see [21; *Lemma 6*].

Remark 2.5 *We try to construct a true "bridge" between our assumption (2.10) and the approximation processes presenting both a particular approximation process for which (2.10) is not valid and some classes of approximation processes which verify (2.10) for an arbitrary number* $q > 1$.

2.5.1. We consider Lupaş operator [16] defined by

$$(S_n f)(x) = \frac{1}{n}\sum_{k=0}^{n}\left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |t-x|\right]_t f\left(\frac{k}{n}\right),$$

$x \in [0,1]$, $f \in C([0,1])$, where, for mutually distinct numbers a, b, c, we denote by $[a,b,c; f(t,x)]_t$ the fact that the divided difference is applied on the variable t.

Actually, $S_n f$ is the piecewise linear continuous function that interpolates f at the points k/n, $k = \overline{0,n}$. Among the properties of S_n

operator, we mention: it is linear positive, $S_n e_i = e_i$ for $i = 0$ and $i = 1$, $S_n e_2 = e_2 + z_n(1 - z_n)/n^2$ where $z_n(x) := nx - [nx]$, $[\cdot]$ indicating the integral part function. In harmony with (2.10), we have $a_{n,k}(x) := n^{-1}[(k-1)/n, k/n, (k+1)/n; |t-x|]_t$ and further on

$$\sum_{k>nx} a_{n,k}(x) = \frac{1}{2} \sum_{k>nx} \{|k+1-nx| + |k-1-nx| - 2|k-nx|\} = z_n(x).$$

Clearly, for any $q > 1$ the relation $|z_n(x) - q^{-1}| = o(1)$ $(n \to \infty)$ is false. Consequently, the operators S_n, $n \in \mathbb{N}$, satisfy (2.1) but (2.10) is not fulfilled.

2.5.2. Following [8] we recall the notion of *bell-shaped function.* A non negative function b belonging to $L_1(\mathbb{R})$ is named bell-shaped if a real number a exists such that b takes a global maximum in $x = a$, b is non-decreasing on $(-\infty, a)$ and non-increasing on $[a, \infty)$. Here a will be called the *center* of the bell-shaped function. The function b may have jump discontinuities. In what follows we consider that supp$b$ is contained in the interval $[-T, T]$, $T > 0$, and the center of b is $a = 0$. Letting $I^* := \displaystyle\int_{-T}^{T} b(t)dt > 0$, the univariate Cardaliaguet-Euvrard neural network operators are given by the formula

$$(F_n f)(x) = \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I^* n^\alpha} b(n^{1-\alpha}(x - k/n)), \quad x \in \mathbb{R},$$

where $0 < \alpha < 1$ and f is continuous and bounded on $\mathbb{R}$.

Based on [4, *Theorem 2.1*], we obtain that (F_n) is a positive approximation process on $L_1(\mathbb{R}) \cap C_B(\mathbb{R})$ with respect to $L_1(\mathbb{R})$, in the sense of Altomare and Campiti [3; *page 264*].

Letting $a_{n,k}(x) := n^{-\alpha} b(n^{1-\alpha}(x - k/n))/I^*$ and following [4; *Eq.(2.10)*] we define $S_n^1(x) := \displaystyle\sum_{k \geq [nx]+1} a_{n,k}(x)$. It was proved, see [4; *Lemma 2.1*] that $S_n^1(x) \to (I^*)^{-1} \displaystyle\int_{-T}^{0} b(t)dt$ as $n \to \infty$. Comparing this result with relation (2.10), it is enough to choose $q = I^* \Big(\displaystyle\int_{-T}^{0} b(t)dt \Big)^{-1}$. Consequently, for a given $q > 1$ we can define a bell-shaped function b such that the above identity holds true. Actually there are an infinity of such functions b, an easy construction being piecewise constant functions.

3 Notation and preliminary results

Since an affine substitution maps (a,b), $-\infty \le a < b \le \infty$ onto $(0,1)$, $(0,\infty)$ or $\mathbb{R}$, in general is enough to consider these intervals as being $Int(J)$. For the first two situations, we are going to present the rate of pointwise convergence of $\Lambda_{n,\alpha}$ operators for functions $f \in BV$ and satisfying the growth condition $|f(t)| \le M_f(1+|t|^N)$, $t \in \mathbb{R}$, where M_f is a positive constant depending on f. We denote briefly this class by BV_N.

For any real numbers $a < b$ set $J^-(a) := (-\infty, a)$, $J(a,b) := [a,b]$ and $J^+(b) := (b,\infty)$.

Throughout the paper we will use the quantities $u_{n,x} := x - xn^{-\beta}$, $v_{n,x} := x + xn^{-\beta}$, $w_{n,x} := x + (1-x)n^{-\beta}$. Next we define the real functions g_x, $\varepsilon_{x,q}$, $\widehat{g}_{1,x}$, $\widehat{g}_{2,x}$, δ_x as follows

$$g_x(t) = \begin{cases} f(t) - f(x-), & t < x, \\ 0, & t = x, \\ f(t) - f(x+), & t > x, \end{cases} \qquad \varepsilon_{x,q}(t) = \begin{cases} -1, & t < x, \\ 0, & t = x, \\ q^\alpha - 1, & t > x, \end{cases}$$

$$\widehat{g}_{1,x}(t) = \begin{cases} g_x(t), & t \ge 0, \\ g_x(0), & t < 0, \end{cases} \qquad \widehat{g}_{2,x}(t) = \begin{cases} g_x(t), & t \le 2x, \\ g_x(2x), & t > 2x, \end{cases}$$

$$\delta_x(t) = \begin{cases} 1, & t = x, \\ 0, & t \ne x. \end{cases}$$

The size of the saltus of f at $x \in Int(J)$ will be denoted by $s_f(x)$, that is $s_f(x) := f(x+) - f(x-)$. For any integer $s \ge 0$ we introduce the s-order central moment of the operator $\Lambda_{n,\alpha}$, $\mu_{n,s}^{\langle\alpha\rangle} := \Lambda_{n,\alpha}\psi_{x,s}$ where $\psi_{x,s}(t) := (t-x)^s$. If $\alpha = 1$ then these moments will be simply denote by $\mu_{n,s}$. For every $(n,s) \in \mathbb{N} \times \mathbb{N}$ and $\alpha \ge 1$ one has

$$\mu_{n,2s}^{\langle\alpha\rangle} \le \alpha \mu_{n,2s}. \tag{3.1}$$

The proof runs as follows. By using the known inequality $b^\alpha - a^\alpha \le \alpha(b-a)$, $0 \le a, b \le 1$, $\alpha \ge 1$, and taking into account (2.6) we get

$$Q_{n,k}^{(\alpha)}(x) = S_{n,k}^\alpha(x) - S_{n,k+1}^\alpha(x) \le \alpha(S_{n,k}(x) - S_{n,k+1}(x)) = \alpha Q_{n,k}^{(1)}(x).$$

Since $\Lambda_{n,\alpha}$ are linear positive operators, for every $\psi_{x,2s} \ge 0$ relation (3.1) holds true.

Some inequalities involving the kernel $K_{n,\alpha}$ will be read as follows.

Lemma 3.1 *If $x \in Int(J)$ then the following relations hold true*
(i) for each $y < x$,

$$\int\limits_{J^-(y)} K_{n,\alpha}(x,t)dt \le \frac{\alpha\mu_{n,2}(x)}{(x-y)^2}, \tag{3.2}$$

(ii) for each $z > x$,

$$\int\limits_{J^+(z)} K_{n,\alpha}(x,t)dt \le \frac{\alpha\mu_{n,2}(x)}{(z-x)^2}. \tag{3.3}$$

Proof. Let $x \in Int(J)$.
If $y < x$, then $1 \le (t-x)^2(x-y)^{-2}$ for each $t \in J^-(y)$.
If $z > x$, then $1 \le (t-x)^2(z-x)^{-2}$ for each $t \in J^+(z)$.
These inequalities combined with (3.1) lead us to the claimed result. □

Further on, we associate with the kernel $K_{n,\alpha}$ the following mapping

$$\lambda_{n,\alpha}(x,t) := \int\limits_{J^-(t)} K_{n,\alpha}(x,u)du, \quad x \in J. \tag{3.4}$$

We are now in position to enunciate some technical results involving the decomposition of the integral $\int\limits_{\mathbb{R}} g_x(t)K_{n,\alpha}(x,t)dt$.

Lemma 3.2 *If*

$$A_{n,\alpha}(x) := \int\limits_{J^-(u_{n,x})} \widehat{g}_{1,x}(t)K_{n,\alpha}(x,t)dt$$

and

$$D_{n,\alpha}(x) := \int\limits_{J^+(v_{n,x})} \widehat{g}_{2,x}(t)K_{n,\alpha}(x,t)dt,$$

then the following statements hold true

$$|A_{n,\alpha}(x)| \le \frac{\alpha\mu_{n,2}(x)}{x^2}\Big(\bigvee_0^x(g_x) + \sum_{k=1}^{n-1}((k+1)^{2\beta} - k^{2\beta})\bigvee_{u_{k,x}}^x(g_x)\Big), \quad n \ge 2, \tag{3.5}$$

$$|D_{n,\alpha}(x)| \le \frac{\alpha\mu_{n,2}(x)}{x^2}\Big(\bigvee_x^{2x}(g_x) + \sum_{k=1}^{n-1}((k+1)^{2\beta} - k^{2\beta})\bigvee_x^{v_{k,x}}(g_x)\Big), \quad n \ge 2. \tag{3.6}$$

Proof. Recalling (3.4), integrating by parts and using the definition of $\widehat{g}_{1,x}$ we have

$$A_{n,\alpha}(x) = \widehat{g}_{1,x}(t)\lambda_{n,\alpha}(x,t)\Big|_{t\in J^-(u_{n,x})} + \int_{J(u_{n,x})} \lambda_{n,\alpha}(x,t)d_t(-\widehat{g}_{1,x}(t))$$

$$= g_x(u_{n,x}-)\lambda_{n,\alpha}(x,u_{n,x}-) + \int_0^{u_{n,x}} \lambda_{n,\alpha}(x,t)d_t(-g_x(t)). \quad (3.7)$$

We can write $|g_x(u_{n,x}-)| = |g_x(u_{n,x}-)-g_x(x)| \le \bigvee_{u_{n,x}}^{x}(g_x)$. For $t < u_{n,x} < x$, applying (3.2) we get

$$\lambda_{n,\alpha}(x,u_{n,x}-) \le \frac{\alpha\mu_{n,2}(x)}{x^2}n^{2\beta}, \quad \lambda_{n,\alpha}(x,t) \le \frac{\alpha\mu_{n,2}(x)}{(x-t)^2}.$$

Since the mappings $t \mapsto \bigvee_t^x(g_x) \pm g_x(t)$ are decreasing for $t < x$, we have $|d_t(-g_x(t))| \le d_t\Big(-\bigvee_t^x(g_x)\Big)$. Gathering these relations, identity (3.7) implies

$$|A_{n,\alpha}(x)| \le \alpha\mu_{n,2}(x)\Big(n^{2\beta}x^{-2}\bigvee_{u_{n,x}}^{x}(g_x) - \int_0^{u_{n,x}}(x-t)^{-2}d_t\Big(\bigvee_t^x(g_x)\Big)\Big)$$

$$= \alpha\mu_{n,2}(x)\Big(x^{-2}\bigvee_0^x(g_x) + 2\int_0^{u_{n,x}}(x-t)^{-3}\bigvee_t^x(g_x)dt\Big). \quad (3.8)$$

In the last integral making the change $t = x - x/y^{\beta}$, one gets $1 \le y < n$ $(n \ge 2)$ and it becomes

$$\frac{\beta}{x^2}\int_1^n y^{2\beta-1}\bigvee_{x-xy^{-\beta}}^{x}(g_x)dy = \frac{\beta}{x^2}\sum_{k=1}^{n-1}\int_k^{k+1}\bigvee_{x-xy^{-\beta}}^{x}(g_x)y^{2\beta-1}dy$$

$$\le \frac{\beta}{x^2}\sum_{k=1}^{n-1}\int_k^{k+1}\bigvee_{u_{k,x}}^{x}(g_x)y^{2\beta-1}dy = \frac{1}{2x^2}\sum_{k=1}^{n-1}((k+1)^{2\beta}-k^{2\beta})\bigvee_{u_{k,x}}^{x}(g_x).$$

We took into account that $y \in [k,k+1)$ implies $[x-xy^{-\beta},x) \subset [u_{k,x},x)$. Returning to (3.8) we obtain inequality (3.5).

In a similar manner (3.6) can be proved, consequently we omit it. □

Lemma 3.3 *Let $f \in BV_N$ and $Int(J) = (0, \infty)$. If*

$$B_{n,\alpha}(x) := \int\limits_{J^-(0)} (g_x(t) - g_x(0)) K_{n,\alpha}(x,t) dt$$

and

$$E_{n,\alpha}(x) := \int\limits_{J^+(2x)} (g_x(t) - g_x(2x)) K_{n,\alpha}(x,t) dt,$$

then the following statements hold true

$$|B_{n,\alpha}(x)| \leq \alpha M_f \left(x^{-2} \mu_{n,2}(x) + \sqrt{\alpha^{-1} \mu_{n,2N}(x)} \right), \tag{3.9}$$

$$|E_{n,\alpha}(x)| \leq \alpha M_f \left((2 + (2x)^N) x^{-2} \mu_{n,2}(x) + 2^N \sqrt{\alpha^{-1} \mu_{n,2N}(x)} \right). \tag{3.10}$$

Proof. Clearly, both proofs follow the same line. We put the second one in detail. Because of $t > 2x$ and $f \in BV_N$ we obtain

$$|g_x(t) - g_x(2x)| = |f(t) - f(2x)| \leq M_f((1 + t^N) + (1 + 2^N x^N)).$$

Further on,

$$|E_{n,\alpha}(x)| \leq M_f \Big((2 + 2^N x^N) \int\limits_{J^+(2x)} K_{n,\alpha}(x,t) dt + \int\limits_{J^+(2x)} t^N K_{n,\alpha}(x,t) dt \Big).$$

For the first integral we apply (3.3) and for the second we use Schwarz inequality

$$\int\limits_{J^+(2x)} t^N K_{n,\alpha}(x,t) dt \leq 2^N \int\limits_{J^+(2x)} (t-x)^N K_{n,\alpha}(x,t) dt$$

$$\leq 2^N \Big\{ \int\limits_{J^+(2x)} (t-x)^{2N} K_{n,\alpha}(x,t) dt \Big\}^{1/2} \Big\{ \int\limits_{J^+(2x)} K_{n,\alpha}(x,t) dt \Big\}^{1/2}$$

$$\leq 2^N \sqrt{\mu_{n,2N}^{\langle \alpha \rangle}(x)},$$

because of $t > 2x$, $J^+(2x) \subset J$ and (2.8). Relation (3.1) finishes the proof. □

Lemma 3.4 *If* $C_{n,\alpha}(x) := \int\limits_{J(u_{n,x},v_{n,x})} g_x(t)K_{n,\alpha}(x,t)dt$, *then one has*

$$|C_{n,\alpha}(x)| \le \bigvee_{u_{n,x}}^{v_{n,x}} (g_x). \tag{3.11}$$

The same relation is true if we substitute $v_{n,x}$ *by* $w_{n,x}$.

Proof. For $t \in J(u_{n,x}, v_{n,x})$ one has $|g_x(t)| = |g_x(t) - g_x(x)| \le \bigvee_{u_{n,x}}^{v_{n,x}} (g_x)$ and knowing that $0 \le \int\limits_{J(u_{n,x},v_{n,x})} K_{n,\alpha}(x,t)dt \le 1$, inequality (3.11) is proved.
The last assertion of our lemma is evident. □

Lemma 3.5 *Under the hypothesis (2.10) one has*

$$|(\Lambda_{n,\alpha}\varepsilon_{x,q})(x)| \le C_{\alpha,q,\psi}(\widetilde{a}(n,x) + \phi_n(x)), \quad x \in Int(J), \tag{3.12}$$

where $C_{\alpha,q,\psi}$ *depends on* α, q *and the length of the set* $\mathrm{supp}(\psi)$.

Proof. We split I_n into three subsets. $I_{n,1} := \{k \in I_n : \ k < n^\beta x + e - t$ for every $t \in \mathrm{supp}(\psi)\}$, $I_{n,2} := \{k \in I_n : \ k > n^\beta x + e - t$ for every $t \in \mathrm{supp}(\psi)\}$ and $I_{n,3} := I_n \setminus (I_{n,1} \cup I_{n,2})$. Taking into account (2.5), (2.2) and (2.8) we obtain

$$\Lambda_{n,\alpha}\varepsilon_{x,q} = -\sum_{k\in I_{n,1}} Q_{n,k}^{(\alpha)} + (q^\alpha - 1)\sum_{k\in I_{n,2}} Q_{n,k}^{(\alpha)}$$

$$+ \sum_{k\in I_{n,3}} Q_{n,k}^{(\alpha)} \int\limits_{\mathrm{supp}(\psi)} \psi(t)\varepsilon_{x,q}\left(\frac{t+k-e}{n^\beta}\right) dt$$

$$= q^\alpha \sum_{k\in I_{n,2}} Q_{n,k}^{(\alpha)} - 1 + \sum_{k\in I_{n,3}} Q_{n,k}^{(\alpha)}\Big(1 + \int\limits_{\mathrm{supp}(\psi)} \psi(t)\varepsilon_{x,q}\left(\frac{t+k-e}{n^\beta}\right) dt\Big).$$

Letting $k_0 := \min(I_{n,2})$, it is evident that any $k > k_0$, $k \in I_n$, will belong to $I_{n,2}$. Taking also into account (2.6), we get $\sum_{k\in I_{n,2}} Q_{n,k}^{(\alpha)} = S_{n,k_0}^{\alpha}$. Using again the inequality $|a^\alpha - 1| \le \alpha|a-1|$, $\alpha \ge 1$, $a \in [0,1]$, and observing that $|\varepsilon_{x,q}| < q^\alpha$, with the help of (2.6) and (2.9) we can write

$$|\Lambda_{n,\alpha}\varepsilon_{x,q}| \le \alpha q \Big| \sum_{\substack{j\ge k_0\\ j\in I_n}} a_{n,j} - \frac{1}{q}\Big| + \alpha(1+q^\alpha)\sum_{k\in I_{n,3}} a_{n,k}. \tag{3.13}$$

Let $\widetilde{l}_\psi$ be the length of the bounded interval $\mathrm{supp}(\psi)$. The construction of $I_{n,3}$ implies $Card(I_{n,3}) \le \lceil \widetilde{l}_\psi \rceil$, where $\lceil s \rceil$ indicates the ceiling of the number s. If $j \ge k_0$, then $j > n^\beta x + e - t$ for every $t \in \mathrm{supp}(\psi)$. At the same time, for every $t \in \mathrm{supp}(\psi)$ one has $|e-t| = \Big| \int\limits_{\mathrm{supp}(\psi)} (u-t)\psi(u)du \Big| \le \widetilde{l}_\psi$.
All these facts combined with (2.10) and (2.1) allow us to get from (3.13) to (3.12). □

4 Main results

The focus of this section is to present the rate of pointwise convergence of $\Lambda_{n,\alpha}$ operators for functions of bounded variation. At first we discuss the case $Int(J) = (0,\infty)$.

Theorem 4.1 *Let $Int(J) = (0,\infty)$. Let a function f belong to BV_N and $\Lambda_{n,\alpha}$ be defined by (2.5) such that (2.10) is fulfilled. For every real number $\beta > 0$, $x > 0$ and integer $n \ge 2$, the following inequality*

$$|(\Lambda_{n,\alpha}f)(x) - q^{-\alpha}f(x+) - (1-q^{-\alpha})f(x-)| \le \frac{\alpha\mu_{n,2}(x)}{x^2}\Delta_n(\beta,f;x)$$

$$+ \bigvee_{x-x/n^\beta}^{x+x/n^\beta}(g_x) + M_{\alpha,f}\mu_{n,2N}^{1/2}(x) + C_{\alpha,q,\psi}|s_f(x)|(\widetilde{a}(n,x) + \phi_n(x))$$

holds true, where

$$\Delta_n(\beta,f;x) = \bigvee_0^{2x}(g_x) + M_f(3+(2x)^N) + \sum_{k=1}^{n-1}((k+1)^{2\beta} - k^{2\beta}) \bigvee_{x-x/k^\beta}^{x+x/k^\beta}(g_x)$$

and $M_f, M_{\alpha,f}, C_{\alpha,q,\psi}$ are positive constants depending only on the quantities which appear as their indices.

Proof. Having in mind the notation introduced in Section 3, for each $t \in \mathbb{R}$ we can write

$$f(t) = q^{-\alpha}f(x+) + (1-q^{-\alpha})f(x-) + g_x(t) + q^{-\alpha}s_f(x)\varepsilon_{x,q}(t)$$
$$+\delta_x(t)(f(t) - q^{-\alpha}f(x+) - (1-q^{-\alpha})f(x-)).$$

In the above we apply the linear operator $\Lambda_{n,\alpha}$. Since $\Lambda_{n,\alpha}$ reproduces the constants and $\Lambda_{n,\alpha}\delta_x$ is null, we get

$$|(\Lambda_{n,\alpha}f)(x) - q^{-\alpha}f(x+) - (1-q^{-\alpha})f(x-)|$$
$$\le |(\Lambda_{n,\alpha}g_x)(x)| + q^{-\alpha}|s_f(x)||(\Lambda_{n,\alpha}\varepsilon_{x,q})(x)|. \tag{4.1}$$

Further on, examining Lemma 3.2, Lemma 3.3 and Lemma 3.4 as well, we can write $\Lambda_{n,\alpha}g_x = A_{n,\alpha} + B_{n,\alpha} + C_{n,\alpha} + D_{n,\alpha} + E_{n,\alpha}$.

Using the statements of these mentioned lemmas together with Lemma 3.5, after some arrangements, from (4.1) we arrive at the claimed result. □

In order to discuss the case $Int(J) = (0,1)$ we need the function $\widehat{g}_{3,x}$ defined as follows

$$\widehat{g}_{3,x}(t) = \begin{cases} g_x(t), & t \le 1, \\ g_x(1), & t > 1. \end{cases}$$

Lemma 4.2 *Let $f \in BV_N$ and $Int(J) = (0,1)$. If*

$$F_{n,\alpha}(x) := \int_{J^+(w_{n,x})} \widehat{g}_{3,x}(t)K_{n,\alpha}(x,t)dt$$

and

$$G_{n,\alpha}(x) := \int_{J^+(1)} (g_x(t) - g_x(1))K_{n,\alpha}(x,t)dt,$$

then the following statements hold true

$$|F_{n,\alpha}(x)| \le \frac{\alpha\mu_{n,2}(x)}{(1-x)^2}\Big(\bigvee_x^1(g_x) + \sum_{k=1}^{n-1}((k+1)^{2\beta} - k^{2\beta})\bigvee_x^{w_{k,x}}(g_x)\Big), \quad n \ge 2, \tag{4.2}$$

$$|G_{n,\alpha}(x)| \le \alpha M_f\left((3 + 2^{N-1}x^N)(1-x)^{-2}\mu_{n,2}(x) + 2^{N-1}\sqrt{\alpha^{-1}\mu_{n,2N}(x)}\right). \tag{4.3}$$

Proof. At first step, integrating by parts, we have

$$F_{n,\alpha}(x) = \widehat{g}_{3,x}(t)\lambda_{n,\alpha}(x,t)\Big|_{t\in J^+(w_{n,x})} - \int\limits_{J^+(w_{n,x})} \lambda_{n,\alpha}(x,t)d_t(\widehat{g}_{3,x}(t))$$

$$= g_x(1) - g_x(w_{n,x}+)\lambda_{n,\alpha}(x,w_{n,x}+) - \int_{w_{n,x}}^{1} \lambda_{n,\alpha}(x,t)d_t(g_x(t)).$$

Since

$$g_x(1) = g_x(w_{n,x}+) + \int\limits_{w_{n,x}}^{1} d_t(g_x(t)) \text{ and } 1 - \lambda_{n,\alpha}(x,z) = \int\limits_{J^+(z)} K_{n,\alpha}(x,u)du$$

for any real z, we get

$$F_{n,\alpha}(x) = g_x(w_{n,x}+) \int\limits_{J^+(w_{n,x})} K_{n,\alpha}(x,u)du$$

$$+ \int\limits_{w_{n,x}}^{1} \Big(\int\limits_{J^+(t)} K_{n,\alpha}(x,u)du\Big) d_t(g_x(t)).$$

Taking into account the following inequalities

$$|g_x(w_{n,x}+)| = |g_x(w_{n,x}+) - g_x(x)| \le \bigvee_x^{w_{n,x}} (g_x), \quad |d_t(g_x(t))| \le d_t\Big(\bigvee_x^{t}(g_x)\Big)$$

and the relation (3.3) as well, one obtains

$$|F_{n,\alpha}(x)| \le \alpha\mu_{n,2}(x)\Big\{\frac{1}{(w_{n,x}-x)^2}\bigvee_x^{w_{n,x}}(g_x) + \int_{w_{n,x}}^{1} \frac{1}{(t-x)^2} d_t\Big(\bigvee_x^{t}(g_x)\Big)\Big\}$$

$$= \alpha\mu_{n,2}(x)\Big\{\frac{1}{(1-x)^2}\bigvee_x^{1}(g_x) + 2\int_{w_{n,x}}^{1} (t-x)^{-3}\bigvee_x^{t}(g_x)dt\Big\}.$$

In the integral, replacing $t = x+(1-x)z^{-\beta}$ and using $[x, x+(1-x)z^{-\beta}] \subset [x, w_{k,x}]$ for any $z \in [k, k+1]$, $k = \overline{1, n-1}$, $n \ge 2$, we have

$$\int_{w_{n,x}}^{1} \bigvee_x^{t}(g_x)\frac{dt}{(t-x)^3} \le \frac{1}{2(1-x)^2}\sum_{k=1}^{n-1}((k+1)^{2\beta} - k^{2\beta}) \bigvee_x^{w_{k,x}}(g_x).$$

Assembling all relations, we obtain (4.2).

The proof of the second statement follows the same line as the proof of (3.10). There is, though, a slight modification. This time, for $x < 1 < t$, we use the following inequality $t^N \le 2^{N-1}((t-x)^N + x^N)$. □

As regards the case $Int(J) = (0,1)$, we decompose $\Lambda_{n,\alpha} g_x = A_{n,\alpha} + B_{n,\alpha} + C_{n,\alpha} + F_{n,\alpha} + G_{n,\alpha}$, $n \ge 2$, and using relations (3.5), (3.9), (3.11), (4.2), (4.3) we obtain the following result.

Theorem 4.3 *Let $Int(J) = (0,1)$. Let a function f belong to BV_N and $\Lambda_{n,\alpha}$ be defined by (2.5) such that (2.10) is fulfilled. For every real number $\beta > 0$, $x \in (0,1)$ and integer $n \ge 2$, the following inequality*

$$|(\Lambda_{n,\alpha} f)(x) - q^{-\alpha} f(x+) - (1-q^{-\alpha}) f(x-)| \le \alpha \mu_{n,2}(x) \tau(x) \Delta_n(\beta, f; x)$$

$$+ \bigvee_{x-x/n^\beta}^{x+(1-x)/n^\beta} (g_x) + M_{\alpha,f} \mu_{n,2N}^{1/2}(x) + C_{\alpha,q,\psi} |s_f(x)| (\widetilde{a}(n,x) + \phi_n(x))$$

holds true, where

$$\Delta_n(\beta, f; x) = \bigvee_0^1 (g_x) + M_f(4 + 2^{N-1} x^N) + \sum_{k=1}^{n-1} ((k+1)^{2\beta} - k^{2\beta}) \bigvee_{x-x/k^\beta}^{x+(1-x)/k^\beta} (g_x),$$

$\tau(x) = \max\{x^{-2}, (1-x)^{-2}\}$ *and* $M_f, M_{\alpha,f}, C_{\alpha,q,\psi}$ *are positive constants depending only on the quantities which appear as their indices.*

Remark 4.4 *From the point of view of Approximation Theory, we are interested in those sequences $(\Lambda_{n,\alpha})_n$ which form an approximation process, see [3; page 264]. In this respect, it is natural to assume that $(\Lambda_n)_n$ enjoys this property. On the other hand, continuity of g_x at x implies that $\bigvee_{x-z_1}^{x+z_2} (g_x) \to 0$ as $z_1, z_2 \to 0^+$. By the virtue of Theorem 4.1 these facts allow us to state the following.*

If

(i) $\phi_n(x) = o(1)$ $(n \to \infty)$ *and*

(ii) $\mu_{n,2}(x) \sum_{k=1}^{n-1} ((k+1)^{2\beta} - k^{2\beta}) \bigvee_{x-x/k^\beta}^{x+x/k^\beta} (g_x) = o(1)$ $(n \to \infty)$

then $\lim_{n\to\infty} (\Lambda_{n,\alpha} f)(x) = \dfrac{f(x+) + (q^\alpha - 1) f(x-)}{q^\alpha}$ *for every* $f \in BV_N$, *where* $x \in (0,\infty)$.

A similar statement is true for the case $Int(J) = (0,1)$.

Remark 4.5 *If x is a continuity point of f then Theorems 4.1 and 4.3 establish an upper bound of the expression $|(\Lambda_{n,\alpha}f)(x) - f(x)|$. This time $s_f(x) = 0$ and the proved inequalities have a simpler form.*

Particular case. In what follows we consider that the random variable X is uniformly distributed in the interval $I_\theta := (0, 2\theta)$, $\theta > 0$. Since $\psi(t) = (2\theta)^{-1}$ for $t \in I_\theta$ and $\psi(t) = 0$ otherwise, we find $E(X) = \theta$. In this special case, denoting by $\Lambda^*_{n,\alpha}$ the operators defined at (2.5), we get

$$(\Lambda^*_{n,\alpha}f)(x) = \frac{n^\beta}{2\theta} \sum_{k \in I_n} Q^{(\alpha)}_{n,k}(x) \int_{I_{k,n,\theta}} f(y)dy, \quad x \in J, \tag{4.4}$$

where $I_{k,n,\theta} := [(k-\theta)n^{-\beta}, (k+\theta)n^{-\beta}]$, $k \in I_n$.

For $\alpha = 1$, $(\Lambda^*_{n,1})_n$ represents a known class of the Kantorovich type operators. It is self-evident that as discrete operators L_n, see (2.1), should be chosen Bernstein, Szasz or Baskakov operators. For a similar sequence, the rate of pointwise convergence for locally bounded functions f measurable on an interval I was studied by G. Aniol. It is fair to notice that those operators [5; *Eq. (2)*] are constructed by using more general subintervals as ours. However, the established results are not the same because of a different approach. For $\theta = 1/2$ and $\beta = 1$, the operators defined by (4.4) represent the Bézier variant of the classical generalized Kantorovich operators, in a slight modified form.

Consequently, for $\Lambda^*_{n,\alpha}$ our main results given at this section can be applied.

Final remark. I am thankful to the anonymous referee for his pertinent suggestions. Moreover, he indicated a probabilistic look of the operators $\Lambda_{n,\alpha}$ as follows.

$$(\Lambda_{n,\alpha}f)(x) = Ef\left(\frac{X + Z^{(\alpha)}_n(x) - e}{n^\beta}\right),$$

where $Z_n(x)$ is a variable independent of X verifying

$$P(Z^{(\alpha)}_n(x) = k) = P^\alpha(Z_n(x) \geq k) - P^\alpha(Z_n(x) \geq k+1)$$

and $P(Z_n(x) = k) = a_{n,k}(x)$. Certainly, manipulating probabilistic tools, similar results as ours can be given in terms of the numerical characteristics of the above random variables.

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