Existence Theory for Nonlinear Operator Equations of Hammerstein Type in Banach Spaces

Donal O'Regan
Department of Mathematics
National University of Ireland
Galway, Ireland

Radu Precup
Department of Applied Mathematics
"Babeş-Bolyai" University
Cluj, Romania

Abstract

We use topological methods to develop an existence theory for nonlinear operator equations of Hammerstein type in Banach spaces. In particular our theory yields existence results to initial and boundary value problems for functional-differential equations in abstract spaces.

2000 AMS SUBJECT CLASSIFICATIONS: 47H30, 47J35, 45N05.
KEY WORDS: Abstract Hammerstein equation, Nonlinear integral equation, Integro-differential equation, Evolution equation.

1 Introduction

One of the most efficient tools for proving the existence of solutions to the nonlinear operator equation

$$u = N(u) \tag{1.1}$$

in a Banach space X is the following Leray–Schauder type principle of Mönch [10] (see also [6], p 204 and [14], p. 67):

Theorem 1.1 *Let X be a Banach space, K a closed convex sunset of X, U a relatively open subset of K and $N : \overline{U} \to K$ continuous. Assume that for some $u_0 \in U$ the following conditions are satisfied:*

(i) *if* $D \subset \overline{U}$ *is countable and* $D \subset \overline{conv}\left(\{u_0\} \cup N(D)\right)$, *then* $\overline{D}$ *is compact;*

(ii) $u \neq (1-\lambda)\, u_0 + \lambda N(u)$ *on* ∂U *for all* $\lambda \in]0,1[$.

Then N *has at least one fixed point in* $\overline{U}$.

In this paper we are concerned with Eq. (1.1) when $X = L^p(0,T;E)$, and N splits as

$$N = SF,$$

where both operators F and S are nonlinear and

$$\begin{aligned} F &: K \subset L^p(0,T;E) \to L^q(0,T;E), \\ S &: L^q(0,T;E) \to L^p(0,T;E). \end{aligned}$$

Here $0 < T < \infty$ and E is a Banach space. In this case we say that (1.1) is an *abstract Hammerstein type equation*. The results of this paper are as follows. First, from Theorem 1.1, we deduce an existence principle for abstract Hammerstein type equations. Next, as applications, we obtain an existence result of Brezis–Browder type for abstract Hammerstein type equations when F is the Nemyskii's superposition operator associated to a given function $f : [0,T] \times E \to E$, and existence results for equations of Volterra type.

The main assumptions on S and some useful lemmas are those from Couchouron–Kamenski [4] and Couchouron–Precup [5], while the main ideas in proving the existence results are adapted from O'Regan–Precup [15], [13] and O'Regan [11].

In particular:

1. if S is a linear integral operator, we can deduce existence results for functional-integral equations of the form

$$u(t) = h(t) + \int_0^T k(t,s)\, F(u)(s)\, ds. \tag{1.2}$$

Here $S(v)(t) = h(t) + \int_0^T k(t,s)\, v(s)\, ds$; when F is the superposition operator associated to a function f we reobtain some of the results in [13] and [15];

2. if S is the *mild solution operator* of a Cauchy problem, we obtain existence results for functional-differential (in particular, integro-differential) evolution equations of the form

$$\begin{cases} u'(t) \in Au(t) + F(u)(t), \text{ a.e. on } [0,T], \\ u(0) = u_0. \end{cases}$$

Here $A : D(A) \subset E \to 2^E$ is an m-dissipative operator (possibly nonlinear and multi-valued), and for any $v \in L^p(0,T;E)$, $S(v) = u$ the unique solution to the Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + v(t), \text{ a.e. on } [0,T], \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Sufficient conditions for such a S which satisfy conditions (S1)-(S2) below are given in [4]. Thus from our theory we reobtain the existence results in [4]; in addition for this type of problems we provide an alternative approach to that presented in [2], p. 252 and [9];

3. when S is the solution operator associated to a boundary value problem, we derive existence results for perturbed boundary value problems in abstract spaces. For example if for any $v \in L^2(0,T;E)$ we let $S(v)$ be the unique continuous solution u of the Dirichlet problem

$$\begin{cases} u''(t) \in Au(t) + v(t), \text{ a.e. on } [0,T], \\ u(0) = u(T) = 0 \end{cases} \tag{1.4}$$

(in the Hilbert space E, with A maximal monotone and $0 \in A(0)$) [5], then we can derive existence results for the functional-differential equation

$$\begin{cases} u''(t) \in Au(t) + F(u)(t), \text{ a.e. on } [0,T], \\ u(0) = u(T) = 0; \end{cases}$$

4. our results apply to nonlinear partial differential equations.

The results of this paper improve, extend and complement those established in [1], [4], [5], [8], [12], [13], [15] and [16].

2 Preliminaries

Throughout this paper E will be a real Banach space with norm $|.|$. By $(.,.)_+$ we denote the *semi inner product* on E,

$$(u,v)_+ = |u| \lim_{t \to 0^+} t^{-1} (|u + tv| - |u|).$$

Recall that $(u, v + w)_+ \leq (u,v)_+ + (u,w)_+$ and $\left|(u,v)_+\right| \leq |u|\,|v|$ for all $u, v, w \in E$.

Let $0 < T < \infty$. A function $u : [0,T] \to E$ is said to be *strongly measurable* on $[0,T]$ if there exists a sequence of finitely-valued functions u_n with

$$u_n(t) \to u(t) \text{ as } n \to \infty, \text{ a.e. on } [0,T].$$

By $\int_0^T u(t)\,dt$ we mean the Bochner integral of u, assuming it exists. Recall that a strongly measurable function $u(.)$ is Bochner integrable if and only if $|u(.)|$ is Lebesgue integrable.

For any real $p \in [1, \infty)$, we consider the space $L^p(0, T; E)$ of all strongly measurable functions $u : [0, T] \to E$ such that $|u|^p$ is Lebesgue integrable on $[0, T]$. $L^p(0, T; E)$ is a Banach space under the norm

$$|u|_p = \left(\int_0^T |u(s)|^p \, ds \right)^{\frac{1}{p}}.$$

Also for $p = \infty$, we let $L^\infty(0, T; E)$ be the space of all strongly measurable function $u : [0, T] \to E$ which are essentially bounded, i.e.,

$$\operatorname{ess} \sup_{t \in [0,T]} |u(t)| := \inf \{a \geq 0 : \; |u(t)| \leq a \text{ a.e. on } [0, T]\} < \infty.$$

$L^\infty(0, T; E)$ is a Banach space under the norm $|u|_\infty = \operatorname{ess} \sup_{t \in [0,T]} |u(t)|$. When $E = \mathbf{R}$, the space $L^p(0, T; \mathbf{R})$ is simply denoted by $L^p(0, T)$. By $|.|_\infty$ we also denote the max-norm on the space $C(0, T; E)$ of all continuous functions $u : [0, T] \to E$.

Next we state a *Weak Compactness Criterion* in $L^p(0, T; E)$ which follows from the results of Diestel–Ruess–Schachermayer [7].

Theorem 2.1 *Let $p \in [1, \infty[$. Let $M \subset L^p(0, T; E)$ be countable and there exists a $\nu \in L^p(0, T)$ with $|u(t)| \leq \nu(t)$ a.e. on $[0, T]$ for all $u \in M$. If $M(t)$ is relatively compact in E for a.e. $t \in [0, T]$, then M is weakly relatively compact in $L^p(0, T; E)$.*

Recall the definition of the *Kuratowski measure of noncompactness* and the *Hausdorff ball measure of noncompactness* of E. For any bounded set $D \subset E$ let

$$\begin{aligned} \alpha(D) &= \inf \left\{ \varepsilon > 0 : \; D \subset \bigcup_{k=1}^{m} D_k, \; \operatorname{diam}(D_k) \leq \varepsilon \right\}, \\ \beta(D) &= \inf \left\{ \varepsilon > 0 : \; D \subset \bigcup_{k=1}^{m} B_\varepsilon(u_k), \text{ where } u_k \in E \right\}. \end{aligned}$$

Here $\operatorname{diam}(D) = \sup\{|u - v| : \; u, v \in D\}$ and $B_\varepsilon(u) = \{v \in E : \; |u - v| < \varepsilon\}$. When it will be necessary to avoid any confusion, we shall write α_E, β_E instead of α and β, respectively. Recall

$$\beta(D) \leq \alpha(D) \leq 2\beta(D) \quad \text{for } D \subset E \text{ bounded}$$

and

$$\beta_E(D) \leq \beta_{E_0}(D) \leq \alpha_{E_0}(D) = \alpha_E(D) \quad \text{for } D \subset E_0 \text{ bounded,}$$

and any linear subspace E_0 of E.

We have the following result concerning the interchange between the measure of noncompactness and the integral.

Theorem 2.2 (Heinz) *(a) If E is a separable Banach space and $M \subset L^1(0,T;E)$ countable with $|u(t)| \leq \nu(t)$ for a.e. $t \in [0,T]$ and every $u \in M$, where $\nu \in L^1(0,T)$, then the function $\psi(t) = \beta(M(t))$ belongs to $L^1(0,T)$ and*

$$\beta\left(\int_0^T M(s)\,ds\right) \leq \int_0^T \beta(M(s))\,ds.$$

(b) If E is a Banach space (not necessarily separable) and $M \subset L^1(0,T;E)$ countable with $|u(t)| \leq \nu(t)$ for a.e. $t \in [0,T]$ and every $u \in M$, where $\nu \in L^1(0,T)$, then the function $\varphi(t) = \alpha(M(t))$ belongs to $L^1(0,T)$ and satisfies

$$\alpha\left(\int_0^T M(s)\,ds\right) \leq 2\int_0^T \alpha(M(s))\,ds.$$

Notice since $|u(t)| \leq \nu(t)$ for every $u \in M$, one has $\beta(M(t)) \leq \nu(t)$; thus, if $\nu \in L^p(0,T)$ then the function ψ in Theorem 2.2 (a) belongs to $L^p(0,T)$. The same is true for the function φ in Theorem 2.2 (b).

We shall use the following definition. A map $\psi : [0,T] \times D \to Y$, where $D \subset X$ and $(X, |.|_X)$, $(Y, |.|_Y)$ are two Banach spaces, is said to be (q,p)-*Carathéodory* $(1 \leq q \leq \infty,\ 1 \leq p \leq \infty)$ if

(C1) $\psi(.,x)$ is strongly measurable for each $x \in D$,

(C2) $\psi(t,.)$ is continuous for a.e. $t \in [0,T]$,

(C3) (a) if $1 \leq p < \infty$: there exists $a \in L^q(0,T;\mathbf{R}_+)$ and $b \in \mathbf{R}_+$ such that

$$|\psi(t,x)|_Y \leq a(t) + b\,|x|_X^p$$

a.e. on $[0,T]$ for all $x \in D$;

(b) if $p = \infty$: for each $r > 0$ there exists $a_r \in L^q(0,T;\mathbf{R}_+)$ such that

$$|\psi(t,x)|_Y \leq a_r(t)$$

a.e. on $[0,T]$ for all $x \in D$ with $|x|_X \leq r$.

(C1)-(C2) are said to be *Carathéodory conditions.*

By $\mathcal{L}(E)$ we shall denote the space of all linear continuous operators from E to E, endowed with the norm $|.|_{\mathcal{L}(E)}$.

For a function $u : [0,T] \to E$ and a positive number r, the notations $(|u| \leq r)$ and $(|u| \geq r)$ stand for the level sets $\{t \in [0,T] : |u(t)| \leq r\}$ and $\{t \in [0,T] : |u(t)| \geq r\}$, respectively.

3 The Existence Principle

Let $0 < T < \infty$, E a real Banach space with norm $|.|$, $p \in [1, \infty]$ and $q \in [1, \infty[$. Let $r \in]1, \infty]$ be the conjugate exponent of q, that is $\frac{1}{q} + \frac{1}{r} = 1$. Consider the abstract Hammerstein type equation

$$u = SF(u), \quad u \in K, \tag{3.1}$$

in a closed convex subset K of $L^p(0,T;E)$. Here

$$F : K \to L^q(0,T;E), \quad S : L^q(0,T;E) \to K.$$

We look for solutions in a bounded subset $\overline{U}$ of K, with U open in K.

Theorem 3.1 *Assume that the following conditions are satisfied:*

(S1) *there exists a function* $k : [0,T]^2 \to \mathbf{R}_+$ *such that* $k \in L^p(0,T;L^r(0,T))$ *(i.e.,* $k(t,.) \in L^r(0,T)$ *, and the function* $t \longmapsto |k(t,.)|_r$ *belongs to* $L^p(0,T)$*), and*

$$|S(w_1)(t) - S(w_2)(t)| \leq \int_0^T k(t,s)\,|w_1(s) - w_2(s)|\,ds \tag{3.2}$$

a.e. on $[0,T]$ *, for all* $w_1, w_2 \in L^q(0,T;E)$ *;*

(S2) *for every compact convex subset* C *of* E*,* S *is sequentially continuous from* $L^q_w(0,T;C)$ *to* $L^p(0,T;E)$ *. Here* $L^q_w(0,T;C)$ *stands for the set* $L^q(0,T;C)$ *endowed with the weak topology of* $L^q(0,T;E)$ *;*

(F1) F *is a continuous and for any constant* $a \geq 0$ *there exists* $\nu_a \in L^q(0,T)$ *such that for any* $u \in K$ *with* $|u|_p \leq a$ *we have* $|F(u)(t)| \leq \nu_a(t)$ *a.e. on* $[0,T]$*;*

(F2) *for every separable closed subspace* E_0 *of* E, *there exists a mapping* $\Psi : L^p(0,T;\mathbf{R}_+) \to L^q(0,T;\mathbf{R}_+)$ *such that* $\Psi(0) = 0$ *and*

$$\beta_{E_0}(F(M)(t) \cap E_0) \leq \Psi(\beta_{E_0}(M(.)))(t) \tag{3.3}$$

a.e. on $[0,T]$ *for every countable set* $M \subset \overline{U}$ *with* $M(t) \subset E_0$ *a.e. on* $[0,T]$, *for which there exists* $\nu \in L^p(0,T)$ *with* $|u(t)| \leq \nu(t)$ *a.e. on* $[0,T]$ *for any* $u \in M$. *In addition* $\varphi = 0$ *is the unique solution in* $L^p(0,T;\mathbf{R}_+)$ *to the inequality*

$$\varphi(t) \leq \int_0^T k(t,s)\,\Psi(\varphi)(s)\,ds \quad a.e.\ on\ [0,T]; \tag{3.4}$$

(L–S) *there exists an element* $u_0 \in U$ *such that*

$$u \neq (1-\lambda)u_0 + \lambda SF(u)$$

on ∂U *for any* $\lambda \in]0,1[$.

Then (3.1) has at least one solution in $\overline{U}$.

Proof. We apply Theorem 1.1 to $X = L^p(0,T;E)$, $Y = L^q(0,T;E)$ and $N = SF$. First notice that from (3.2) we have

$$|S(w_1) - S(w_2)|_p \leq ||k(t,.)|_r|_p\, |w_1 - w_2|_q,$$

which shows that S is continuous (in fact Lipschitz continuous). Consequently, also using the continuity of F assumed in (F1), we have that SF is continuous. Next in order to prove that condition (i) holds we use the following two lemmas from [5].

Lemma 3.1 *Let* $S : L^q(0,T;E) \to L^p(0,T;E)$ *satisfies (S1)-(S2),* $q \in [1,\infty[$ *and* $p \in [1,\infty]$. *Let* $M \subset L^q(0,T;\,E)$ *be countable with*

$$|u(t)| \leq \nu(t)$$

a.e. on $[0,T]$, *for all* $u \in M$, *where* $\nu \in L^q(0,T)$. *Let* E_0 *be a separable closed subspace of* E *with* $u(t) \in E_0$ *a.e. on* $[0,T]$, *for every* $u \in M \cup S(M)$. *Then the function* $\varphi(t) = \beta_{E_0}(M(t))$ *belongs to* $L^q(0,T)$ *and satisfies*

$$\beta_{E_0}(S(M)(t)) \leq \int_0^T k(t,s)\,\varphi(s)\,ds \quad a.e.\ on\ [0,T].$$

Lemma 3.2 *Assume (S1) and (S2). Let M be a countable subset of $L^q(0,T;E)$ such that $M(t)$ is relatively compact for a.e. $t \in [0,T]$ and there is a function $\nu \in L^q(0,T)$ with $|u(t)| \le \nu(t)$ a.e. on $[0,T]$, for every $u \in M$. Then the set $S(M)$ is relatively compact in $L^p(0,T;E)$. In addition S is continuous from M equipped with the relative weak topology of $L^q(0,T;E)$ to $L^p(0,T;E)$ equipped with the strong topology.*

Assume $D \subset \overline{U}$ is countable and

$$D \subset \overline{\text{conv}}\left(\{u_0\} \cup SF(D)\right). \tag{3.5}$$

Since both $F(D)$ and $SF(D)$ are countable sets of strongly measurable functions, we may assume that their values as well as the values of u_0 belong to a separable closed subspace E_0 of E. Clearly the same is true for the values of all functions in $\overline{\text{conv}}\left(\{u_0\} \cup SF(D)\right)$, and so for the values of all functions in D. Since U is bounded there exists $a \ge 0$ with $|u|_p \le a$ for any $u \in \overline{U}$. Then from (F1) we deduce that $|F(u)(t)| \le \nu_a(t)$ a.e. on $[0,T]$, for every $u \in \overline{U}$ and some $\nu_a \in L^q(0,T)$. Now according to Lemma 3.1, the function $\beta_{E_0}(F(D)(t))$ belongs to $L^q(0,T)$ and

$$\beta_{E_0}(SF(D)(t)) \le \int_0^T k(t,s)\, \beta_{E_0}(F(D)(s))\, ds \ \text{ a.e. on } [0,T]. \tag{3.6}$$

Also, for any $u \in \overline{U}$ we have

$$\begin{aligned} |SF(u)(t)| &\le |S(0)(t)| + \int_0^T k(t,s)\, |F(u)(s)|\, ds \\ &\le |S(0)(t)| + |\nu_a|_q\, |k(t,.)|_r =: \widetilde{\nu}(t), \end{aligned}$$

with $\widetilde{\nu} \in L^p(0,T)$. Then using (3.5) we deduce that the function φ given by $\varphi(t) = \beta_{E_0}(D(t))$ belongs to $L^p(0,T;\mathbf{R}_+)$. Now (3.5), (3.6) and (3.3) imply

$$\begin{aligned} \beta_{E_0}(D(t)) &\le \beta_{E_0}(SF(D)(t)) \le \int_0^T k(t,s)\, \beta_{E_0}(F(D)(s))\, ds \quad (3.7) \\ &\le \int_0^T k(t,s)\, \Psi\left(\beta_{E_0}(D(.))\right)(s)\, ds. \end{aligned}$$

Then by (F2), $\varphi = 0$. Now (3.3) together with $\Psi(0) = 0$ guarantees $\beta_{E_0}(F(D)(t)) = 0$. Let $(u_k)_{k \ge 1}$ be any sequence of elements of D. By the Weak Compactness Criterion $(F(u_k))_{k \ge 1}$ has a weakly convergent subsequence in $L^q(0,T;E)$. Next by Lemma 3.2 the corresponding subsequence of $(SF(u_k))_{k\geq 1}$ converges in $L^p(0,T;E)$. Hence $SF(D)$ is relatively compact in $L^p(0,T;E)$. Now Mazur's Lemma guarantees $\overline{\text{conv}}\,(\{u_0\}\cup SF(D))$ is compact and so $\overline{D}$ is compact too. Thus Theorem 1.1 applies and the proof is complete. ■

Remark 3.1 *(a) If the values of S are in $C(0,T;E)$ then any solution of (3.1) in $K\subset L^p(0,T;E)$ $(1\leq p\leq\infty)$ belongs to $C(0,T;E)$.*

(b) The existence theory in $C(0,T;E)$ appears as a particular case, where $p=\infty$ and $K\subseteq C(0,T;E)$.

Remark 3.2 *Assume $q\leq p$ and $\Psi(\varphi)(t)=\delta(t)\varphi(t)$ for all $t\in[0,T]$ and $\varphi\in L^p(0,T;\mathbf{R}_+)$, where $\delta\in L^{\frac{pq}{p-q}}(0,T)$. Here $\frac{pq}{p-q}=q$ if $p=\infty$ and $\frac{pq}{p-q}=\infty$ if $p=q$. Then $\varphi=0$ is the unique solution in $L^p(0,T;\mathbf{R}_+)$ of (3.4) if*

$$|\delta|_{\frac{pq}{p-q}}\left||k(t,.)|_r\right|_p<1. \tag{3.8}$$

Indeed, from

$$\varphi(t)\leq\int_0^T k(t,s)\,\delta(s)\,\varphi(s)\,ds \tag{3.9}$$

and $\frac{1}{r}+\frac{p-q}{pq}+\frac{1}{p}=1$, by Hölder's inequality one has

$$|\varphi|_p\leq\left||k(t,.)|_r\right|_p|\delta|_{\frac{pq}{p-q}}|\varphi|_p,$$

whence by (3.8) $|\varphi|_p=0$ and so $\varphi=0$.

Remark 3.3 *Under the assumptions of Remark 3.2, if in addition S is of Volterra type, i.e., $k(t,s)=0$ for $t<s$, then condition (3.8) for guaranteeing $\varphi=0$ is superfluous.*

Indeed, if S is of Volterra type, (3.9) can be written as

$$\varphi(t)\leq\int_0^t k(t,s)\,\delta(s)\,\varphi(s)\,ds.$$

Hence

$$\varphi(t)\leq|k(t,.)|_r\,|\delta|_{\frac{pq}{p-q}}\left(\int_0^t\varphi(s)^p\,ds\right)^{\frac{1}{p}}.$$

Then

$$\int_0^t\varphi(\tau)^p\,d\tau\leq c\int_0^t\left(|k(\tau,.)|_r^p\int_0^\tau\varphi(s)^p\,ds\right)d\tau.$$

Now Gronwall's inequality guarantees $\int_0^t\varphi(s)^p\,ds=0$ for all $t\in[0,T]$. So $\varphi=0$.

Remark 3.4 *Assume S is of Volterra type and*

$$\Psi(\varphi)(t) = \int_0^t \delta(s)\varphi(s)\,ds$$

for all $t \in [0,T]$ and $\varphi \in L^p(0,T;\mathbf{R}_+)$, where $\delta \in L^{p'}(0,T;\mathbf{R}_+)$ with $p' > \frac{p}{p-1}$. Here $\frac{p}{p-1} = 1$ if $p = \infty$ and $\frac{p}{p-1} = \infty$ if $p = 1$. Then $\varphi = 0$ is the unique solution in $L^p(0,T;\mathbf{R}_+)$ of (3.4)

Indeed, for any $\theta > 0$, from (3.4) we have

$$\begin{aligned}\varphi(t) &\leq \int_0^t k(t,s)\int_0^s \delta(\tau)\varphi(\tau)\,d\tau ds \\ &= \int_0^t k(t,s)\int_0^s \delta(\tau)e^{\theta\tau}\varphi(\tau)e^{-\theta\tau}\,d\tau ds.\end{aligned}$$

Hölder's inequality for $\frac{1}{p'} + \frac{p'(p-1)-p}{pp'} + \frac{1}{p} = 1$ yields

$$\begin{aligned}\varphi(t) &\leq \int_0^t k(t,s)\,|\delta|_{p'}\left|\varphi e^{-\theta\tau}\right|_p \left(\int_0^s e^{\frac{\theta pp'\tau}{p'(p-1)-p}}d\tau\right)^{\frac{p'(p-1)-p}{pp'}} ds \\ &= |\delta|_{p'}\left|\varphi e^{-\theta\tau}\right|_p \left(\frac{p'(p-1)-p}{\theta pp'}\right)^{\frac{p'(p-1)-p}{pp'}} \int_0^t k(t,s)\,e^{\theta s}ds.\end{aligned}$$

Use again Hölder's inequality to obtain

$$\varphi(t) \leq |\delta|_{p'}\left|\varphi e^{-\theta\tau}\right|_p \left(\frac{p'(p-1)-p}{\theta pp'}\right)^{\frac{p'(p-1)-p}{pp'}} |k(t,.)|_r \left(\frac{1}{\theta q}\right)^{\frac{1}{q}} e^{\theta t}.$$

Then

$$\left|\varphi e^{-\theta\tau}\right|_p \leq |\delta|_{p'}\left|\varphi e^{-\theta\tau}\right|_p \left(\frac{p'(p-1)-p}{\theta pp'}\right)^{\frac{p'(p-1)-p}{pp'}} \left(\frac{1}{\theta q}\right)^{\frac{1}{q}} \left||k(t,.)|_r\right|_p.$$

Clearly we can choose $\theta > 0$ so large that the above inequality implies $\left|\varphi e^{-\theta\tau}\right|_p = 0$, whence $\varphi = 0$.

4 Applications

In this section we are mainly concerned with the Leray–Schauder boundary condition (L–S) in Theorem 3.1.

4.1 A Brezis–Browder Type Result

First we present a Brezis–Browder [3] type result for the case where F is the Nemytskii's superposition operator associated to a given function $f : [0,T] \times E \to E$, given by

$$F(u)(t) = f(t, u(t)). \tag{4.1}$$

The result extends a theorem established in [11].

Theorem 4.1 *Assume (S1)-(S2) hold. In addition assume the following conditions:*

(f1) *f satisfies the Carathéodory conditions;*

(f2) *for every separable closed subspace E_0 of E, there exists a $(q, p/q)$-Carathéodory function $\psi : [0,T] \times \mathbf{R}_+ \to \mathbf{R}_+$ such that $\psi(t, 0) = 0$ and*

$$\beta_{E_0}(f(t, M) \cap E_0) \leq \psi(t, \beta_{E_0}(M))$$

a.e. on $[0,T]$ for every bounded countable set $M \subset E_0$. In addition $\varphi = 0$ is the unique solution in $L^p(0,T; \mathbf{R}_+)$ to the inequality

$$\varphi(t) \leq \int_0^T k(t,s)\, \psi(s, \varphi(s))\, ds \ \text{ a.e. on } [0,T];$$

(B1) *there exists $R_0 > 0$ and $a_0 > 0$ such that*

$$(f(t,u), u)_+ \geq a_0 |u| \, |f(t,u)|$$

a.e. on $[0,T]$ for all $u \in E$ with $|u| \geq R_0$;

(B2) *there exists $\gamma \geq \max\left\{q-1, (p-1)^{-1}\right\}$, $\eta > 0$ and $\phi \in L^r(0,T; \mathbf{R}_+)$ such that*

$$\eta\, |f(t,u)|^\gamma \leq \phi(t) + |u|$$

a.e. on $[0,T]$ for all $u \in E$ with $|u| \geq R_0$;

(B3) *there exists $h \in L^p(0,T;E)$ and $B_0 \in \mathbf{R}_+$ with*

$$\int_0^T (v(t), S(v)(t) - h(t))_+ \, dt \leq B_0 \tag{4.2}$$

for all $v \in L^q(0,T;E)$.

Then there exists a solution $u \in K$ to the equation $u(.) = S(f(., u(.)))$.

Proof. Notice first that from (f1), (B2) and $\gamma \geq \max\left\{q-1, \frac{1}{p-1}\right\}$ it follows that f is a $\left(\gamma+1, \frac{p}{\gamma+1}\right)$-Carathéodory function. Also any $\left(\gamma+1, \frac{p}{\gamma+1}\right)$-Carathéodory function is $\left(q, \frac{p}{q}\right)$-Carathéodory since $\gamma+1 \geq q$ and so $L^{\gamma+1}(0,T;E) \subset L^q(0,T;E)$. Consequently, it is easy to see that (F1)-(F2) hold. Here

$$\Psi(\varphi)(t) = \psi(t, \varphi(t)).$$

We shall prove the existence of a number $R > 0$ such that

$$|u|_p < R \tag{4.3}$$

for any solution $u \in K$ of the equation

$$u = \lambda SF(u) \tag{4.4}$$

and any $\lambda \in]0,1[$. Recall that here F is given in (**??**). This will guarantee (L–S) for $U = \left\{u \in K : |u|_p < R\right\}$ and $u_0 = 0$.

Let u be any solution of (4.4) for some $\lambda \in]0,1[$. Then using (3.2) and Hölder's inequality we obtain

$$\begin{aligned} |u(t)| &\leq |SF(u)(t) - S(0)(t)| + |S(0)(t)| \\ &\leq \int_0^T k(t,s)|F(u)(s)|\,ds + |S(0)(t)| \\ &\leq \alpha(t) + |k(t,.)|_r |F(u)|_q. \end{aligned}$$

Here $\alpha(t) = |S(0)(t)|$. It follows that there is a constant $b > 0$ not depending of u such that

$$\begin{aligned} |u|_p &\leq |\alpha|_p + ||k(t,.)|_r|_p |F(u)|_q \\ &\leq |\alpha|_p + b|F(u)|_{\gamma+1}. \end{aligned} \tag{4.5}$$

On the other hand,

$$\begin{aligned} |F(u)|_{\gamma+1} &= \left(\int_0^T |F(u)(t)|^{\gamma+1}\,dt\right)^{\frac{1}{\gamma+1}} \\ &\leq c + \left(\int_{(|u|\geq R_0)} |F(u)(t)|^{\gamma+1}\,dt\right)^{\frac{1}{\gamma+1}}, \end{aligned}$$

where $c>0$ is independent of u. Returning to (4.5) we deduce

$$|u|_p \leq a + b\left(\int_{(|u|\geq R_0)} |F(u)(t)|^{\gamma+1}\, dt\right)^{\frac{1}{\gamma+1}}. \tag{4.6}$$

Here $a = |\alpha|_p + bc$. Next we shall estimate the last term in the above inequality. From (B3) we have

$$\begin{aligned}
\int_0^T (F(u), u)_+\, dt &= \int_0^T (F(u), \lambda SF(u))_+\, dt \\
&= \lambda \int_0^T (F(u), SF(u) - h)_+\, dt + \lambda \int_0^T (F(u), h)_+\, dt \\
&\leq B_0 + \int_0^T |h|\, |F(u)|\, dt.
\end{aligned}$$

Also from (B1) and (B2), one has

$$\begin{aligned}
\int_{(|u|\geq R_0)} (F(u), u)_+\, dt &\geq a_0 \int_{(|u|\geq R_0)} |u|\, |F(u)|\, dt \\
&\geq a_0\eta \int_{(|u|\geq R_0)} |F(u)|^{\gamma+1}\, dt - a_0 \int_{(|u|\geq R_0)} \phi\, |F(u)|\, dt.
\end{aligned}$$

Consequently

$$\begin{aligned}
& a_0\eta \int_{(|u|\geq R_0)} |F(u)|^{\gamma+1}\, dt \\
\leq\ & a_0 \int_{(|u|\geq R_0)} \phi\, |F(u)|\, dt \\
& + B_0 + \int_0^T |h|\, |F(u)|\, dt + \int_{(|u|\leq R_0)} |u|\, |F(u)|\, dt \\
\leq\ & B_1 + a_0 \int_{(|u|\geq R_0)} \phi\, |F(u)|\, dt + \int_{(|u|\geq R_0)} |h|\, |F(u)|\, dt,
\end{aligned}$$

with some constant B_1. Apply Hölder's inequality taking into account the inclusions $L^r(0,T;E) \subset L^{\frac{\gamma+1}{\gamma}}(0,T;E)$, $L^p(0,T;E) \subset L^{\frac{\gamma+1}{\gamma}}(0,T;E)$, to obtain

$$\begin{aligned} & a_0\eta \int_{(|u|\geq R_0)} |F(u)|^{\gamma+1} dt \\ \leq \quad & B_1 + a_0 |\phi|_{(\gamma+1)/\gamma} \left(\int_{(|u|\geq R_0)} |F(u)|^{\gamma+1} dt \right)^{\frac{1}{\gamma+1}} \\ & + |h|_{(\gamma+1)/\gamma} \left(\int_{(|u|\geq R_0)} |F(u)|^{\gamma+1} dt \right)^{\frac{1}{\gamma+1}} . \end{aligned}$$

It follows that there exists a constant B_2 with

$$\int_{(|u|\geq R_0)} |F(u)|^{\gamma+1} dt \leq B_2.$$

Returning to (4.6) we obtain

$$|u|_p \leq a + bB_2^{\frac{1}{\gamma+1}} =: R_1.$$

Finally take any $R > R_1$ to guarantee the strict inequality (4.3). ■

Remark 4.1 *For Eq. (1.2), condition (4.2) with $B_0 = 0$ says that the kernel k is of negative type. It is of interest to note that such a condition also holds for the solution operator associated with (1.4).*

Indeed, if u solves (1.4) then $v = u'' - w$, where $w(t) \in Au(t)$. Then, since A is monotone and $0 \in A(0)$, we have that $(w(t), u(t)) \geq 0$, and so

$$\begin{aligned} (v(t), u(t)) &= \big(u''(t) - w(t), u(t)\big) \\ &= -\left|u'(t)\right|^2 + \frac{1}{2}\left(|u(t)|^2\right)'' - (w(t), u(t)) \\ &\leq \frac{1}{2}\left(|u(t)|^2\right)''. \end{aligned}$$

Integrate over $[0, T]$ using $\left(|u(t)|^2\right)' = 2(u(t), u'(t))$, to deduce

$$\int_0^T (v(t), S(v)(t))\, dt = \int_0^T (v(t), u(t))\, dt \leq 0,$$

Hence (4.2) holds with $h(t) = 0$ and $B_0 = 0$.

4.2 An Existence Result for Volterra Type Equations

The second application concerns Eq. (3.1) with S of Volterra type. The result extends a theorem proved in [5].

Theorem 4.2 *Assume (S1), (S2), (F1), (F2) hold with* $1 \leq q = p < \infty$, $k \in L^p\left(0, T; L^{r'}(0,T)\right)$ *for some* $r' > r$, *and* S *is of Volterra type. In addition assume that*

(F3) *one has*

$$|F(u)(t)| \leq a(t) + b|u(t)| + \int_0^t c(s)|u(s)|\,ds$$

a.e. on $[0,T]$ *for any* $u \in K$. *Here* $a \in L^p(0,T;\mathbf{R}_+)$, $b \in \mathbf{R}_+$ *and* $c \in L^{r'}(0,T;\mathbf{R}_+)$.

Then (3.1) has solutions in K.

Proof. We shall apply Theorem 3.1 to $U = \{u \in K : \|u\| < R\}$ for any $R > |S(0)|_p$ and a suitable equivalent norm $\|.\|$ on $L^p(0,T;E)$.

Let $u \in K$ be any solution of (4.4) for some $\lambda \in]0,1[$. Then for any $\theta > 0$ we have

$$\begin{aligned}
|u(t)| &\leq \lambda|S(0)(t)| + \lambda\int_0^t k(t,s)|F(u)(s)|\,ds \qquad (4.7)\\
&\leq \lambda|S(0)(t)| + \lambda\int_0^t k(t,s)\left(a(s) + b|u(s)| + \int_0^s c(\tau)|u(\tau)|\,d\tau\right)ds\\
&\leq \lambda|S(0)(t)|\\
&\quad +\lambda\int_0^t k(t,s)\left(e^{\theta s}\left(a + b|u|e^{-\theta s}\right) + \int_0^s ce^{\theta\tau}e^{-\theta\tau}|u|\,d\tau\right)ds.
\end{aligned}$$

Define an equivalent norm on $L^p(0,T;E)$ by $\|u\| = \left|e^{-\theta t}u(t)\right|_p$. Then, since $\frac{1}{r'} + \frac{r'-r}{rr'} + \frac{1}{p} = 1$, Hölder's inequality guarantees

$$\begin{aligned}
&\int_0^t k(t,s)e^{\theta s}\left(a(s) + b|u(s)|e^{-\theta s}\right)ds \qquad (4.8)\\
\leq\ & |k(t,.)|_{r'}\left(|a|_p + b\|u\|\right)\left(\int_0^t e^{\frac{\theta rr'}{r'-r}s}ds\right)^{\frac{r'-r}{rr'}}\\
\leq\ & |k(t,.)|_{r'}\left(|a|_p + b\|u\|\right)\left(\frac{r'-r}{\theta rr'}\right)^{\frac{r'-r}{rr'}} e^{\theta t}.
\end{aligned}$$

Also

$$\begin{aligned} & \int_0^t k(t,s) \int_0^s c(\tau) e^{\theta\tau} e^{-\theta\tau} |u(\tau)| \, d\tau ds & (4.9) \\ \leq \; & |c|_{r'} \|u\| \left(\frac{r'-r}{\theta r r'} \right)^{\frac{r'-r}{rr'}} \int_0^t k(t,s) e^{\theta s} ds \\ \leq \; & |c|_{r'} \|u\| \left(\frac{r'-r}{\theta r r'} \right)^{\frac{r'-r}{rr'}} \left(\frac{1}{\theta r'} \right)^{\frac{1}{r'}} e^{\theta t}. \end{aligned}$$

From (4.7), (4.8) and (4.9) it follows that we may choose a sufficiently large $R > 0$ such that $\|u\| < R$. ■

The next result contains an example of operator F for which (F1)-(F3) hold, and is extremely useful when discussing existence for integro-differential equations.

Theorem 4.3 *Assume (S1)-(S2) hold with* $1 \leq q = p < \infty$, $k \in L^p(0,T; L^{r'}(0,T))$ *for some* $r' > r$, *and* S *is of Volterra type. Let* $\kappa \in L^{r'}(0,T; \mathcal{L}(E))$ *and* $P : E \to E$ *be continuous such that*

$$|P(u)| \leq a_0 + b_0 |u|$$

for all $u \in E$ *and some* $a_0, b_0 \in \mathbf{R}_+$, *and*

$$\alpha(P(D)) \leq c_0 \alpha(D)$$

for any bounded set $D \subset E$ *and some* $c_0 \in \mathbf{R}_+$. *Let* F *be given by*

$$F(u)(t) = \int_0^t \kappa(s) P(u(s)) \, ds.$$

Then (3.1) has at least one solution $u \in K$.

Proof. First notice that

$$|F(u)(t)| \leq \int_0^t |\kappa(s)|_{\mathcal{L}(E)} (a_0 + b_0 |u(s)|) \, ds. \tag{4.10}$$

Hence $|F(u)(t)| \leq |a_0 + b_0 u|_p |\kappa|_r$ so (F1) trivially holds. Inequality (4.10) also implies that (F3) holds .To check (F2) let E_0 be a separable closed subspace of E, and M a bounded countable subset of K with $M(t) \subset E_0$

a.e. on $[0,T]$, for which there exists $\nu \in L^p(0,T)$ such that $|u(t)| \leq \nu(t)$ a.e. on $[0,T]$ for any $u \in M$. Then

$$\begin{aligned}\beta_{E_0}(F(M)(t) \cap E_0) &\leq 2\beta(F(M)(t) \cap E_0) \leq 2\beta(F(M)(t)) \\ &\leq 2\alpha(F(M(t))) \leq 4\int_0^t \alpha(\kappa(s) P(M(s)))\, ds \\ &\leq 4\int_0^t |\kappa(s)|_{\mathcal{L}(E)}\, \alpha(P(M(s)))\, ds \\ &\leq 4c_0 \int_0^t |\kappa(s)|_{\mathcal{L}(E)}\, \alpha(M(s))\, ds \\ &\leq 8c_0 \int_0^t |\kappa(s)|_{\mathcal{L}(E)}\, \beta_{E_0}(M(s))\, ds.\end{aligned}$$

Hence (3.3) holds with

$$\Psi(\varphi)(t) = 8c_0 \int_0^t |\kappa(s)|_{\mathcal{L}(E)}\, \varphi(s)\, ds.$$

Here the function $\delta(t) := 8c_0 |\kappa(t)|_{\mathcal{L}(E)}$ belongs to $L^{r'}(0,T)$ and $r' > r = \frac{p}{p-1}$. Thus according to Remark 3.4, (F2) is satisfied. Now Theorem 3.1 finishes the proof. ■

Remark 4.2 *A similar result is true for*

$$F(u)(t) = \int_0^t \kappa(t-s) P(u(s))\, ds.$$

Example. Let $A : D(A) \subset E \to 2^E$ be an m-dissipative mapping and $u_0 \in \overline{D(A)}$. Assume that the mild solution operator S given by $S(v) = u$ where u is the unique solution of (1.3) satisfies (S1) and (S2). Let $\kappa \in L^{r'}(0,T;\mathcal{L}(E))$ for some $r' > 1$ and let P be as in Theorem 4.3. Then the problems

$$\begin{cases} u'(t) \in Au(t) + \int_0^t \kappa(s) P(u(s))\, ds, & \text{a.e. on } [0,T], \\ u(0) = u_0 \end{cases}$$

and

$$\begin{cases} u'(t) \in Au(t) + \int_0^t \kappa(t-s) P(u(s))\, ds, & \text{a.e. on } [0,T], \\ u(0) = u_0 \end{cases}$$

have solutions in $C(0,T;E)$.

In this case $k(t,s) = \begin{cases} m \text{ (a constant)}, & s < t \\ 0, & s > t \end{cases}$ (see [4]) and S has values in $C(0,T;E)$. The result follows from Theorem 4.3, Remark 4.2 and Remark 3.1 (a) if we choose $1 \leq q = p < \infty$ such that $r' > r = \frac{p}{p-1}$.

References

[1] Agarwal, R.P. and O'Regan, D., Existence criteria for operator inclusions in abstract spaces, *J. Comput. Appl. Math.* **113** (2000), 183-193.

[2] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces,* Editura Academiei–Noordhoff International Publishing, Bucureşti–Leyden, 1976.

[3] Brezis, H. and Browder, F.E., Existence theorems for nonlinear integral equations of Hammerstein type, *Bull. Amer. Math. Soc.* **81** (1975), 73-78.

[4] Couchouron, J-F. and Kamenski, M., An abstract topological point of view and a general averaging principle in the theory of differential inclusions, *Nonlinear Anal.* **42** (2000), 1101-1129.

[5] Couchouron, J-F. and Precup, R., Existence principles for inclusions of Hammerstein type involving noncompact acyclic multivalued maps, *Electron. J. Differential Equations* **2002** (2002), No. 04, 1-21.

[6] Deimling, K., *Nonlinear Functional Analysis,* Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1985.

[7] Diestel, J., Ruess, W.M. and Schachermayer, W., Weak compactness in $L^1(\mu, X)$, *Proc. Amer. Math. Soc.* **118** (1993), 447-453.

[8] Guo, D., Lakshmikantham, V. and Liu, X., *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht–Boston–London, 1996.

[9] Mitidieri, E. and Vrabie, I.I., Existence for nonlinear functional differential equations, *Hiroshima Math. J.* **17** (1987), 627-649.

[10] Mönch, H., Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[11] O'Regan, D., Volterra and Urysohn integral equations in Banach spaces, *J. Appl. Math. Stochastic Anal.* **11** (1998), 449-464.

[12] O'Regan, D. and Meehan, M., *Existence Theory for Nonlinear Integral and Integrodifferential Equations,* Kluwer Academic Publishers, Dordrecht–Boston–London, 1998.

[13] O'Regan, D. and Precup, R., Existence criteria for integral equations in Banach spaces, *J. Inequal. Appl.* **6** (2001), 77-97.

[14] O'Regan, D. and Precup, R., *Theorems of Leray-Schauder Type and Applications,* Gordon and Breach Science Publishers, Amsterdam, 2001.

[15] O'Regan, D. and Precup, R., Integrable solutions of Hammerstein integral inclusions in Banach spaces, *Dynamics Cont., Discrete Impulsive Systems* **9** (2002), 165-176.

[16] Precup, R., *Methods in Nonlinear Integral Equations,* Kluwer Academic Publishers, Dordrecht–Boston–London, 2002.