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On the variation detracting property of a class of operators

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Abstract

This work is focused upon the study of a general class of linear positive operators of discrete type. We show that, under suitable assumptions, the sequence enjoys the variation detracting property.

Keywords: Linear positive operators; Variation detracting property; Convergence in variation; Baskakov-type operators

1. Introduction

The present note is strongly motivated by the paper [1] which deals with the variation detracting property and the rate of approximation of the Bernstein, Kantorovich and Szász–Mirakjan operators, as well as of general singular convolution operators both in the one-dimensional periodic case and in the multivariate setting. The contents of the article is organized as follows. Section 2 is devoted to some preliminaries regarding the variation detracting property and the notion of convergence in variation. In Section 3 we study a slight modification of a known general class of linear positive operators introduced by Baskakov [2], proving that, under additional assumptions, these operators possess the variation detracting property and also are bounded with respect to a certain norm. A particular case is presented, too.

2. Notation and preliminaries

Let I be a bounded or unbounded interval. Throughout the work, $V_I[f]$ stands for the total Jordan variation of the real-valued function f defined on I. The class of all functions of bounded variation on I will be denoted by $BV(I)$. This space can be endowed both with a seminorm $|\cdot|_{BV(I)}$ and with a BV-norm, $\|\cdot\|_{BV(I)}$, where

$$|f|_{BV(I)} := V_I[f], \quad \|f\|_{BV(I)} := V_I[f] + |f(a)|,$$

$f \in BV(I)$, a being any fixed point of I. The qualitative leap from a bounded interval to an unbounded interval $[\alpha, \infty)$ can be achieved via the formula $V_{[\alpha,\infty)}[f] = \lim_{\lambda\to\infty} V_{[\alpha,\lambda]}[f]$, for any $f \in BV([\alpha, \infty))$.

A linear operator L acting on the space $BV(I)$ possesses the *variation detracting* property if the following inequality $V_I[Lf] \le V_I[f]$ holds, for each $f \in BV(I)$.

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Further on, we consider a sequence of linear operators $(L_n)_{n\ge1}$ acting on a function space $\mathcal{S}$ such that $BV(I) \subset \mathcal{S}$. For a given $f \in BV(I)$, the sequence $(L_n)_{n\ge1}$ converges *in variation* to f if

$$\lim_{n\to\infty} V_I[L_n f - f] = 0 \tag{1}$$

holds.

This represents the BV*-approximation* of a function f by the sequence $(L_n f)_{n\ge1}$.

Set $AC(I)$, the space of all absolutely continuous real-valued functions defined on I. If $\lim_{n\to\infty} V_I[g_n - f] = 0$ for a sequence $(g_n)_{n\ge1}$ in $AC(I)$, then also $f \in AC(I)$ and

$$V_I[g_n - f] = \int_I |g_n'(u) - f'(u)|\mathrm{d}u; \tag{2}$$

see [1, page 301]. Thus, convergence in variation of $(g_n)_{n\ge1} \subset AC(I)$ to f means the convergence of $(g_n)_{n\ge1}$ to f' in the $L^1(I)$-norm.

The first research devoted to the variation detracting property and the convergence in variation of a sequence of linear operators was due to Lorentz. In [3, Section 1.7] he proved that the Bernstein operators B_n verify $V_{[0,1]}[B_n f] \le V_{[0,1]}[f]$, $n \in \mathbb{N}$. Moreover, for $f \in BV([0,1])$ the necessary and sufficient condition for $\lim_{n\to\infty} V_{[0,1]}[B_n f - f] = 0$ to take place is $f \in AC([0,1])$.

Considering the Szász–Mirakjan operators S_n, $n \in \mathbb{N}$,

$$(S_n f)(x) := \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad s_{n,k}(x) := \mathrm{e}^{-nx}\frac{(nx)^k}{k!}, \quad x \ge 0, \tag{3}$$

in [1, Proposition 4.1] it has been shown that these operators are variation detracting and also are bounded with respect to the BV-norm.

Our aim is to enrich the register of examples of linear positive discrete-type operators which enjoy the above properties.

3. Main result

Letting $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, by $C^{k,m}(\mathbb{R}_+)$, $0 \le k, m < \infty$, we denote the set of all functions f k-times continuously differentiable on $\mathbb{R}_+$ with the property $f^{(k)}(x) = \mathcal{O}(x^m)(x \to \infty)$. Baskakov [2] introduced the operators $L_n : C^{0,0}(\mathbb{R}_+) \to C(J)$, $n \in \mathbb{N}$,

$$(L_n f)(x) := \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) f\left(\frac{k}{n}\right), \tag{4}$$

which are generated by a sequence of functions $(\varphi_n)_{n\ge1}$, $\varphi_n : \mathbb{C} \to \mathbb{C}$, having the following properties:

$(P_1)\varphi_n$, $n \in \mathbb{N}$, are analytic on a domain D containing the disc $\{z \in \mathbb{C} : |z - R| \le R\}$ and $J := [0, R]$.

(P_2) $\varphi_n(0) = 1$, $n \in \mathbb{N}$.

(P_3) φ_n, $n \in \mathbb{N}$, are completely monotone on J, i.e., $(-1)^k \varphi_n^{(k)}(x) \ge 0$ for $x \in J$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$.

(P_4) There exists a positive integer $m(n)$ such that

$$\varphi_n^{(k)}(x) = -n\varphi_{m(n)}^{(k-1)}(x)(1 + \alpha_{k,n}(x)), \quad x \in J,\ (n,k) \in \mathbb{N}\times\mathbb{N},$$

where $\alpha_{k,n}(x)$ converges to zero uniformly in k and x on J for n tending to infinity.

(P_5) $\lim_{n\to\infty} \frac{n}{m(n)} = 1$.

Baskakov showed that every function $f \in C^{0,0}(\mathbb{R}_+)$ is approximated uniformly on the interval J by the sequence $(L_n f)_{n\ge1}$ and Schurer [4] proved that this property holds for the larger class $C^{0,2}(\mathbb{R}_+)$. We consider the following variant of (P_1) and (P_4) properties.

(P_1') φ_n, $n \in \mathbb{N}$, are real functions on $J := \mathbb{R}_+$ which are infinitely differentiable on $\mathbb{R}_+$ and property (P_4) takes place in the particular case $\alpha_{k,n} = 0$, $(k,n) \in \mathbb{N}\times\mathbb{N}$.

Theorem. *Let L_n be defined by (4) such that the properties (P_1'), $(P_2$–$P_5)$ hold and the function φ_n verifies the following conditions:*

$$n\int_0^\infty \varphi_{m(n)}(x)\mathrm{d}x \le 1, \tag{5}$$

$$\varphi_n^{(k)}(x) = o(x^{-k}) \quad (x\to\infty),\ k\in\mathbb{N}_0, \tag{6}$$

for any $n\in\mathbb{N}$. If $f\in BV(\mathbb{R}_+)$, then

$$V_{\mathbb{R}_+}[L_n f] \le V_{\mathbb{R}_+}[f], \tag{7}$$

$$\|L_n f\|_{BV(\mathbb{R}_+)} \le \|f\|_{BV(\mathbb{R}_+)}. \tag{8}$$

Proof. Following Martini [5, Theorem 1] we get

$$(L_n f)'(x) = -\sum_{k=0}^\infty \frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x)\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right). \tag{9}$$

Set $I_k := \int_0^\infty x^k\varphi_n^{(k+1)}(x)\mathrm{d}x$, $k\in\mathbb{N}$. Integrating $(k-1)$ times by parts and using each time (6) we obtain $I_k = (-1)^{k-1}k(k-1)\dots 2I_1$. On the basis of (P_1'), (P_4) and relation (6), clearly $I_1 = n\int_0^\infty\varphi_{m(n)}(x)\mathrm{d}x$. Taking into account the above relations, (P_3) and (5) we can write

$$\begin{aligned}\int_0^\infty |(L_n f)'(x)|\mathrm{d}x &\le \sum_{k=0}^\infty \left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| n\int_0^\infty \varphi_{m(n)}(x)\mathrm{d}x \\ &\le \sum_{k=0}^\infty \left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| \le V_{\mathbb{R}_+}[f],\end{aligned}$$

for each $f\in BV(\mathbb{R}_+)$. This implies (7).

Since $(L_n f)(0) = f(0)$ (see (P_2)), relation (8) is a result of (7). □

Considering a Kantorovich variant of L_n defined by

$$(L_n^* f)(x) := \sum_{k=0}^\infty\left(n\int_{k/n}^{(k+1)/n} f(t)\mathrm{d}t\right)\phi_{n,k}(x), \quad n\in\mathbb{N},\ x\ge 0,$$

where $\phi_{n,k}(x) := \frac{(-x)^k}{k!}\varphi_{m(n)}^{(k)}(x)$, relation (9) and (P_4) guarantee that $(L_n f)' = L_n^* f'$. If $f\in AC(\mathbb{R}_+)$, following Orlicz's result (see e.g. [3, Theorem 2.1.2]) and on the basis of (2) we get

$$V_{\mathbb{R}_+}[L_n f - f] = \int_0^\infty |(L_n^* f')(x) - f'(x)|\mathrm{d}x \to 0, \quad n\to\infty.$$

The condition $f\in AC(\mathbb{R}_+)$ is also necessary for the convergence described by (1), since $L_n f\in AC(\mathbb{R}_+)$ yields $f\in AC(\mathbb{R}_+)$. This statement holds because $AC(\mathbb{R}_+)$ is a closed subspace of $BV(\mathbb{R}_+)$ in the variation seminorm; see the work of C. Bardaro, P.L. Butzer, R.L. Stens, G. Vinti [1, page 304].

Choosing $\varphi_n(x) = \mathrm{e}^{-nx}$, we get $m(n) = n$ and L_n turns into Szász–Mirakjan operators given by (3). The requirements (5) and (6) are fulfilled; consequently (7) and (8) hold.

Choosing $\varphi_n(x) = (1+x)^{-n}$, again (P_1'), $(P_2$–$P_5)$ are satisfied with $m(n) = n+1$. In this case, the operator L_n is called the classical nth Baskakov operator. By direct computation, we prove that (5) and (6) apply; consequently (7) and (8) hold.

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