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ARONSZAJN TYPE THEOREMS FOR INTEGRAL EQUATIONS ON UNBOUNDED DOMAINS VIA MAXIMAL SOLUTIONS

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. This paper discusses the topological structure of the solution set of a general Volterra integral equation. Under natural conditions we show that the solution set is an R_δ set.

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1. Introduction

In this paper we discuss the solution set of the Volterra integral equation

$$y(t) = h(t) + \int_0^t k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0, \infty) \tag{1.1}$$

where $f : [0,\infty) \times \mathbf{R}^M \to \mathbf{R}^M$ and the matrix valued function $k : \{(s,t) : 0 \le s \le t < n\} \to L_{M\times M}[0,n]$ for each $n \in N = \{1, 2, ...\}$; here $M \in N$. Basically we show that if f is bounded by a L^1_{loc}-Carathéodory function and if a sequence of differential equations have maximal solutions then the solution set of (1.1) is an R_δ set. Two approaches will be presented in this paper and our results extend those in [1-4].

2. Solution set

The following result can be found in [6].

Theorem 2.1. *Let X be a closed set in a Fréchet space (E,d), and $F : X \to E$ a continuous compact operator. Assume there exists a sequence $\{U_n\}_n$ of closed convex sets in E such that*

$$\forall n \in N = \{1,2,....\},\ 0 \in U_n; \tag{2.1}$$

$$\lim_{n\to\infty} diam(U_n) = 0, \tag{2.2}$$

and there exists a sequence $\{F_n\}_n$ of operators $F_n : X \to E$, such that

$$\forall n \in N,\ \forall x \in X,\ F(x) - F_n(x) \in U_n; \tag{2.3}$$

$$I - F_n \text{ is a homeomorphism of the set } (I-F_n)^{-1}(U_n) \text{ onto } U_n. \tag{2.4}$$

Then $\{x \in X : \ x = F(x)\}$ is an R_δ set.

Remark 2.1. Recall a nonempty set A is contractible provided there exists $x_0 \in A$ and a homotopy $H : A \times [0,1] \to A$ such that $H(x,1) = x$ and $H(x,0) = x_0$ for every $x \in A$. A set A is called an R_δ set provided there exists a decreasing sequence $\{A_n\}_1^\infty$ of nonempty compact, contractible sets such that $A = \cap\{A_n : \ n = 1,2,...\}$.

In this section $E = C([0,\infty), \mathbf{R}^M)$ will be the space of continuous functions defined on the interval $[0,\infty)$ with values in $\mathbf{R}^M$. Now E is a Fréchet space with the topology given by the complete family of seminorms $\{p_m\}_{m\geq 1}$, or, equivalently, by the distance d defined by

$$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)},$$

for $x,y \in E$; here $p_m(x) = \sup\{|x(t)| : t \in [0,m]\} \equiv |x|_m$ for $x \in E$.

Let X be a nonempty closed subset of E. We consider an operator $F : E \to E$. Define the sequence of operators $\{F_n\}_n$, $F_n : E \to E$, as follows:

$$F_n(x)(t) = F(x)(r_n(t)), \text{ for } x \in E \text{ and } t \geq 0, \tag{2.5}$$

where

$$r_n(t) = \begin{cases} 0, \text{ if } t \in [0, 1/n]; \\ t - \frac{1}{n}, \text{ if } t > 1/n. \end{cases} \tag{2.6}$$

In [4] (see also [2, Chapter 6]) we established the following result using Theorem 2.1.

Theorem 2.2. *Assume that $E = C([0,\infty), \mathbf{R}^M)$ and that $F : X \to E$ is a continuous and compact operator. Also assume that the following conditions hold:*

(i) $\exists u_0 \in \mathbf{R}^M$ such that $F(x)(0) = u_0$, for all $x \in X$;

(ii) $\forall \epsilon > 0$, $\forall x, y \in X$, if $x(t) = y(t)$, $\forall t \in [0, \epsilon]$, then $F(x)(t) = F(y)(t)$, $\forall t \in [0, \epsilon]$ (i.e. F is an abstract Volterra operator);

(iii) if $y \in E$ satisfies $y = F(y)$, then $y \in X$;

(iv) $\exists \eta > 0$ such that $\forall n$, if $y \in E$ satisfies $y = F_n(y) + z$, where $p_m(z) \leq \eta, \forall m$, then $y \in X$.

Then $Fix(F)$ is an R_δ set.

Now we discuss the topological structure of the solution set of the Volterra integral equation

$$y(t) = h(t) + \int_0^t k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0, \infty). \tag{2.7}$$

Throughout $f : [0,\infty) \times \mathbf{R}^M \to \mathbf{R}^M$ and the matrix valued function $k : \{(s,t) : 0 \leq s \leq t < n\} \to L_{M\times M}[0,n]$ for each $n \in N = \{1, 2, ...\}$.

We now use Theorem 2.2 to obtain a new existence result for (2.7). First recall a function $g : [0,n] \times \mathbf{R} \to \mathbf{R}$ ($n \in N$ fixed) is a L^1-Carathéodory function if

(a). the map $t \mapsto g(t,y)$ is measurable for all $y \in \mathbf{R}$,

(b). the map $y \mapsto g(t,y)$ is continuous for a.e. $t \in [0,n]$,

and

(c). for any $r > 0$, $\exists \mu_r \in L^1[0,n]$ such that $|y| \leq r$ implies $|g(t,y)| \leq \mu_r(t)$ for a.e. $t \in [0,n]$.

A function $g : [0,\infty) \times \mathbf{R} \to \mathbf{R}$ is a L^1_{loc}-Carathéodory function if (a), (b) and (c) above hold when g is restricted to $[0,n] \times \mathbf{R}$ for any $n \in N$.

Theorem 2.3. *Let* $k : \{(s,t) : 0 \le s \le t < n\} \to L_{M\times M}[0,n]$ *for each* $n \in N = \{1,2,...\}$ *and* $f : [0,\infty) \times \mathbf{R}^M \to \mathbf{R}^M$ *and suppose the following conditions are satisfied:*

$$h \in C([0,\infty),\mathbf{R}^M) \tag{2.8}$$

$$y \mapsto f(t,y) \ \text{ is continuous for a.e. } \ t \in [0,\infty) \tag{2.9}$$

$$t \mapsto f(t,y) \ \text{ is measurable for every } \ y \in \mathbf{R}^M \tag{2.10}$$

$$\begin{cases} \text{for each } \ n \in N, \ \text{for each } \ t \in [0,n] \ \text{ we have} \\ \text{that } \ k(t,s) \ \text{ is measurable on } \ [0,t] \ \text{ and } \ k(t) \\ = \operatorname{ess\,sup} |k(t,s)|, \, 0 \le s \le t, \ \text{ is bounded on } \ [0,n] \end{cases} \tag{2.11}$$

$$\begin{cases} \text{for each } \ n \in N, \ \text{ the map } \ t \mapsto k_t \ \text{ is continuous} \\ \text{from } \ [0,n] \ \text{ to } \ L^\infty([0,n], L_{M\times M}[0,n]); \\ \text{here } \ k_t(s) = k(t,s) \end{cases} \tag{2.12}$$

$$\begin{cases} \text{there exists a } \ L^1_{loc}[0,\infty) - \text{Carathéodory function} \\ g : [0,\infty) \times [0,\infty) \to [0,\infty) \ \text{ such that} \\ |f(t,x)| \le g(t,|x|) \ \text{ for a.e. } \ t \in [0,\infty) \\ \text{and all } \ x \in \mathbf{R}^M \end{cases} \tag{2.13}$$

$$g(t,x) \ \text{ is nondecreasing in } \ x \ \text{ for a.e. } \ t \in [0,\infty) \tag{2.14}$$

and

$$\begin{cases} \text{there exists } \ \eta > 0, \ \text{ such that for each } \ n \in N, \ \text{ the problem} \\ \begin{cases} v'(t) = \left(\sup_{t\in[0,n]} k(t)\right) g(t,v(t)) \ \text{ a.e. } \ t \in [0,n] \\ v(0) = |h|_n + \eta \end{cases} \\ \text{has a maximal solution } \ r_n(t) \ \text{ on } \ [0,n] \ \text{ (here } \ r_n \in C[0,n]) \end{cases} \tag{2.15}$$

Then the solution set of (2.7) *is an* R_δ *set.*

Remark 2.2. Recall a subset A of $C([0,\infty),\mathbf{R}^M)$ is bounded if and only if there exists there exists a positive continuous function $\phi : [0,\infty) \to \mathbf{R}$ with $|x(t)| \le \phi(t)$ for all $t \in [0,\infty)$, $x \in A$.

PROOF: Let $E = C([0,\infty),\mathbf{R}^M)$,

$$X = \{y \in C([0,\infty) : |y(t)| \le r_m(t), \ t \in [0,m], \ \forall m \in N\}$$

and let $F : E \to E$ be defined by

$$F(y)(t) = h(t) + \int_0^t k(t,s)\, f(s,y(s))ds \text{ for } t \in [0,\infty).$$

Notice X is a closed subset of E. Conditions (2.8) – (2.12) ensure that F is well defined, F is a Volterra operator,

$$F(y)(0) = h(0),\ \forall y \in C([0,\infty), \mathbf{R}^M),$$

and the restriction $F : C([0,m], \mathbf{R}^M) \to C([0,m], \mathbf{R}^M)$ is continuous (see [2]). In fact, $F : X \to C([0,\infty), \mathbf{R}^M)$ is continuous, because if $\{y_j\}_{j\in N}$ is a sequence in X and $y_0 \in C([0,\infty), \mathbf{R}^M)$ is such that $y_j \to y_0$ in $C([0,\infty), \mathbf{R}^M)$ as $j \to \infty$, then $y_j \to y_0$ in $C([0,m], \mathbf{R}^M)$ as $j \to \infty$, for all m. Since $F : C([0,m], \mathbf{R}^M) \to C([0,m], \mathbf{R}^M)$ is continuous, we then have that $F(y_j) \to F(y_0)$ in $C([0,m], \mathbf{R}^M)$ as $j \to \infty$, for all m. This implies that $F(y_j) \to F(y_0)$ in $C([0,\infty), \mathbf{R}^M)$ as $j \to \infty$.

We show now that $F : X \to C([0,\infty), \mathbf{R}^M)$ is compact, and that (iii) and (iv) of Theorem 2.2 hold. First we show that

$$F : X \to C([0,\infty), \mathbf{R}^M) \ \text{ is compact.}$$

Let $\{y_j\}_{j\in N}$ be a sequence in X and consider the sequence $\{F(y_j)\}_{j\in N}$ in $F(X)$. Now $X|_{[0,m]}$ is bounded in $C([0,m], \mathbf{R}^M)$ for all m (see the definition of X); here

$$X|_{[0,m]} = \{y|_{[0,m]} : y \in X\}.$$

The restriction $F : X|_{[0,m]} \to C([0,m], \mathbf{R}^M)$, is compact (see [2] or [5] where it follows easily from the Arzela-Ascoli theorem), therefore $F(X|_{[0,m]})$ is relatively compact in $C([0,m], \mathbf{R}^M)$. For $m = 1$, there exists a subsequence N_1 of N, and there exists a $z_1 \in C([0,1], \mathbf{R}^M)$, such that

$$F(y_j)|_{[0,1]} \to z_1 \ \text{ in } \ C([0,1], \mathbf{R}^M) \ \text{ as } \ j \to \infty \ \text{ in } \ N_1.$$

Now consider the sequence $\{F(y_j)\}_{j\in N_1}$, restricted to $[0,2]$. Since $F(X|_{[0,2]})$ is relatively compact in $C([0,2], \mathbf{R}^M)$, there exists a subsequence N_1 of N_2, and there exists a $z_2 \in C([0,2], \mathbf{R}^M)$, such that

$$F(y_j)|_{[0,2]} \to z_2 \text{ in } \ C([0,2], \mathbf{R}^M) \ \text{ as } \ j \to \infty \ \text{ in } \ N_2.$$

In addition,

$$z_2|_{[0,1]} = z_1 \text{ on } [0,1].$$

By induction, assume the sequence $\{F(y_j)\}_{j\in N_k}$ and $z_k \in C([0,k], \mathbf{R}^M)$ are found such that $N_k \subseteq N_{k-1} \subseteq ... \subseteq N_1 \subseteq N$,

$$F(y_j)|_{[0,k]} \to z_k \ \text{ in } \ C([0,k], \mathbf{R}^M) \ \text{ as } \ j \to \infty \ \text{ in } N_k,$$

and

$$z_k|_{[0,1]} = z_{k-1} \text{ on } [0, k-1].$$

Since $F(X|_{[0,k+1]})$ is relatively compact in $C([0,k+1],\mathbf{R}^M)$, there exists a subsequence N_{k+1} of N_k, and there exists a $z_{k+1} \in C([0,k+1],\mathbf{R}^N)$, such that

$$F(y_j)|_{[0,k+1]} \to z_{k+1} \text{ in } C([0,k+1],\mathbf{R}^M) \text{ as } j \to \infty \text{ in } N_{k+1}.$$

In addition,

$$z_{k+1}|_{[0,k]} = z_k \text{ on } [0,k].$$

Now define $z \in C([0,\infty),\mathbf{R}^M)$ by

$$z(t) = z_k(t),\ t \in [k-1,k),\ k = 1,2,\ldots .$$

The induction above shows that the sequence $\{F(y_j)\}_{j\in N}$ contains a subsequence which converges in $C([0,\infty),\mathbf{R}^M)$ to $z \in C([0,\infty),\mathbf{R}^M)$. Therefore $F(X)$ is relatively compact in $C([0,\infty),\mathbf{R}^M)$, and the operator $F : X \to C([0,\infty),\mathbf{R}^M)$ is compact.

To see that (iv) of Theorem 2.2 is satisfied let $\eta > 0$ be given as in (2.15). Now let $n \in N$ and let $y \in C([0,\infty),\mathbf{R}^M)$ be such that $y(t) = F_n(y)(t)+z(t)$, $t \in [0,\infty)$, where z is such that $p_m(z) \le \eta$, $\forall m$, and

$$F_n(y)(t) = \begin{cases} h(0), & \text{if } t \in [0,1/n) \\ h\left(t-\frac{1}{n}\right) + \int_0^{t-1/n} k\left(t-\frac{1}{n},s\right) g(s,y(s))ds, & \text{if } t \in [1/n,\infty). \end{cases}$$

Let $m \in \mathbf{N}$ be arbitrary. Then we have for $x \in [0,m]$ that

$$|y(x)| \le |h|_m + \left(\sup_{s\in[0,m]} k(s)\right) \int_0^x g(s,|y(s)|)\,ds + \eta \equiv v(x).$$

Now (2.14) implies

$$v'(x) = \left(\sup_{s\in[0,m]} k(s)\right) g(x,|y(x)|) \le \left(\sup_{s\in[0,m]} k(s)\right) g(x,v(x))$$

for almost everywhere $x \in [0,m]$, so

$$\begin{cases} v'(x) \le \left(\sup_{s\in[0,m]} k(s)\right) g(x,v(x)) & \text{for a.e. } x \in [0,m] \\ v(0) = |h|_m + \eta. \end{cases}$$

Now [7, Theorem 1.10.2] guarantees that $v(x) \le r_m(x)$ for $x \in [0,m]$, so $|y(x)| \le v(x) \le r_m(x)$ for $x \in [0,m]$. We can do this argument for all $m \in N$. Consequently (iv) of Theorem 2.2 holds.

To see that (iii) of Theorem 2.2 is satisfied let $y \in C([0,\infty),\mathbf{R}^M)$ be such that $y(t) = F(y)(t)$ for $t \in [0,\infty)$. Let $m \in \mathbf{N}$ be arbitrary. Then we have for $x \in [0,m]$ that

$$|y(x)| \le |h|_m + \left(\sup_{s\in[0,m]} k(s)\right) \int_0^x g(s,|y(s)|)\,ds \equiv w(x).$$

Now (2.14) implies

$$w'(x) = \left(\sup_{s\in[0,m]} k(s)\right) g(x,|y(x)|) \le \left(\sup_{s\in[0,m]} k(s)\right) g(x,w(x))$$

for almost everywhere $x \in [0,m]$ and $w(0) = |h|_m \le |h|_m + \eta$ where η is as in (2.15), so

$$\begin{cases} w'(x) \le \left(\sup_{s\in[0,m]} k(s)\right) g(x,w(x)) & \text{for a.e. } \ x \in [0,m] \\ w(0) \le |h|_m + \eta. \end{cases}$$

Now [7, Theorem 1.10.2] guarantees that $w(x) \le r_m(x)$ for $x \in [0,m]$, so $|y(x)| \le w(x) \le r_m(x)$ for $x \in [0,m]$. We can do this argument for all $m \in N$. Consequently (iii) of Theorem 2.2 holds.

Now all the conditions in Theorem 2.2 are satisfied so the solution set of (2.7) is an R_δ set. $\square$

Remark 2.3. A special case of (2.7) is first order differential equations. In fact in this case assumption (2.14) can be removed in Theorem 2.3 (see the ideas in [1]).

An alternate approach to solution sets can be found in [3]. It is based on Theorem 2.2 (so on Theorem 2.1) when $X = E$. For completeness we discuss this approach now. In [3] we established the following result.

Theorem 2.4. *Let $F : C([0,\infty),\mathbf{R}^M) \to C([0,\infty),\mathbf{R}^M)$ be a continuous, compact map. Also assume that the following conditions hold:*

(i) $\exists u_0 \in \mathbf{R}^M$ with $F(x)(0) = u_0$, for all $x \in C([0,\infty),\mathbf{R}^M)$;

(ii) $\forall\epsilon > 0$, $\forall x, y \in C([0,\infty), \mathbf{R}^M)$, *if* $x(t) = y(t)$, $\forall t \in [0,\epsilon]$, *then* $F(x)(t) = F(y)(t)$, $\forall t \in [0,\epsilon]$ *(i.e. F is an abstract Volterra operator).*

Then $Fix(F)$ is an R_δ set.

We remark that in application (see (2.7)

$$F : C([0,\infty), \mathbf{R}^M) \to C([0,\infty), \mathbf{R}^M)$$

is usually continuous, and completely continuous but it is rarely compact. As a result we would like to relax the compactness assumption on F in Theorem 2.4. In applications we usually encounter the nonlinear operator equation

$$y(t) = L\,Fy(t) \quad \text{for} \quad t \in [0,\infty); \tag{2.16}$$

here L is an affine map. We will assume the following conditions are satisfied:

$$L\,F : C([0,\infty), \mathbf{R}^M) \to C([0,\infty), \mathbf{R}^M) \tag{2.17}$$

$$\exists u_0 \in \mathbf{R}^M \quad \text{with} \quad L\,F(x)(0) = u_0, \quad \text{for all} \quad x \in C([0,\infty), \mathbf{R}^M) \tag{2.18}$$

$$\begin{cases} \forall\epsilon > 0,\ \forall x, y \in C([0,\infty), \mathbf{R}^M), & \text{if} \quad x(t) = y(t)\ \forall t \in [0,\epsilon] \\ \text{then} \quad L\,F\,(x)(t) = L\,F\,(y)(t)\ \forall t \in [0,\epsilon] \end{cases} \tag{2.19}$$

and

$$\begin{cases} \exists \text{ a continuous function } \phi : [0,\infty) \to [0,\infty) \\ \text{such that } |y(t)| \le \phi(t) \text{ for } t \in [0,\infty), \text{ for any} \\ \text{posible solution } y \in C([0,\infty), \mathbf{R}^M) \text{ to (2.16).} \end{cases} \tag{2.20}$$

Let $\epsilon > 0$ be given and let $\tau_\epsilon : \mathbf{R}^M \to [0,1]$ be the Urysohn function for

$$\left(\overline{B}(0,1)\,,\ \mathbf{R}^M \setminus B(0,1+\epsilon)\right)$$

such that

$$\tau_\epsilon(x) = 1 \ \text{ if } \ |x| \le 1 \quad \text{and} \quad \tau_\epsilon(x) = 0 \ \text{ if } \ |x| \ge 1+\epsilon.$$

Let the operator F_ϵ be defined by

$$F_\epsilon\,(y)(t) = \tau_\epsilon\left(\frac{y(t)}{\phi(t)+1}\right) F(y)(t); \quad \text{here} \quad y \in C([0,\infty), \mathbf{R}^M).$$

Consider the operator equation

$$y(t) = L\,F_\epsilon\,y(t) \quad \text{for} \quad t \in [0,\infty). \tag{2.21}$$

The next result follows easily from Theorem 2.4 (see [3]).

Theorem 2.5. *Suppose* (2.17)-(2.20) *hold. Let* $\epsilon > 0$ *be given and assume the following conditions are satisfied:*

$$\begin{cases} |w(t)| \leq \phi(t) \ \ \textit{for} \ \ t \in [0, \infty), \ \ \textit{for any possible} \\ \textit{solution} \ \ w \in C([0,\infty), \mathbf{R}^M) \ \ \textit{to (2.21)} \end{cases} \tag{2.22}$$

and

$$L\, F_\epsilon : C([0,\infty), \mathbf{R}^M) \to C([0,\infty), \mathbf{R}^M) \ \textit{is continuous and compact.} \tag{2.23}$$

Then the solution set of (2.16) *is an* R_δ *set.*

Now its easy to apply Theorem 2.5 to establish results for (2.7) (see the ideas in [8]).

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