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The Nonlinear Heat Equation via Fixed Point Principles

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Abstract. Starting with the existence and uniqueness result of J.L. Lions for the non-homogenous heat equation with the source term in $H^{-1}(\Omega)$, we present existence results for the nonlinear perturbed heat equation via Banach, Schauder and Leray-Schauder principles.

Key Words: Parabolic equation, Nonlinear operator, Sobolev space.
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1 Introduction. The non-homogenous heat equation in $H^{-1}(\Omega)$

We start with an existence and uniqueness result of J.L. Lions (see [4] and [3]) for the non-homogenous heat equation with the source term in $H^{-1}(\Omega)$. We include a proof adapted from Temam [9] for completeness.

Theorem 1.1 ([4]) *If* $f \in L^2\left(0,T;H^{-1}(\Omega)\right)$ *and* $g_0 \in L^2(\Omega)$, *then there exists a unique function* u *such that*

(1.1)
$$u \in L^2\left(0,T;H_0^1(\Omega)\right) \cap C\left([0,T];L^2(\Omega)\right), \quad u' \in L^2\left(0,T;H^{-1}(\Omega)\right)$$

(1.2) $$\begin{cases} (u'(t), v) + (u(t), v)_{H_0^1} = (f(t), v) \\ \qquad a.e.\ on\ [0,T], \quad for\ all\ v \in H_0^1(\Omega) \\ u(0) = g_0. \end{cases}$$

Proof. We look for a solution in the form

(1.3) $$u(t) = \sum_{k=1}^{\infty} u_k(t)\, \phi_k.$$

We shall denote by λ_k and ϕ_k the eigenvalues and eigenfunctions of laplacean. Hence

$$\begin{cases} \Delta\phi_k + \lambda_k \phi_k = 0 \\ \phi_k \in H_0^1(\Omega), \ |\phi_k|_{L^2(\Omega)} = 1. \end{cases}$$

If we formally replace into (1.2) we obtain

$$u_k(t) = e^{-\lambda_k t} g_0^k + \int_0^t e^{-\lambda_k (t-s)} f_k(s)\, ds$$

where $f_k(t) = (f(t), \phi_k)$.

Let us now consider a partial sum of series (1.3), i.e.,

$$s_m(t) = \sum_{k=1}^{m} u_k(t)\, \phi_k.$$

Clearly $s_m \in C\left([0,T]; H_0^1(\Omega)\right)$, $s'_m \in L^2\left(0,T; H_0^1(\Omega)\right)$ and

(1.4) $$\begin{cases} (s'_m(t), \phi_j) + (s_m(t), \phi_j)_{H_0^1(\Omega)} = (f(t), \phi_j), \\ \qquad j = 1, 2, ..., m \\ s_m(0) = g_{0m} \end{cases}$$

where g_{0m} is the orthogonal projection in $L^2(\Omega)$ of g_0 on the space spanned by $\phi_1, \phi_2, ..., \phi_m$, i.e., $g_{0m} = \sum_{k=1}^m g_0^k \phi_k$, with $g_0^k = (g_0, \phi_k)_{L^2(\Omega)}$. Thus s_m is the solution of the projection of problem (1.2) on the finite-dimensional space spanned by $\phi_1, \phi_2, ..., \phi_m$.

From (1.4) we deduce

$$\left(s_m'(t), s_m(t)\right) + |s_m(t)|_{H_0^1}^2 = (f(t), s_m(t)),$$

that is

$$\frac{1}{2}\frac{d}{dt}|s_m|_{L^2}^2 + |s_m|_{H_0^1}^2 = (f, s_m).$$

Integration gives

$$\frac{1}{2}|s_m(t)|_{L^2}^2 - \frac{1}{2}|g_{0m}|_{L^2}^2 + \int_0^t |s_m(\tau)|_{H_0^1}^2 d\tau \tag{1.5}$$

$$= \int_0^t (f(\tau), s_m(\tau)) d\tau \leq \int_0^t |f(\tau)|_{H^{-1}} |s_m(\tau)|_{H_0^1} d\tau.$$

Using Hölder's inequality and $|g_{0m}|_{L^2} \leq |g_0|_{L^2}$, we obtain

$$|s_m|_{L^2\left(0,T;H_0^1(\Omega)\right)}^2 - \frac{1}{2}|g_0|_{L^2}^2 \leq |f|_{L^2(0,T;H^{-1}(\Omega))} |s_m|_{L^2\left(0,T;H_0^1(\Omega)\right)}.$$

This implies that the sequence (s_m) is bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$. Now from (1.5), we obtain

$$|s_m(t)|_{L^2}^2 \leq c, \quad t \in [0,T]$$

which shows that (s_m) is bounded in $L^\infty\left(0,T;L^2(\Omega)\right)$, the dual space of $L^1\left(0,T;L^2(\Omega)\right)$.

Recall that the closed unit ball of any reflexive Banach space (Hilbert space, in particular) is compact with respect to the weak topology. Also, the closed unit ball of the dual of a Banach space is compact in the weak-star topology (see Brezis [2, Théorèmes III.16 şi III.15]). Thus, there exists a subsequence of (s_m), also denoted by (s_m) and functions $u \in L^2\left(0,T;H_0^1(\Omega)\right)$ and $\bar{u} \in L^\infty\left(0,T;L^2(\Omega)\right)$ with

$$\begin{aligned} s_m &\to u \text{ in } L^2\left(0,T;H_0^1(\Omega)\right) \text{ weakly} \\ s_m &\to \bar{u} \text{ in } L^\infty\left(0,T;L^2(\Omega)\right) \text{ weakly-star.} \end{aligned}$$

Furthermore, it is well-known that if a sequence is weakly convergent then there is a sequence of convex combinations of its elements which strongly converges to the same limit. Let (s_m^c) be this sequence of convex combinations of elements from (s_m) which strongly converges to u in $L^2\left(0,T;H_0^1(\Omega)\right)$. Passing eventually to a new subsequence, we may assume that $s_m^c(t) \to u(t)$ in $H_0^1(\Omega)$ (consequently in $L^2(\Omega)$) for a.e. $t \in [0,T]$ and $s_m^c \to \bar{u}$ in $L^\infty\left(0,T;L^2(\Omega)\right)$ weakly-star. Then

$$(s_m^c, v) \to (\bar{u}, v) \text{ for all } v \in L^1\left(0,T;L^2(\Omega)\right)$$

that is

$$(s_m^c - \bar{u}, v)_{L^2(\Omega)} \to 0 \text{ in } L^1(0,T)$$

whence $(s_m^c(t) - \bar{u}(t), v(t))_{L^2(\Omega)} \to 0$ a.e. on $[0,T]$, for at least a subsequence. In addition

$$(s_m^c(t) - \bar{u}(t), v(t))_{L^2(\Omega)} \to (u(t) - \bar{u}(t), v(t))_{L^2(\Omega)}.$$

It follows that $(u(t) - \bar{u}(t), v(t))_{L^2(\Omega)} = 0$ a.e. on $[0,T]$, for every $v \in L^1\left(0,T;L^2(\Omega)\right)$. Hence

$$(u - \bar{u}, v)_{L^2(0,T;L^2(\Omega))} = 0 \text{ for all } v \in L^2\left(0,T;L^2(\Omega)\right)$$

whence $u = \bar{u}$ and

$$\begin{aligned} s_m &\to u \text{ in } L^2\left(0,T;H_0^1(\Omega)\right) \text{ weakly} \\ s_m &\to u \text{ in } L^\infty\left(0,T;L^2(\Omega)\right) \text{ weakly-star.} \end{aligned} \tag{1.6}$$

This means that

$$\begin{aligned} (h, s_m) &\to (h,u) \text{ for every } h \in L^2\left(0,T;H^{-1}(\Omega)\right) \\ (s_m, v) &\to (u,v) \text{ for every } v \in L^1\left(0,T;L^2(\Omega)\right). \end{aligned} \tag{1.7}$$

From (1.4) we deduce

$$(1.8)\qquad \left(s_m\left(t\right),\phi_j\right)-\left(g_{0m},\phi_j\right)+\int_0^t\left(s_m\left(\tau\right),\phi_j\right)_{H_0^1}d\tau = \int_0^t\left(f\left(\tau\right),\phi_j\right)d\tau.$$

If we pass to the limit with $m\to\infty$ and we use (1.7) we obtain

$$\left(u\left(t\right),\phi_j\right)-\left(g_0,\phi_j\right)+\int_0^t\left(u\left(\tau\right),\phi_j\right)_{H_0^1}d\tau = \int_0^t\left(f\left(\tau\right),\phi_j\right)d\tau.$$

This implies

$$\left(u'\left(t\right),\phi_j\right)+\left(u\left(t\right),\phi_j\right)_{H_0^1}=\left(f\left(t\right).\phi_j\right)\ \text{for a.e. } t\in[0,T]\,, j=1,2,\ldots.$$

Then

$$\left(u'\left(t\right),v\right)+\left(u\left(t\right),v\right)_{H_0^1}=\left(f\left(t\right),v\right)$$

for a.e. $t\in[0,T]$, all $v\in H_0^1\left(\Omega\right)$. If we set $v=\varphi\in C_0^\infty\left(\Omega\right)$, we deduce that $u'\left(t\right)=\Delta u\left(t\right)+f\left(t\right)$ in $\mathcal{D}'\left(\Omega\right)$. Since f and Δu belong to $L^2\left(0,T;H^{-1}\left(\Omega\right)\right)$, it follows that u' also belongs to that space.

To prove that $u\in C\left([0,T];L^2\left(\Omega\right)\right)$ it suffices to note that

$$\frac{d}{dt}\left|u\right|_{L^2}^2=2\left(u',u\right).$$

Then, since $u\in L^2\left(0,T;H_0^1\left(\Omega\right)\right)$ and $u'\in L^2\left(0,T;H^{-1}\left(\Omega\right)\right)$, we see that $\frac{d}{dt}\left|u\right|_{L^2}^2\in L^1\left(0,T\right)$, whence we derive the continuity of $\left|u\right|_{L^2}^2$. Similarly, for each $t_0\in[0,T]$, the function $\left|u\left(t\right)-u\left(t_0\right)\right|_{L^2}$ is continuous. In particular, $\left|u\left(t\right)-u\left(t_0\right)\right|_{L^2}\to 0$ as $t\to t_0$. Therefore $u\in C\left([0,T];L^2\left(\Omega\right)\right)$.

Now if we pass to the limit in $s_m\left(0\right)=g_{0m}$ we obtain $u\left(0\right)=g_0$. Thus u satisfies (1.2).

For uniqueness, assume that u and v are two solutions, that is functions satisfying (1.1) and (1.2). Then for $w = u - v$ one has

$$\frac{1}{2}\frac{d}{dt}|w|_{L^2}^2 + |w(t)|_{H_0^1}^2 = 0 \text{ a.e. on } [0,T],$$

whence we derive $d|w|_{L^2}^2/dt \le 0$. This shows that $|w|_{L^2}^2$ is decreasing on $[0,T]$. In addition $w(0) = 0$. Hence $w = 0$, that is $u = v$.

Notice that the uniqueness of solution implies that (1.6) holds for the entire sequence (s_m) not only for one of its subsequences. ■

By a (weak or generalized) *solution* of the Cauchy–Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \text{in } Q := \Omega \times (0,\infty) \\ u(x,0) = g_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \Sigma := \partial\Omega \times (0,\infty) \end{cases} \tag{1.9}$$

where $f \in L^2(0,T;H^{-1}(\Omega))$ and $g_0 \in L^2(\Omega)$, we mean a function u which satisfies (1.1) and (1.2).

2 Nonlinear heat equation

According to Theorem 1.1, one can associate to the Cauchy–Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \text{in } Q \\ u(x,0) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Sigma \end{cases} \tag{2.1}$$

the solution operator

$$S : L^2(0,T;H^{-1}(\Omega)) \to L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)),$$

given by $Sf = u$, where u is the solution of problem (2.1). The next estimation theorem gives, on the one hand, the continuously dependence on f and g_0 of the solution u of problem (1.1)-(1.2) and, on the other hand, guarantees the nonexpansivity of the solution operator S from

$L^2\left(0,T;H^{-1}(\Omega)\right)$ to $L^2\left(0,T;H_0^1(\Omega)\right)$.

Theorem 2.1 *Let $f\in L^2\left(0,T;H^{-1}(\Omega)\right)$ and $g_0\in L^2(\Omega)$. If u is the solution of problem* (1.1)-(1.2), *then for every $t\in[0,T]$ one has*

$$|u|_{L^2\left(0,t;H_0^1(\Omega)\right)}\le\frac{1}{2}\left(|f|+\sqrt{|f|^2+2|g_0|^2}\right)\tag{2.2}$$

where $|f|=|f|_{L^2(0,t;H^{-1}(\Omega))}$ and $|g_0|=|g_0|_{L^2(\Omega)}$.

In particular, for $g_0=0$,

$$|u|_{L^2\left(0,t;H_0^1(\Omega)\right)}\le|f|_{L^2(0,t;H^{-1}(\Omega))},\quad t\in[0,T].$$

Proof. If we set $v=u(t)$ in (1.2) we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|_{L^2}^2+|u(t)|_{H_0^1}^2=(f(t),u(t)).$$

By integration

$$\frac{1}{2}|u(t)|_{L^2}^2-\frac{1}{2}|g_0|^2+|u|_{L^2\left(0,t;H_0^1(\Omega)\right)}^2\le|f|\;|u|_{L^2\left(0,t;H_0^1(\Omega)\right)}.\tag{2.3}$$

Then

$$|u|_{L^2\left(0,t;H_0^1\right)}^2-|f|\;|u|_{L^2\left(0,t;H_0^1\right)}-\frac{1}{2}|g_0|^2\le 0$$

whence the conclusion is immediate. ■

Let us now consider the nonlinear problem

$$\begin{cases}\frac{\partial u}{\partial t}-\Delta u=\Phi(u) & \text{in } Q\\ u(x,0)=g_0(x) & \text{in } \Omega\\ u=0 & \text{on } \Sigma.\end{cases}\tag{2.4}$$

The following existence and uniqueness result is established by means of Banach fixed point theorem.

Theorem 2.2 *Let $g_0\in L^2(\Omega)$ and*

$$\Phi:C\left([0,T];L^2(\Omega)\right)\to L^2\left(0,T;H^{-1}(\Omega)\right)$$

be a map for which there exists a constant $a \in \mathbf{R}_+$ *such that the following inequality holds for all* $u, v \in C\left([0,T]; L^2(\Omega)\right)$

$$|\Phi(u)(t) - \Phi(v)(t)|_{H^{-1}(\Omega)} \leq a\,|u(t) - v(t)|_{L^2(\Omega)} \quad a.e. \ on \ [0,T].$$

Then there exists a unique solution u *to problem* (2.4), *i.e.,*

$$u \in L^2\left(0,T; H_0^1(\Omega)\right) \cap C\left([0,T]; L^2(\Omega)\right), \quad u' \in L^2\left(0,T; H^{-1}(\Omega)\right)$$

and

$$\begin{cases} (u'(t), v) + (u(t), v)_{H_0^1} = (\Phi(u)(t), v) \\ \qquad a.e. \ on \ [0,T], \ for\ all \ v \in H_0^1(\Omega) \\ u(0) = g_0. \end{cases}$$

Proof. Let u_0 be the solution of problem (2.4) corresponding to $\Phi = 0$. We have to solve the fixed point problem

$$u = u_0 + (S \circ \Phi)(u), \quad u \in C\left([0,T]; L^2(\Omega)\right).$$

The conclusion will follow from Banach fixed point theorem once we have showed that the operator

$$A : C\left([0,T]; L^2(\Omega)\right) \to C\left([0,T]; L^2(\Omega)\right), \quad A(u) = u_0 + (S \circ \Phi)(u)$$

is a contraction with respect to a suitable norm on $C\left([0,T]; L^2(\Omega)\right)$. Let $u, v \in C\left([0,T]; L^2(\Omega)\right)$. We have $A(u)(0) - A(v)(0) = 0$ and

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|A(u)(t) - A(v)(t)|_{L^2}^2 + |A(u)(t) - A(v)(t)|_{H_0^1}^2 \\ = \ & (\Phi(u)(t) - \Phi(v)(t), A(u)(t) - A(v)(t)). \end{aligned}$$

It follows that

$$\begin{aligned} & |A(u)(t) - A(v)(t)|_{L^2}^2 \\ \leq \ & 2\int_0^t |\Phi(u)(s) - \Phi(v)(s)|_{H^{-1}} |A(u)(s) - A(v)(s)|_{H_0^1} ds \\ \leq \ & 2a\int_0^t |u(s) - v(s)|_{L^2} |A(u)(s) - A(v)(s)|_{H_0^1} ds. \end{aligned}$$

Let $\theta > a^2$ be a fixed number. Consider the norm on $C\left([0,T]\,;L^2\left(\Omega\right)\right)$

$$\|u\| = \max_{t\in[0,T]}\left(|u\left(t\right)|_{L^2(\Omega)}\,e^{-\theta\,t}\right).$$

Then

$$\begin{aligned}
& \left|A\left(u\right)\left(t\right) - A\left(v\right)\left(t\right)\right|_{L^2}^2 \\
\leq\ & 2a\left\|u - v\right\|\int_0^t \left|A\left(u\right)\left(s\right) - A\left(v\right)\left(s\right)\right|_{H_0^1} e^{\theta\,s}ds \\
\leq\ & \frac{2a}{\sqrt{2\theta}}e^{\theta\,t}\left\|u - v\right\|\left(\int_0^t \left|A\left(u\right)\left(s\right) - A\left(v\right)\left(s\right)\right|_{H_0^1}^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $A\left(u\right) - A\left(v\right) = S\left(\Phi\left(u\right) - \Phi\left(v\right)\right)$ and S is nonexpansive from $L^2\left(0,t;H^{-1}\left(\Omega\right)\right)$ to $L^2\left(0,t;H_0^1\left(\Omega\right)\right)$, we deduce that

$$\begin{aligned}
& \left|A\left(u\right)\left(t\right) - A\left(v\right)\left(t\right)\right|_{L^2}^2 \\
\leq\ & \frac{2a}{\sqrt{2\theta}}e^{\theta\,t}\left\|u - v\right\|\left(\int_0^t \left|\Phi\left(u\right) - \Phi\left(v\right)\right|_{H^{-1}}^2 ds\right)^{\frac{1}{2}} \\
\leq\ & \frac{2a^2}{\sqrt{2\theta}}e^{\theta\,t}\left\|u - v\right\|\left(\int_0^t \left|u\left(s\right) - v\left(s\right)\right|_{L^2}^2 ds\right)^{\frac{1}{2}} \\
\leq\ & \frac{a^2}{\theta}e^{2\theta\,t}\left\|u - v\right\|^2.
\end{aligned}$$

Divide by $e^{2\theta\,t}$ and take the maximum over $[0,T]$ to obtain

$$\left\|A\left(u\right) - A\left(v\right)\right\| \leq \frac{a}{\sqrt{\theta}}\left\|u - v\right\|.$$

Since $a/\sqrt{\theta} < 1$ the operator A is a contraction with respect to the norm $\|.\|$. ■

Examples

1. Let $\Psi : L^2(\Omega) \to H^{-1}(\Omega)$ be a map for which there exists a constant $a \in \mathbf{R}_+$ with

$$|\Psi(u) - \Psi(v)|_{H^{-1}(\Omega)} \le a\,|u - v|_{L^2(\Omega)}, \quad u, v \in L^2(\Omega). \tag{2.5}$$

Then the map $\Phi : C([0,T]; L^2(\Omega)) \to L^2(0,T; H^{-1}(\Omega))$ given by

$$\Phi(u)(t) = \Psi(u(t)) \quad \left(u \in L^2(0,T; L^2(\Omega)),\ t \in [0,T]\right)$$

satisfies all the assumptions of Theorem 2.2.

2. Let $\psi : \Omega \times \mathbf{R} \to \mathbf{R}$ be a function such that $\psi(.,\tau)$ is measurable for each $\tau \in \mathbf{R}$, $\psi(.,0) \in H^{-1}(\Omega)$ and there is a constant $a_0 \in \mathbf{R}_+$ with

$$|\psi(x,\tau_1) - \psi(x,\tau_2)| \le a_0\,|\tau_1 - \tau_2|$$

for a.e. $x \in \Omega$ and all $\tau_1, \tau_2 \in \mathbf{R}$.

Then the operator $\Psi : L^2(\Omega) \to H^{-1}(\Omega)$ defined by

$$\Psi(u) = \psi(., u(.))$$

satisfies all the assumptions of the previous example.

Indeed, we may write

$$\Psi(u) = \psi(.,0) + \sigma(., u(.))$$

where $\sigma(x,\tau) = \psi(x,\tau) - \psi(x,0)$. Notice that $|\sigma(x,\tau)| \le a_0\,|\tau|$ (for a.e. $x \in \Omega$ and all $\tau \in \mathbf{R}$), so the superposition operator $\sigma(., u(.))$ (see [7]) maps $L^2(\Omega)$ into $L^2(\Omega)$ and $|\sigma(., u(.)) - \sigma(., v(.))|_{L^2(\Omega)} \le a_0\,|u - v|_{L^2(\Omega)}$. The imbedding $L^2(\Omega) \subset H^{-1}(\Omega)$ being continuous there exists a constant $a \in \mathbf{R}_+$ such that (2.5) holds.

The next lemmas are used in order to apply Schauder fixed point theorem, more exactly for proving the complete continuity of the solution operator S.

Lemma 2.1 (Ascoli–Arzèla) *Let $(B, |.|_B)$ be a Banach space. A subset F of $C(0,T;B)$ is relatively compact if and only if $F(t) = \{f(t) : f \in$*

$F\}$ is relatively compact in B for each $t \in [0,T]$ and F is equicontinuous, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(t_1) - f(t_2)|_B \leq \varepsilon$ for all $f \in F$ and $t_1, t_2 \in [0,T]$ with $|t_1 - t_2| \leq \delta$.

See O'Regan and Precup [5, p. 72] for a proof.

Lemma 2.2 (Lions) *Let X, B and Y be Banach spaces such that the following imbeddings hold:*

$$X \subset B \text{ compactly and } B \subset Y \text{ continuously.}$$

Then, for every $\eta > 0$ there is an $N \geq 0$ with

$$|u|_B \leq \eta\, |u|_X + N\, |u|_Y \quad \text{for all } u \in X. \tag{2.6}$$

Proof. For each $n \in \mathbf{N}$, let $U_n = \{u \in B : |u|_B < \eta + n\,|u|_Y\}$. The sets U_n are open in B, $U_n \subset U_{n+1}$ and $B = \bigcup_{n\in\mathbf{N}} U_n$. The unit ball S of X being relatively compact in B, there exists an N such that $S \subset U_N$. Hence $|v|_B < \eta\,|v|_X + N\,|v|_Y$ for every $v \in X$ with $|v|_X = 1$. The inequality for any $u \in X$ can be immediately derived if we let $v = u/\,|u|_X$. ■

Lemma 2.3 *Let X, B and Y be as in Lemma 2.2. If a set F is bounded in $L^p(0,T;X)$ and relatively compact in $L^p(0,T;Y)$, where $1 \leq p \leq \infty$, then F is relatively compact in $L^p(0,T;B)$.*

Proof. For a given $\varepsilon > 0$, there exists a sequence $\{f_j\}$ of elements of F such that for every $f \in F$, there is a f_j with $|f - f_j|_{L^p(0,T;Y)} \leq \varepsilon$. Inequality (2.6) implies

$$\begin{aligned} |f - f_j|_{L^p(0,T;B)} &\leq \eta\,|f - f_j|_{L^p(0,T;X)} + N\,|f - f_j|_{L^p(0,T;Y)} \\ &\leq \eta\, c + N\varepsilon \end{aligned}$$

where c is the diameter of F in $L^p(0,T;X)$. For any $\varepsilon' > 0$, if we choose $\eta = \varepsilon'/(2c)$ and $\varepsilon = \varepsilon'/(2N)$, we obtain $|f - f_j|_{L^p(0,T;B)} \leq \varepsilon$. This guarantees that F is relatively compact in $L^p(0,T;B)$. ■

Theorem 2.3 *The solution operator* S *is completely continuous from* $L^2\left(0,T;H^{-1}(\Omega)\right)$ *to* $L^2\left(0,T;L^p(\Omega)\right)$ *for* $(2^*)' \le p < 2^*$ *if* $n \ge 3$ *and for any* $p \ge 1$ *if* $n = 1$ *or* $n = 2$.

Proof. Under the assumptions on p we have $H_0^1(\Omega) \subset L^p(\Omega) \subset H^{-1}(\Omega)$, where the first imbedding is compact and the second one is continuous (see Adams [1] and Precup [6]). According to Lemma 2.3 it suffices to prove that for any bounded subset M of $L^2\left(0,T;H^{-1}(\Omega)\right)$, the set $S(M)$ is bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$ (which is true by Theorem 2.1) and relatively compact in $L^2\left(0,T;H^{-1}(\Omega)\right)$. We shall prove more, namely that $S(M)$ is relatively compact in $C\left([0,T];H^{-1}(\Omega)\right)$. From inequalities (2.2) and (2.3) we deduce that $S(M)$ is bounded in $C\left([0,T];L^2(\Omega)\right)$. Then, for each $t \in [0,T]$, the set $S(M)(t)$ is bounded in $L^2(\Omega)$ and so relatively compact in $H^{-1}(\Omega)$. It remains to prove that $S(M)$ is equicontinuous in $C\left([0,T];H^{-1}(\Omega)\right)$.

Notice that

$$S(M)' = \{u' : u \in S(M)\}$$

is bounded in $L^2\left(0,T;H^{-1}(\Omega)\right)$. Indeed, if $u = S(f)$, $f \in M$, then $u'(t) = \Delta u(t) + f(t)$, whence $|u'(t)|_{H^{-1}} \le |\Delta u(t)|_{H^{-1}} + |f(t)|_{H^{-1}}$. Since Δ is an isometry between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, we have $|\Delta u(t)|_{H^{-1}} = |u(t)|_{H_0^1}$. Consequently

$$\begin{aligned} |u'|_{L^2(0,T;H^{-1}(\Omega))} &\le |u|_{L^2\left(0,T;H_0^1(\Omega)\right)} + |f|_{L^2(0,T;H^{-1}(\Omega))} \\ &\le 2\,|f|_{L^2(0,T;H^{-1}(\Omega))}. \end{aligned}$$

Thus $S(M)'$ is bounded in $L^2\left(0,T;H^{-1}(\Omega)\right)$.

Furthermore, from

$$u(t_1) - u(t_2) = \int_{t_2}^{t_1} u'(s)\,ds$$

we deduce that

$$
\begin{aligned}
|u(t_1) - u(t_2)|_{H^{-1}} &\leq \left| \int_{t_2}^{t_1} |u'(s)|_{H^{-1}}\, ds \right| \\
&\leq \sqrt{|t_1 - t_2|}\, |u'|_{L^2(0,T;H^{-1}(\Omega))}
\end{aligned}
$$

whence it follows the equicontinuity of $S(M)$ in $C\left([0,T]; H^{-1}(\Omega)\right)$. ■

The next existence result comes from Schauder fixed point theorem. The Lipschitz condition on the nonlinear term Φ in Theorem 2.2 is weakened to a growth condition at most linear.

Theorem 2.4 *Let $g_0 \in L^2(\Omega)$ and*

$$
\Phi : L^2\left(0,T; L^2(\Omega)\right) \to L^2\left(0,T; H^{-1}(\Omega)\right)
$$

be a continuous map for which there is a constant $a \in \mathbf{R}_+$ such that the following inequality holds for all $u \in C\left([0,T]; L^2(\Omega)\right)$

$$
|\Phi(u)(t) - \Phi(0)(t)|_{H^{-1}(\Omega)} \leq a\, |u(t)|_{L^2(\Omega)}, \quad a.e. \ on \ [0,T].
$$

Then there exists at least one solution to problem (2.4).

Proof. We look for a fixed point of the operator

$$
A : L^2\left(0,T; L^2(\Omega)\right) \to L^2\left(0,T; L^2(\Omega)\right), \quad A(u) = u_0 + (S \circ \Phi)(u)
$$

Theorem 2.3 and the boundedness of the operator Φ, guarantee the complete continuity of A. It remains to find a nonempty, bounded, closed and convex subset D of $L^2\left(0,T; L^2(\Omega)\right)$ with $A(D) \subset D$.

Let $u \in C\left([0,T]; L^2(\Omega)\right)$. As in the proof of Theorem 2.2, for $\theta > a^2$ one obtains

$$
\|A(u) - A(0)\| \leq \frac{a}{\sqrt{\theta}} \|u\|.
$$

It follows that

$$
\|A(u)\| \leq \|A(0)\| + \frac{a}{\sqrt{\theta}} \|u\|.
$$

Choosing a $\theta > a^2$ and a positive number $R \geq \|A(0)\| / \left(1 - a/\sqrt{\theta}\right)$ we deduce that $A(D_0) \subset D_0$ for $D_0 = \{u \in C\left([0,T]; L^2(\Omega)\right) : \|u\| \leq R\}$. Let D be the closure of D_0 in $L^2\left(0,T; L^2(\Omega)\right)$. Clearly D is nonempty, bounded, closed and convex in $L^2\left(0,T; L^2(\Omega)\right)$. In addition, using the continuity of A from $L^2\left(0,T; L^2(\Omega)\right)$ to itself, we immediately see that $A(D) \subset D$. Thus, the Schauder fixed point theorem applies. ■

Examples

1. Let $\Psi : L^2(\Omega) \to H^{-1}(\Omega)$ be a continuous map for which there is a constant $a \in \mathbf{R}_+$ with

$$|\Psi(u) - \Psi(0)|_{H^{-1}(\Omega)} \leq a\,|u|_{L^2(\Omega)}\,, \quad u \in L^2(\Omega)\,. \tag{2.7}$$

Then the map $\Phi : L^2\left(0,T; L^2(\Omega)\right) \to L^2\left(0,T; H^{-1}(\Omega)\right)$ given by

$$\Phi(u)(t) = \Psi(u(t)) \quad \left(u \in L^2\left(0,T; L^2(\Omega)\right),\ t \in [0,T]\right)$$

satisfies all the assumptions of Theorem 2.4.

2. Let $\psi : \Omega \times \mathbf{R} \to \mathbf{R}$ be a function such that $\psi(.,\tau)$ is measurable for every $\tau \in \mathbf{R}$, $\psi(x,.)$ is continuous for a.e. $x \in \Omega$, $\psi(.,0) \in H^{-1}(\Omega)$ and there is $a_0 \in \mathbf{R}_+$ with

$$|\psi(x,\tau) - \psi(x,0)| \leq a_0\,|\tau|$$

for a.e. $x \in \Omega$ and all $\tau \in \mathbf{R}$.

Then the operator $\Psi : L^2(\Omega) \to H^{-1}(\Omega)$ defined by

$$\Psi(u) = \psi(., u(.))$$

satisfies all the conditions of the previous example.

We finish with a result concerning a superlinear problem. It is established by means of Leray–Schauder fixed point theorem (see Precup [7]).

Theorem 2.5 *Let $n \geq 3$, $g_0 \in L^2(\Omega)$, $f \in H^{-1}(\Omega)$ and $\psi : \Omega \times \mathbf{R} \to \mathbf{R}$ be a function such that $\psi(.,u)$ is measurable for every $u \in \mathbf{R}$, $\psi(x,.)$ is continuous for a.e. $x \in \Omega$, $\psi(.,0) = 0$ and there are constants $a, \alpha \in \mathbf{R}_+$*

with $1 \leq \alpha < 2^* - 1$ *and*

$$|\psi(x,u)| \leq a\,|u|^{a} \tag{2.8}$$

for a.e. $x \in \Omega$ *and all* $u \in \mathbf{R}$*. In addition assume that*

$$u\,\psi(x,u) \leq 0 \tag{2.9}$$

for a.e. $x \in \Omega$ *and all* $u \in \mathbf{R}$*. Then there exists at least one solution to problem* (2.4)*, where* $\Phi(u)(t) = \psi(.,u(t)) + f$*.*

Proof. Let $p = \alpha\,(2^*)'$. From $1 \leq \alpha < 2^* - 1 = 2^*/(2^*)'$ one has $(2^*)' \leq p < 2^*$. Then Theorem 2.3 guarantees that the solution operator S is completely continuous from $L^2\left(0,T;H^{-1}(\Omega)\right)$ into $L^2(0,T;L^p(\Omega))$. We look for a fixed point of the operator

$$A : L^2(0,T;L^p(\Omega)) \to L^2(0,T;L^p(\Omega)), \quad A(u) = u_0 + (S \circ \Phi)(u).$$

Notice $A(u) = u_0 + S(f) + S(\Phi_0(u))$, where $\Phi_0(u)(t) = N_\psi(u(t))$ and N_ψ is the superposition operator associated to ψ. Hypothesis (2.8) guarantees that N_ψ is well-defined, continuous and bounded from $L^p(\Omega)$ to $L^{(2^*)'}(\Omega)$ and $|N_\psi(v)|_{L^{(2^*)'}} \leq a\,|v|_{L^p}^{\alpha}$. As a result, the operator Φ_0 is well-defined, continuous and bounded from $L^2(0,T;L^p(\Omega))$ to $L^2\left(0,T;L^{(2^*)'}(\Omega)\right)$ and, consequently, from $L^2(0,T;L^p(\Omega))$ to $L^2\left(0,T;H^{-1}(\Omega)\right)$. Therefore A is completely continuous.

Furthermore, we show that there exists a constant $R > 0$ such that $|u|_{L^2(0,T;L^p(\Omega))} < R$ for any solution u of $u = \lambda A(u)$ and any $\lambda \in (0,1)$.

Indeed, if $u = \lambda A(u)$, then

$$\begin{cases} (u'(t),v) + (u(t),v)_{H_0^1} = \lambda\,(\psi(.,u(t)) + f, v), & v \in H_0^1(\Omega) \\ u(0) = \lambda g_0. \end{cases}$$

For $v = u(t)$, using (2.9) we deduce

$$\begin{aligned} &\frac{d}{dt}|u(t)|_{L^2(\Omega)}^2 + |u(t)|_{H_0^1(\Omega)}^2 = \lambda\,(\psi(.,u(t)) + f, u(t)) \\ &\leq \lambda\,(f,u(t)) \leq |f|_{L^{(2^*)'}(\Omega)}\,|u(t)|_{L^{2^*}(\Omega)} \leq c\,|u(t)|_{H_0^1(\Omega)}. \end{aligned}$$

Integration gives

$$\int_0^T |u(t)|^2_{H_0^1(\Omega)}\, dt \le c' \left(\int_0^T |u(t)|^2_{H_0^1(\Omega)}\, dt \right)^{\frac{1}{2}} + c''$$

where constants c, c' do not depend on u and λ. Hence $|u|_{L^2\left(0,T;H_0^1(\Omega)\right)} \le C$, whence, since $H_0^1(\Omega) \subset L^p(\Omega)$, we immediately obtain an estimation of the type $|u|_{L^2(0,T;L^p(\Omega))} < R$.

The Leray–Schauder fixed point theorem finishes the proof. ■

Example

The function $\psi(x,u) = -|u|^{\alpha-1} u \ \ (u \in \mathbf{R})$, where $1 \le \alpha < 2^* - 1$, satisfies all the assumptions of Theorem 2.5.

For related results we refer the reader to Taylor [8].

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