Localization of critical points via mountain pass type theorems

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1 Introduction

The so called mountain pass theorem of Ambrosetti and Rabinowitz [1] is one of the most used tools in studying nonlinear equations having a variational form (see [10], [17] and [20]). It concerns a real-valued C^1 functional $E(u)$ defined on a real Banach space X, for which one desires to find a critical point, i.e., a point u where $E'(u) = 0$.

Theorem 1 (Ambrosetti–Rabinowitz) *Let X be a Banach space and $E \in C^1(X)$. Assume that there exist $u_0, u_1 \in X$ and r with $|u_0| < r < |u_1|$ such that*

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : u \in X, \ |u| = r\}.$$

Let

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = u_0, \ \gamma(1) = u_1\} \tag{1}$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)). \tag{2}$$

Then there exists a sequence of elements $u_k \in X$ such that

$$E(u_k) \to c, \quad E'(u_k) \to 0 \quad as \quad k \to \infty.$$

If, in addition, E satisfies the Palais-Smale condition, i.e. any sequence as above has a convergent subsequence, then there exists an element $u \in X \setminus \{u_0, u_1\}$ with

$$E(u) = c, \quad E'(u) = 0. \tag{3}$$

Notice Γ is the set of all continuous paths joining u_0 and u_1. Roughly speaking, the mountain pass theorem says that if we are at the point u_0 of altitude $E(u_0)$ located in a cauldron surrounded by high mountains, and we wish to reach to a point u_1 of altitude $E(u_1)$, over there the mountains, we can find a path going from u_0 to u_1, through a mountain pass. To find a mountain pass we have to choose a path which mounts the least. Thus we have to consider an optimal path in the set of all continuous paths connecting two given points separated by a "mountain range". A number of authors have been interested to restrict the competing paths to a bounded region in order to locate a critical point. For example, in [6] the authors gave a variant of the mountain pass theorem in a convex set M of the Hilbert space X (identified to its dual), using the Schauder invariance condition $(I - E')(M) \subset M$, while in [18] (see also [19] and [14]) a critical point is located in a ball $\overline{B}_R$ of X under the Leray-Schauder boundary condition for $I - E'$. Here I stands for the identity map of X.

Theorem 2 (Schechter) *Let X be a Hilbert space, $R > 0$ and $E \in C^1\left(\overline{B}_R\right)$. Assume that for some $\nu_0 > 0$,*

$$(E'(u), u) \geq -\nu_0, \quad u \in \partial B_R \tag{4}$$

and that there are $u_0,\, u_1 \in \overline{B}_R$ and r with $|u_0| < r < |u_1|$ such that

$$\max\{E(u_0), E(u_1)\} < \inf\left\{E(u) : u \in \overline{B}_R,\ |u| = r\right\}. \tag{5}$$

Let

$$\Gamma_R = \left\{\gamma \in C\left([0,1]; \overline{B}_R\right) : \gamma(0) = u_0,\ \gamma(1) = u_1\right\}$$

and

$$c_R = \inf_{\gamma \in \Gamma_R} \max_{t \in [0,1]} E(\gamma(t)).$$

Then either there is a sequence of elements $u_k \in \overline{B}_R$ with

$$E(u_k) \to c_R, \quad E'(u_k) \to 0, \tag{6}$$

or there is a sequence of elements $u_k \in \partial B_R$ such that

$$E(u_k) \to c_R,\ E'(u_k) - \frac{(E'(u_k), u_k)}{R^2} u_k \to 0,\ (E'(u_k), u_k) \leq 0. \tag{7}$$

If in addition E satisfies the Schechter-Palais-Smale condition, i.e. any sequence as above has a convergent subsequence, and

$$E'(u) + \mu u \neq 0,\ u \in \partial B_R,\ \mu > 0, \tag{8}$$

then there exists an element $u \in \overline{B}_R \setminus \{u_0, u_1\}$ with

$$E(u) = c_R, \quad E'(u) = 0.$$

Remark 1 (8) is the Leray–Schauder boundary condition (see [11]) for the operator $I - E'$, i.e., it is equivalent to

$$u \neq \lambda (I - E')(u), \quad u \in \partial B_R,\ \lambda \in (0,1).$$

The Schauder and the Leray-Schauder conditions are used to solve the difficult problem of constructing paths which do not leave region M. Such a construction suggested in [8] to introduce the notion of an invariant set of descending flow of E with respect to a pseudogradient of E. Thus the difficult problem is reduced to prove that for a given set M, there exists a pseudogradient with respect to which M is an invariant set of descending flow (a difficult problem as well). Related topics can be found in [3], [5], [9], [12] and [16].

In this paper, we first survey some of our existence results for abstract Hammerstein equations established in [12] and [14], and then we present our recent results [15] concerning the localization of critical points in conical shells with application to a two point boundary value problem.

2 Nontrivial Solvability of Abstract Hammerstein Equations

Here we discuss the abstract Hammerstein equation

$$u = AN(u), \quad u \in Y, \tag{9}$$

where Y is a Banach space, $N : Y \to Y^*$ and $A : Y^* \to Y$ is linear. Assume that A splits into

$$\begin{cases} A = HH^* \text{ with } H : X \to Y \text{ and } H^* : Y^* \to X, \\ \text{where } X \text{ is a Hilbert space.} \end{cases} \tag{10}$$

Then (9) can be converted into an equation in X, namely

$$v = H^*NH(v), \quad v \in X. \tag{11}$$

Indeed, if u solves (9) then $v = H^*N(u)$ is a solution of (11), and conversely if v solves (11) then $u = H(v)$ is a solution of (9). Moreover, H realizes an one-to-one correspondence between the solution sets of the two equations. If, in addition, we assume

$$N = J' \text{ for some } J \in C^1(Y; \mathbf{R}), \ J(0) = 0, \tag{12}$$

and

$$H \text{ is bounded linear and } H^* \text{ is the adjoint of } H, \tag{13}$$

then (11) is equivalent to the critical point problem

$$E'(v) = 0, \quad v \in X$$

for the energy functional

$$E : X \to \mathbf{R}, \quad E(v) = \frac{1}{2}|v|_X^2 - JH(v).$$

Here $|.|_X$ stands for the norm of X. Notice $E \in C^1(X; \mathbf{R})$ and

$$E'(v) = v - H^* N H(v), \quad v \in X.$$

We now state an existence principle for (9) in a ball of X, whose proof is based on Theorem 2.

Theorem 3 *Assume* (10), (12) *and* (13). *Assume that* $N(0) = 0$ *and the functional* $(N(.), .)$ *sends bounded sets into upper bounded sets. In addition assume that there are* $v_1 \in X \setminus \{0\}$, $r \in (0, |v_1|)$ *and* $R \geq |v_1|$ *such that the following conditions are satisfied:*

$$\max\left\{0, |v_1|_X^2/2 - JH(v_1)\right\} < \inf\left\{|v|_X^2/2 - JH(v) : |v|_X = r\right\}, \tag{14}$$

$$v \neq \lambda H^* N H(v) \quad \text{for } |v|_X = R, \ \lambda \in (0,1), \tag{15}$$

$$E \text{ satisfies the Schecter-Palais-Smale condition.} \tag{16}$$

Then there exists a $v \in X \setminus \{0\}$ *with* $|v|_X \leq R$ *such that* $u = H(v)$ *is a non-zero solution of* (9).

Specialized for the Hammerstein integral equation in $\mathbf{R}^n$

$$u(x) = \int_\Omega \kappa(x,y) f(y, u(y)) \, dy \quad \text{a.e. on } \Omega, \tag{17}$$

Theorem 3 gives:

Theorem 4 *Let* $\Omega \subset \mathbf{R}^N$ *be bounded open,* $2 \leq p < p_0 < \infty$, $1/p + 1/q = 1$, $1/p_0 + 1/q_0 = 1$, $\kappa : \Omega^2 \to \mathbf{R}$ *and* $f : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$. *Assume the following conditions are satisfied:*

(*i*) *the operator* $A : L^{q_0}(\Omega; \mathbf{R}^n) \to L^{p_0}(\Omega; \mathbf{R}^n)$ *given by*

$$A(u)(x) = \int_\Omega \kappa(x,y) u(y) \, dy$$

is bounded and its restriction $A : L^2(\Omega; \mathbf{R}^n) \to L^2(\Omega; \mathbf{R}^n)$ *is positive, self-adjoint, and completely continuous;*

(*ii*) f *is* (p,q)*-Carathéodory of potential* F *and* $f(x,0) = 0$ *a.e. on* Ω;

(*iii*) *there are* $v_1 \in L^2(\Omega; \mathbf{R}^n) \setminus \{0\}$ *and* $r \in (0, |v_1|_2)$ *such that*

$$\max\left\{0, \ |v_1|_2^2/2 - \int_\Omega F(y, v_1(y)) \, dy\right\} \tag{18}$$

$$< \ \inf\left\{|v|_2^2/2 - \int_\Omega F(y, v(y)) \, dy : \ v \in L^2(\Omega; \mathbf{R}^n), \ |v|_2 = r\right\};$$

(*iv*) *there is* $R \geq |v_1|_2$ *such that*

$$H(v) \neq \lambda \int_\Omega \kappa(., y) f(y, H(v)(y)) \, dy \tag{19}$$

for every $v \in L^2(\Omega; \mathbf{R}^n)$ *with* $|v|_2 = R$ *and all* $\lambda \in (0,1)$.

Then the Hammerstein equation (17) *has at least one non-zero solution* u *in* $L^p(\Omega; \mathbf{R}^n)$ *of the form* $u = H(v)$ *with* $v \in L^2(\Omega; \mathbf{R}^n)$, $|v|_2 \leq R$.

Proof. Apply Theorem 3 to $N = N_f$ and J given by

$$J(u) = \int_\Omega F(y, u(y))\, dy \quad (u \in L^p(\Omega; \mathbf{R}^n)).$$

Since N_f is a bounded operator, the map $(N_f(.), .)$ sends bounded sets into bounded sets. By (iii), (iv), conditions (14) and (15) hold trivially. It remains to show that the attached functional E satisfies the Schechter-Palais-Smale condition. To do this let (v_k) be any sequence of elements in $L^2(\Omega; \mathbf{R}^n)$ with $0 < |v_k|_2 \leq R$ satisfying

$$E(v_k) \to \mu \in \mathbf{R}, \quad (E'(v_k), v_k)_2 \to \nu \leq 0$$

and

$$E'(v_k) - |v_k|_2^{-2} (E'(v_k), v_k)_2\, v_k \to 0. \tag{20}$$

We may assume that $|v_k|_2 \to a$, for some $a \in [0, R]$. If $a = 0$ we have finished. Assume $a \in (0, R]$. Then

$$|v_k|_2^{-2} (E'(v_k), v_k)_2 \to a^{-2}\nu \in (-\infty, 0].$$

On the other hand, (v_k) being bounded and H being completely continuous, we may assume, passing eventually to a subsequence, that $H(v_k)$ converges. Then since N_f and H^* are continuous we deduce that $H^* N_f H(v_k)$ converges too. Now from (20) which can be written as

$$v_k - H^* N_f H(v_k) - |v_k|_2^{-2} (E'(v_k), v_k)_2\, v_k \to 0$$

we infer that the corresponding subsequence of (v_k) is convergent. Thus E satisfies the Schechter-Palais-Smale condition. ■

Remark 2 If $p > 2$, a sufficient condition for (iii) is the following one:

(iii^*) there is a $v_1 \in L^2(\Omega; \mathbf{R}^n) \setminus \{0\}$ such that

$$\frac{1}{2} |v_1|_2^2 - \int_\Omega F(y, v_1(y))\, dy \leq 0, \tag{21}$$

$$|f(x, z)| \leq a\left(1 + |z|^{p-1}\right) \tag{22}$$

for a.e. $x \in \Omega$ and all $z \in R^n$, where $a \in R_+$, and

$$\lim_{z \to 0} |f(x, z)| / |z| = 0 \tag{23}$$

uniformly for a.e. $x \in \Omega$.

Indeed, (23) implies that for any given $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x, z)| \leq \varepsilon |z| \quad \textit{for a.e. } x \in \Omega \textit{ and } |z| < \delta.$$

For $|z| \geq \delta,$ from (22) we deduce

$$\begin{aligned} |f(x,z)| &\leq a\left(\frac{|z|}{\delta}+|z|^{p-1}\right) \leq a\left(\left(\frac{|z|}{\delta}\right)^{p-1}+|z|^{p-1}\right) \\ &= c(\varepsilon)\,|z|^{p-1}. \end{aligned}$$

Here $c(\varepsilon) = a\left(\delta^{1-p}+1\right) \geq 0.$ Hence

$$|f(x,z)| \leq \varepsilon\,|z| + c(\varepsilon)\,|z|^{p-1}$$

for a.e. $x \in \Omega$ and all $z \in R^n$. Next we obtain

$$\begin{aligned} E(v) &= \frac{1}{2}|v|_2^2 - JH(v) \\ &\geq \frac{1}{2}|v|_2^2 - \int_\Omega |H(v)(x)|\left|f\left(x,\theta_{H(v),0}(x)\,H(v)(x)\right)\right| dx \\ &\geq \frac{1}{2}|v|_2^2 - \int_\Omega |H(v)|\left(\varepsilon\,|H(v)| + c(\varepsilon)\,|H(v)|^{p-1}\right) dx \\ &= \frac{1}{2}|v|_2^2 - \varepsilon\,|H(v)|_2^2 - c(\varepsilon)\,|H(v)|_p^p. \end{aligned}$$

Since

$$\begin{aligned} |H(v)|_2^2 &= (H(v), H(v))_2 = (H^*H(v), v)_2 \\ &\leq |H^*|\,|H|\,|v|_2^2 \end{aligned}$$

and

$$|H(v)|_p \leq c\,|v|_2$$

because H is bounded, we deduce

$$\begin{aligned} E(v) &\geq \frac{1}{2}|v|_2^2 - \varepsilon\,|H^*|\,|H|\,|v|_2^2 - \overline{c}(\varepsilon)\,|v|_2^p \\ &= \left(\frac{1}{2} - \varepsilon\,|H^*|\,|H| - \overline{c}(\varepsilon)\,|v|_2^{p-2}\right)|v|_2^2. \end{aligned}$$

Now we choose any $\varepsilon > 0$ such that

$$\frac{1}{2} - \varepsilon\,|H^*|\,|H| > 0$$

and an $r > 0$ small enough so that

$$r < |v_1|_2, \quad \frac{1}{2} - \varepsilon\,|H^*|\,|H| - \overline{c}(\varepsilon)\,r^{p-2} > 0$$

(recall $p > 2$). Then

$$E(v) > 0 \quad \textit{for} \;\; |v|_2 = r.$$

This together with (21) guarantees (18).

Remark 3 Assume $H : L^2(\Omega; \mathbf{R}^n) \to L^p(\Omega; \mathbf{R}^n)$ is one-to-one. Then, by the theorem on the continuity of the inverse operator, there are constants $\alpha, \beta > 0$ such that

$$\alpha |v|_2 \leq |H(v)|_p \leq \beta |v|_2, \quad v \in L^2(\Omega; \mathbf{R}^n).$$

Now a sufficient condition for (iv) is to exist an $R \geq |v_1|_2$ with

$$|A|_{q,p} \left(|g|_q + c\, \beta^{p-1} R^{p-1} \right) \leq \alpha R.$$

Here $|A|_{q,p}$ is norm of A as operator from $L^q(\Omega; \mathbf{R}^n)$ to $L^p(\Omega; \mathbf{R}^n)$, whilst $g \in L^q(\Omega; \mathbf{R}_+)$ and $c \in R_+$ come from the (p,q)-Carathéodory property of f, i.e.,

$$|f(x,z)| \leq g(x) + c|z|^{p-1}$$

for a.e. $x \in \Omega$ and all $z \in R^n$.

Indeed, if $v \in L^2(\Omega; \mathbf{R}^n)$ satisfies $|v|_2 = R$ then

$$\begin{aligned} |AN_f H(v)|_p &\leq |A|_{q,p} |N_f H(v)|_q \\ &\leq |A|_{q,p} \left(|g|_q + c\, |H(v)|_p^{p-1} \right) \\ &\leq |A|_{q,p} \left(|g|_q + c\, \beta^{p-1} R^{p-1} \right) \\ &\leq \alpha R \\ &\leq |H(v)|_p. \end{aligned}$$

This guarantees (19) for all $\lambda \in (0,1)$.

Similar results for the existence of solutions in a ball of $L^p(\Omega; \mathbf{R}^n)$ are presented in [12], [13] by means of a mountain pass theorem on closed convex sets owed to Guo–Sun–Qi [6].

3 Critical point theorems in conical shells

Let us consider two real Hilbert spaces, X with inner product and norm $(.,.)$, $|.|$, and H with inner product and norm $\langle .,. \rangle$, $\|.\|$, and we assume that $X \subset H$, X is dense in H, the injection being continuous. We shall denote by c_0 the imbedding constant with

$$\|u\| \leq c_0 |u| \quad \text{for all } u \in X.$$

We identify H to its dual H', thanks to the Riesz representation theorem and we obtain

$$X \subset H \equiv H' \subset X'$$

where each space is dense in the following one, the injections being continuous. By $\langle .,. \rangle$ we also denote de natural duality between X and X', that is $\langle x^*, x \rangle = x^*(x)$ for $x \in X$ and $x^* \in X'$. When $x^* \in H$, one has that $\langle x^*, x \rangle$ is exactly the scalar product in H of x and x^*. Let L be the linear continuous operator from X to X' (the canonical isomorphism of X onto X'), given by

$$(u,v) = \langle Lu, v\rangle, \quad \text{for all } u,v \in X$$

and let J from X' into X, be the inverse of L. Then

$$(Ju,v) = \langle u,v\rangle \text{ for all } u \in X', v \in X.$$

This for $u \in H$ implies

$$|Ju|^2 = \langle u, Ju\rangle \leq \|u\| \, \|Ju\| \leq c_0 \|u\| \, |Ju| \, .$$

Hence

$$|Ju| \leq c_0 \|u\| \, . \tag{24}$$

We consider a C^1 real functional E defined on X and we are interested in the equation $E'(u) = 0$.

By a wedge of X we shall understand a convex closed nonempty set $K \subset X$, $K \neq \{0\}$, with $\lambda u \in K$ for every $u \in K$ and $\lambda \geq 0$. Thus K has not necessarily be a cone (when $K \cap (-K) = \{0\}$) and, in particular, K might be the whole space X.

In what follows we shall assume that J is "positive" with respect to K, i.e.,

$$Ju \in K \text{ for every } u \in K.$$

Let R_0, R_1 be such that $0 < R_0 < c_0 R_1$ and let $K_{R_0 R_1}$ be the conical shell

$$K_{R_0 R_1} = \{u \in K : \|u\| \geq R_0, \ |u| \leq R_1\}\, .$$

In applications, $|.|$ is the specific energy norm, while $\|.\|$ is an L^p-norm which can be used instead of $|.|$ because of its monotonicity property with respect to the order relation.

Notice that there exists a number R with $R \leq R_1$ and

$$|Ju| \geq R > 0 \ \text{ for all } u \in K_{R_0 R_1}. \tag{25}$$

Indeed, otherwise, there would be a sequence (u_k) of elements in $K_{R_0 R_1}$ with $|Ju_k| \to 0$ as $k \to \infty$. Now, from

$$R_0^2 \leq \|u_k\|^2 = \langle u_k, u_k\rangle = (Ju_k, u_k) \leq |Ju_k| \, |u_k| \leq R_1 \, |Ju_k|$$

letting $k \to \infty$, we derive the contradiction $R_0^2 \leq 0$.

In [15], starting from the results in [18], [19], we have presented a variant of the mountain pass theorem in the conical shell $K_{R_0 R_1}$ assuming that the operator $I - JE'$ satisfies a compression boundary condition like that in the corresponding fixed point theorem of Krasnoselskii [7]. The localization result immediately yields multiplicity results for functionals with a "wavily relief".

Theorem 5 *Assume that there exist* $u_0, u_1 \in K_{R_0 R_1}$ *and* ν_0, $r > 0$, $|u_0| < r < |u_1|$, *such that the following conditions are satisfied:*

$$u - JE'(u) \in K \text{ for all } u \in K; \tag{26}$$

$$(JE'(u), Ju) \le \nu_0 \text{ for all } u \in K_{R_0 R_1} \text{ with } \|u\| = R_0; \tag{27}$$

$$(JE'(u), u) \ge -\nu_0 \text{ for all } u \in K_{R_0 R_1} \text{ with } |u| = R_1; \tag{28}$$

$$\max\{E(u_0), E(u_1)\} < \inf_{\substack{u \in K_{R_0 R_1} \\ |u| = r}} E(u). \tag{29}$$

Let

$$\Gamma = \{\gamma \in C([0,1]; K_{R_0 R_1}) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)).$$

Then there exists a sequence (u_k) *with* $u_k \in K_{R_0 R_1}$ *such that*

$$E(u_k) \to c \quad \text{as} \quad k \to \infty \tag{30}$$

and one of the following three properties holds:

$$E'(u_k) \to 0 \quad \text{as} \quad k \to \infty; \tag{31}$$

$$\begin{cases} \|u_k\| = R_0, \quad (JE'(u_k), Ju_k) \ge 0 \quad \text{and} \\ JE'(u_k) - \frac{(JE'(u_k), Ju_k)}{|Ju_k|^2} Ju_k \to 0 \ (\text{in } X) \quad \text{as} \quad k \to \infty; \end{cases} \tag{32}$$

$$\begin{cases} |u_k| = R_1, \quad (JE'(u_k), u_k) \le 0 \quad \text{and} \\ JE'(u_k) - \frac{(JE'(u_k), u_k)}{R_1^2} u_k \to 0 \ (\text{in } X) \ \text{as} \quad k \to \infty. \end{cases} \tag{33}$$

If in addition, any sequence (u_k) *as above has a convergent (in* X*) subsequence and* E *satisfies the boundary conditions*

$$JE'(u) - \lambda Ju \ne 0 \text{ for } u \in K_{R_0 R_1},\ \|u\| = R_0,\ \lambda > 0 \tag{34}$$

$$JE'(u) + \lambda u \ne 0 \text{ for } u \in K_{R_0 R_1},\ |u| = R_1,\ \lambda > 0, \tag{35}$$

then there exists $u \in K_{R_0 R_1}$ *with*

$$E'(u) = 0 \text{ and } E(u) = c.$$

Remark 4 Let $N(u) := u - JE'(u)$. Conditions (34), (35) can be written under the form

$$N(u) + \lambda Ju \ne u \quad \text{for } \|u\| = R_0,\ \lambda > 0 \tag{36}$$

$$N(u) \ne (1 + \lambda) u \quad \text{for } |u| = R_1,\ \lambda > 0. \tag{37}$$

The next critical point result (together with the Remark which follows) can be compared to the fixed point Theorem 20.2 in [2].

Theorem 6 *Assume that there exist* $u_0, u_1 \in K_{R_0 R_1}$ *and* $\nu_0,\ r > 0,\ |u_0| < r < |u_1|$, *such that conditions* (26), (29), (34) *and* (35) *hold. In addition assume that* $N := I - JE'$ *and* J *are compact from* X *to* X. *Then there exists a point* $u \in K_{R_0 R_1}$ *with* $E'(u) = 0$ *and* $E(u) = c$.

Remark 5 In case that $X = H$, when $|.| = \|.\|$ and $J = I$, the conclusion of Theorem 6 is also true even though I is not compact, if we add the condition

$$\inf \{|N(u)| : u \in K,\ |u| = R_0\} > 0. \tag{38}$$

The following result is the compression type mountain pass theorem accompanying the corresponding fixed point theorem of Krasnoselskii [7] (see also [4, p. 325]).

Theorem 7 *Assume that there exist* $u_0, u_1 \in K_{R_0 R_1}$ *and* $\nu_0,\ r > 0,\ |u_0| < r < |u_1|$, *such that conditions* (26) *and* (29) *hold. In addition assume that norm* $\|.\|$ *is increasing with respect to* K, *i.e.,*

$$\|u + v\| > \|u\| \text{ for all } u, v \in K,\ v \neq 0,$$

the maps J *and* $N := I - JE'$ *are compact from* X *to* X, *and*

(a) $\|N(u)\| \geq \|u\|$ *for* $\|u\| = R_0$,

(b) $|N(u)| \leq |u|$ *for* $|u| = R_1$.

Then there exists a point $u \in K_{R_0 R_1}$ *with* $E'(u) = 0$ *and* $E(u) = c$.

A similar result holds for critical points of minimum type.

Theorem 8 *Assume that conditions* (26), (27), (28) *are satisfied and that*

$$m := \inf_{K_{R_0 R_1}} E > -\infty. \tag{39}$$

Then there exists a sequence (u_k) *with* $u_k \in K_{R_0 R_1}$ *such that*

$$E(u_k) \to m \quad \text{as} \quad k \to \infty \tag{40}$$

and one of conditions (31), (32), (33) *holds. If in addition, any sequence* (u_k) *as above has a convergent subsequence and* (34), (35) *hold, then there exists* $u \in K_{R_0 R_1}$ *with*

$$E'(u) = 0 \text{ and } E(u) = m.$$

Remark 6 If both conditions (29) and (39) are satisfied, then Theorems 5 and 8 guarantee the existence of two distinct critical points of E in $K_{R_0 R_1}$.

4 Application

Consider the two-point boundary value problem

$$\begin{cases} u''(t) + f(u(t)) = 0, \ t \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{41}$$

Here f is a continuous function from $\mathbf{R}$ into $\mathbf{R}$, with $f(\mathbf{R}_+) \subset \mathbf{R}_+$. Let $X = H_0^1(0,1)$ with inner product and norm

$$(u,v) = \int_0^1 u'v'dt, \quad |u| = \left(\int_0^1 u'^2 dt\right)^{1/2}$$

and let $H = L^2(0,1)$ with inner product and norm

$$\langle u,v\rangle = \int_0^1 u\,v\,dt, \quad \|u\| = \left(\int_0^1 u^2 dt\right)^{1/2}.$$

We also denote by $|u|_\infty$ the max norm in $C[0,1]$ and by $|u|_{L^2(a,b)}$ the usual norm of $L^2(a,b)$.

Here $E : H_0^1(0,1) \to \mathbf{R}$ is given by

$$E(u) = \int_0^1 \left(\frac{1}{2}u'(t)^2 - F(u(t))\right) dt, \ u \in H_0^1(0,1),$$

where $F(u) = \int_0^u f(\tau)\,d\tau$. One has that $E'(u) = -u'' - f(u)$ in $H^{-1}(0,1)$,

$$(Jv, w) = \langle v, w\rangle \ \text{ for all } v \in H^{-1}(0,1), \ w \in H_0^1(0,1)$$

and $Jv = \int_0^1 G(t,s)\,v(s)\,ds$ for $v \in L^2(0,1)$, where $G(t,s)$ is the corresponding Green's function given by

$$G(t,s) = \begin{cases} s(1-t), \ \text{for } 0 \le s \le t \le 1 \\ t(1-s), \ \text{for } 0 \le t \le s \le 1. \end{cases}$$

Also $N(u) := u - JE'(u) = Jf(u)$ and

$$Jf(u) = \int_0^1 G(t,s)\,f(u(s))\,ds.$$

Notice, since the imbedding of $H_0^1(0,1)$ into $C[0,1]$ is compact, N and J are compact from $H_0^1(0,1)$ to itself. Also note that

$$G(t,s) \le G(s,s) \quad \text{for all } t,s \in [0,1] \tag{42}$$

and for every interval $[a,b]$ with $0 < a < b < 1$, there is a constant $M > 0$ with

$$MG(s,s) \le G(t,s) \quad \text{for all } s \in [0,1], \ t \in [a,b]. \tag{43}$$

These properties of Green's function guarantee that for every nonnegative function $v \in L^2(0,1)$, one has

$$(Jv)(t) \geq M \|Jv\| \text{ for all } t \in [a,b]. \tag{44}$$

Indeed, if $v \geq 0$ on $[0,1]$, $t \in [a,b]$ and $t^* \in [0,1]$, then from (42), (43), we obtain

$$\begin{aligned}(Jv)(t) &= \int_0^1 G(t,s)\,v(s)\,ds \geq M \int_0^1 G(s,s)\,v(s)\,ds \\ &\geq M \int_0^1 G(t^*,s)\,v(s)\,ds = M\,(Jv)(t^*).\end{aligned}$$

This proves (44) if we choose t^* with $(Jv)(t^*) = |Jv|_\infty$ and we take into account that $|u|_\infty \geq \|u\|$ for all $u \in C[0,1]$.

Now we consider a cone K in $H_0^1(0,1)$, defined by

$$K = \left\{u \in H_0^1(0,1) : u \geq 0 \text{ on } [0,1] \text{ and } u(t) \geq M\|u\| \text{ for } t \in [a,b]\right\}.$$

If $u \geq 0$ on $[0,1]$, then $f(u) \geq 0$ on $[0,1]$ since $f(\mathbf{R}_+) \subset \mathbf{R}_+$ and so, according to (44), $Jf(u) \in K$. Consequently, $u - JE'(u) \in K$ for every $u \in K$.

Before we state our hypotheses, we recall that constant c_0 is such that $\|u\| \leq c_0|u|$ for all $u \in H_0^1(0,1)$ and we denote by c_∞ the imbedding constant of the inclusion $H_0^1(0,1) \subset C[0,1]$, i.e. $|u|_\infty \leq c_\infty |u|$ for all $u \in H_0^1(0,1)$. Also, for the subinterval $[a,b]$ of $[0,1]$, we let $\chi_{[a,b]}$ be the characteristic function of $[a,b]$, i.e., $\chi_{[a,b]}(t) = 1$ if $t \in [a,b]$, $\chi_{[a,b]}(t) = 0$ otherwise.

Our assumptions are as follows:

(H1) There exist R_0, R_1 with $0 < R_0 < c_0 R_1$ such that

$$\frac{\min\limits_{\tau \in [MR_0, c_\infty R_1]} f(\tau)}{R_0} \geq \frac{1}{\left|J\chi_{[a,b]}\right|_{L^2(a,b)}} \tag{45}$$

$$\frac{\max\limits_{\tau \in [0, c_\infty R_1]} f(\tau)}{R_1} \leq \frac{1}{c_0}. \tag{46}$$

(H2) There are $u_0, u_1 \in K_{R_0 R_1}$ and r such that $|u_0| < r < |u_1|$ and

$$\max\{E(u_0), E(u_1)\} < \inf_{\substack{u \in K_{R_0R_1} \\ |u| = r}} E(u).$$

Remark 7 If f is nondecreasing on $[0, c_\infty R_1]$, then (45) and (46) become

$$\frac{f(MR_0)}{MR_0} \geq \frac{1}{M\left|J\chi_{[a,b]}\right|_{L^2(a,b)}} \tag{47}$$

and respectively

$$\frac{f(c_\infty R_1)}{c_\infty R_1} \leq \frac{1}{c_0 c_\infty}. \tag{48}$$

Therefore, in this case, in order to guarantee (45) and (46), we only need to know how the nonlinearity f behaves at two points MR_0 and $c_\infty R_1$.

Theorem 7 and Theorem 8 immediately yield the following two solutions existence result.

Theorem 9 *Assume that* (H1) *and* (H2) *hold. Then* (41) *has at least two positive solutions in* $K_{R_0 R_1}$.

Example. Let

$$f(u) = \begin{cases} \frac{1}{2}\sqrt{u} & \text{if } 0 \leq u \leq 1 \\ \frac{1}{2}u^2 & \text{if } 1 < u \leq b \\ \frac{1}{2}\left(\sqrt{u-b} + b^2\right) & \text{if } u > b. \end{cases} \tag{49}$$

Here $b > 2$ and will be specified later. Obviously f is increasing on $\mathbf{R}_+$ and

$$F(u) = \begin{cases} \frac{1}{3}u^{3/2} & \text{if } 0 \leq u \leq 1 \\ \frac{1}{6}\left(u^3 + 1\right) & \text{if } 1 < u \leq b. \end{cases}$$

First note that if we choose $r = 2$, then $\inf\limits_{\substack{u \in K \\ |u| = r}} E(u) \geq \frac{1}{2}$. Indeed, if $u \in K$ and $|u| = 2$, then since $|u|_\infty \leq |u|$, we have that $0 \leq u(t) \leq 2$ and so $F(u(t)) \leq \frac{3}{2}$ for all $t \in [0,1]$. Hence

$$E(u) = \frac{1}{2}|u|^2 - \int_0^1 F(u(t))\,dt \geq 2 - \frac{3}{2} = \frac{1}{2}.$$

Let $u_0 = \phi$, where ϕ is the positive eigenfunction corresponding to the first eigenvalue λ_1, i.e.

$$\begin{aligned} \phi'' + \lambda_1 \phi &= 0,\ t \in (0,1) \\ \phi(0) &= \phi(1) = 0, \end{aligned}$$

$\phi \geq 0$ and $|\phi| = 1$. Then $|u_0| = 1 < r$ and

$$E(u_0) = \frac{1}{2}|\phi|^2 - \int_0^1 F(\phi(t))\,dt = \frac{1}{2} - \int_0^1 F(\phi(t))\,dt < \frac{1}{2}.$$

Next we take $u_1 := b\phi$ and we assume that $b > \frac{1}{|\phi|_\infty}$. We have $|u_1| = b > r$ and

$$E(u_1) < \frac{1}{2}b^2 - \frac{1}{6}\int_{(b\phi(t)>1)} (b\phi(t))^3\,dt. \tag{50}$$

Since the limit of the right side of (50) as $b \to \infty$ is equal to $-\infty$, we may choose b large enough that $E(u_1) < \frac{1}{2}$. Hence condition (H2) is satisfied. Finally, since

$$\lim_{\tau \to 0} \frac{f(\tau)}{\tau} = \frac{1}{2}\lim_{\tau \to 0} \frac{\sqrt{\tau}}{\tau} = \infty \text{ and } \lim_{\tau \to \infty} \frac{f(\tau)}{\tau} = \frac{1}{2}\lim_{\tau \to \infty} \frac{\sqrt{\tau - b} + b^2}{\tau} = 0,$$

we may find R_0, R_1 such that $u_0, u_1 \in K_{R_0 R_1}$ and (47), (48) hold.

Therefore, according to Theorem 9, problem (41) with f given by (49) and b sufficiently large has two positive solutions.

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