Automation
Computers
Applied Mathematics
ISSN 1221–437X
Vol. 17 (2008) no. 2
pp. 191–199

On Approximating Operators Preserving Certain Polynomials

Octavian Agratini and Saddika Tarabie

Octavian Agratini: Faculty of Mathematics
and Computer Science, Babeş-Bolyai University, 400084
Cluj-Napoca, Romania

agratini@math.ubbcluj.ro

Saddika Tarabie: Faculty of Sciences, Tishrin University,
1267 Latakia, Syria

sadikatorbey@yahoo.com

Abstract: The paper centers around a general class of discrete linear positive operators depending on a real parameter $\alpha \geq 0$ and preserving both the constants and the polynomial $x^2 + \alpha x$. Under some given conditions, this sequence of operators forms an approximation process for certain real valued functions defined on an interval J. Two cases are investigated: $J = [0, 1]$ and $J = [0, \infty)$, respectively. Quantitative estimates are proved in different normed spaces and some particular cases are presented.

Key Words: positive linear operators, Popoviciu-Bohman-Korovkin criterion, Bernstein polynomials, Szász-Mirakjan operators, Baskakov operators, polynomial weight spaces

MSC 2000: 41A36, 41A25

Received: Oct 28 2008

1 Introduction

Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators defined on the Banach space $C([a,b])$. The well-known Popoviciu-Bohman-Korovkin criterion asserts: if $(L_n e_k)_{n\geq 1}$ converges to e_k uniformly on $[a,b]$, $k\in\{0,1,2\}$, for the test functions $e_0(x)=1$, $e_1(x)=x$, $e_2(x)=x^2$, then $(L_n f)_{n\geq 1}$ converges to f uniformly on $[a,b]$, for each $f\in C([a,b])$. Many classical linear positive operators have the degree of exactness one, this meaning they preserve the monomials e_0 and e_1. The most known and investigated operators of this kind are Bernstein operators,

$$B_n : C([0,1])\to C([0,1]),\quad (B_n f)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),\quad x\in[0,1]. \tag{1.1}$$

The present note is motivated by the following previous results.

Firstly, J.P. King [8] has presented an example of linear and positive operators $V_n : C([0,1])\to C([0,1])$ given as follows

$$(V_n f)(x)=\sum_{k=0}^{n}\binom{n}{k}(r_n^*(x))^k(1-r_n^*(x))^{n-k}f\left(\frac{k}{n}\right),\quad x\in[0,1], \tag{1.2}$$

where $r_n^* : [0,1]\to[0,1]$ are defined by

$$r_n^*(x)=\begin{cases} x^2, & n=1,\\ -\dfrac{1}{2(n-1)}+\sqrt{\dfrac{n}{n-1}x^2+\dfrac{1}{4(n-1)^2}}, & n=2,3,\ldots\end{cases}$$

This sequence preserves the test functions e_0, e_2 and $(V_n e_1)(x)=r_n^*(x)$ holds. Replacing r_n^* by e_1, one reobtains Bernstein operators (1.1). Further results regarding V_n operator have been recently obtained by H. Gonska and P. Piţul [7]. By using A-statistical convergence, an analog of King's result has been proved by O. Dumand and C. Orhan [5]. In [1] we indicated a general technique to construct sequences of univariate operators of discrete type with the same property as in King's example, i.e., their degree of exactness is null, but they reproduce the third function of the celebrated criterion. This way we gave the modified variants of Szász, Baskakov and Bernstein-Cholovsky operators. The iterates of this general class have been studied in [2].

Secondly, in [4] the authors introduced and investigated a family $(B_{n,\alpha})_{n\geq 2}$ of sequences of Bernstein-type operators depending on a real parameter $\alpha\geq 0$ and defined by

$$(B_{n,\alpha}f)(x)=\sum_{k=0}^{n}\binom{n}{k}r_{n,\alpha}^k(x)(1-r_{n,\alpha}(x))^{n-k}f\left(\frac{k}{n}\right),\quad f\in C([0,1]),\ x\in[0,1], \tag{1.3}$$

where

$$r_{n,\alpha}(x)=-\frac{n\alpha+1}{2(n-1)}+\sqrt{\frac{(n\alpha+1)^2}{4(n-1)^2}+\frac{n(\alpha x+x^2)}{n-1}},\quad n\geq 2,\ x\in[0,1].$$

This class reproduces both e_0 and the polynomial $e_2+\alpha e_1$. Clearly, for $\alpha=0$, $B_{n,0}$ becomes V_n operator ($n\geq 2$), see (1.2). Moreover, for each $n\geq 2$ and $x\in[0,1]$, if α runs to infinity, then $r_{n,\alpha}(x)$ tends to x and $B_{n,\alpha}f$ becomes the classical Bernstein polynomials $B_n f$, see (1.1).

Our aim is to introduce a general class of discrete type operators with the same property as in [4], to reproduce e_0 and $e_2 + \alpha e_1$. This family is defined on certain subspaces of $C(J)$, $J \subset \mathbb{R}$, the space of all real-valued continuous functions on J. We take into account two kinds of intervals: $J = [0, 1]$ and $J = \mathbb{R}_+ := [0, \infty)$, respectively. For the first case, the local and global rate of convergence is established by using the classical modulus of continuity $\omega(f; \cdot)$ associated to any function $f \in C([0, 1])$. As usual, this space is endowed with the sup-norm $\| \cdot \|_{C([0,1])}$. For the second case, the approximation property of our class is given in the frame of spaces of functions with polynomial growth. The involved spaces are defined via certain weights. More precisely, for a given $p \geq 2$, we consider the weight w_p, $w_p(x) = (1 + x^p)^{-1}$, $x \geq 0$, and the corresponding space

$$C_p(\mathbb{R}_+) := \{f \in C(\mathbb{R}_+);\ w_p(x)f(x) \text{ is convergent as } x \text{ tends to } \infty\},$$

endowed with the norm $\| \cdot \|_{C_p}$, $\|f\|_{C_p} = \sup_{x \geq 0} w_p(x)|f(x)|$.

We notice, since $p \geq 2$, the test function e_j, $j \in \{0, 1, 2\}$, belong to $C_p(\mathbb{R}_+)$.

2 Construction of the class $(L^*_{n,\alpha})_{n \geq 2}$

For each $n \geq 2$, let $\Delta_n := (x_{n,k})_{k \in I_n}$ be a net on the interval J, where $I_n \subset \mathbb{N}$ is a set of indices consistent with J, this meaning $\{x_{n,k} :\ k \in I_n\} \subset J$. We consider the operators L_n having the form

$$(L_n f)(x) = \sum_{k \in I_n} u_{n,k}(x) f(x_{n,k}), \quad x \in J, \tag{2.1}$$

where $u_{n,k} \in C(J)$, $u_{n,k} \geq 0$, for every $(n, k) \in \{2, 3, \dots\} \times I_n$ and $f \in \mathcal{F}(J) := \{g \in C(J) :$ the series in (2.1) is convergent$\}$.

Clearly, each L_n is a linear positive operator. We also mention that the right-hand side of (2.1) could be a finite sum. In this case, $\mathcal{F}(J)$ is just $C(J)$.

Further on, we assume that the following identities

$$(L_n e_0)(x) = 1, \quad (L_n e_1)(x) = x, \quad (L_n e_2)(x) = a_n x^2 + b_n x, \quad x \in J, \tag{2.2}$$

are fulfilled for each $n \geq 2$. Moreover, we assume

$$a_n > 0, \quad b_n > 0, \quad \lim_n a_n = 1, \quad \lim_n b_n = 0. \tag{2.3}$$

On the basis of (2.2) and (2.3) according to Popoviciu-Bohman-Korovkin theorem, one has $\lim_n \|L_n f - f\|_{C(K)} = 0$, on any compact $K \subset J$. Since $u_{n,k} \geq 0$ $(k \in I_n)$ and $\sum_{k \in I_n} u_{n,k} = e_0$, we deduce that each $u_{n,k}$ belongs to $C_B(J)$, the space of all real-valued continuous and bounded functions on J.

Let $\alpha \geq 0$ be fixed. For each $n = 2, 3, \dots$ setting

$$c_{n,\alpha} := \frac{b_n + \alpha}{2a_n},$$

we define the functions $v_{n,\alpha}: J \to \mathbb{R}_+$,

$$v_{n,\alpha}(x) = -c_{n,\alpha} + \sqrt{c_{n,\alpha}^2 + \frac{x^2 + \alpha x}{a_n}}, \quad x \in J. \tag{2.4}$$

Clearly, $v_{n,\alpha} \in C(J)$. Taking into account (2.1), we consider the linear and positive operators

$$(L_{n,\alpha}^* f)(x) = \sum_{k \in I_n} u_{n,k}(v_{n,\alpha}(x)) f(x_{n,k}), \quad x \in J, \tag{2.5}$$

where $f \in \mathcal{F}(J)$.

Theorem 2.1 *Let $L_{n,\alpha}^*$, $n = 2, 3, \ldots$, be defined by (2.5). The following relations hold.*

(i) $L_{n,\alpha}^* e_0 = e_0$, $L_{n,\alpha}^* e_1 = v_{n,\alpha}$, $L_{n,\alpha}^*(e_2 + \alpha e_1) = e_2 + \alpha e_1$.

(ii) $(L_{n,\alpha}^* \varphi_x^2)(x) = (2x + \alpha)(x - v_{n,\alpha}(x))$, $x \in J$, where $\varphi_x : J \to \mathbb{R}$ is defined by $\varphi_x(t) = t - x$.

Proof. (i) Since $(L_{n,\alpha}^* f)(x) = (L_n f)(v_{n,\alpha}(x))$, on the basis of (2.2), for $f = e_0$ and $f = e_1$, the first two claimed identities are obvious. With the help of the same identities (2.2) we also can write

$$(L_{n,\alpha}^*(e_2 + \alpha e_1))(x) = (L_{n,\alpha}^* e_2)(x) + \alpha (L_{n,\alpha}^* e_1)(x) = a_n v_{n,\alpha}^2(x) + (b_n + \alpha) v_{n,\alpha}(x).$$

Taking into account (2.4), by direct computation the result follows.

(ii) Since $L_{n,\alpha}^* \varphi_x^2 = L_{n,\alpha}^*(e_2 + \alpha e_1) - (2x + \alpha) L_{n,\alpha}^* e_1 + x^2 L_{n,\alpha}^* e_0$, by using the identities established in the first part of this theorem, we obtain the desired result. □

We indicate two elementary properties of the functions introduced to (2.4).

Lemma 2.2 *Let $v_{n,\alpha}$, $n = 2, 3, \ldots$ be defined by (2.4). For each $x \in J$ one has*

(i) $0 \le v_{n,\alpha}(x) \le x$,

(ii) $\lim_n v_{n,\alpha}(x) = x$.

Proof. (i) Since φ_x^2 is a non-negative function and $L_{n,\alpha}^*$ is a positive operator, the inequality $v_{n,\alpha}(x) \le x$ is implied by Theorem 2.1 part (ii).

(ii) Taking into account the assumptions (2.3), the identity is obtained by a straight computation. □

3 Examples

Starting from some known approximation processes of the form (2.1) and verifying conditions (2.2), we focus our attention on obtaining modified processes of $L_{n,\alpha}^*$-type.

Case 1: $J = [0, 1]$. We consider $I_n = \{0, 1, \ldots, n\}$ and the net $\Delta_n = (k/n)_{k=\overline{0,n}}$.

Example 3.1 *Bernstein-type operators.* Choosing $u_{n,k} = \binom{n}{k} x^k (1 - x)^{n-k}$, L_n becomes Bernstein operator B_n, see (1.1). We have $a_n = 1 - \dfrac{1}{n}$ and $b_n = \dfrac{1}{n}$, consequently, for $n \ge 2$, hypotheses (2.3) take place. Taking $\alpha = 0$, our operator $L^*_{n,0}$ turns into V_n operator introduced by King, see (1.2). Keeping $\alpha \ge 0$ arbitrary, $L^*_{n,\alpha}$ becomes $B_{n,\alpha}$ operator specified at (1.3).

Case 2: $J = [0,\infty)$. We consider $I_n = \mathbb{N}$ and the equidistant net $\Delta_n = (k/n)_{k\ge 0}$.

Example 3.2 *Szász-Mirakjan-Favard modified operators.* We consider $u_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$. In this case we get $a_n = 1$ and $b_n = 1/n$. Consequently,

$$(L^*_{n,\alpha}f)(x) = \sum_{k=0}^{\infty} e^{-nv_{n,\alpha}(x)} \frac{(nv_{n,\alpha}(x))^k}{k!} f\left(\frac{k}{n}\right), \tag{3.1}$$

where $v_{n,\alpha}(x) = (\sqrt{(n\alpha+1)^2 + 4n^2(x^2+\alpha x)} - n\alpha - 1)/2n$, $x \ge 0$, $n \ge 2$.

Example 3.3 *Baskakov modified operators.* Choosing $u_{n,k}(x) = \binom{n+k-1}{k} x^k(1+x)^{-n-k}$, we deduce $a_n = 1 + \dfrac{1}{n}$ and $b_n = \dfrac{1}{n}$. Hypotheses (2.3) are verified. The corresponding operators have the form

$$(L^*_{n,\alpha}f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{v^k_{n,\alpha}(x)}{(1+v_{n,\alpha}(x))^{n+k}} f\left(\frac{k}{n}\right), \tag{3.2}$$

where, on the basis of (2.4), one has

$$v_{n,\alpha}(x) = \frac{\sqrt{(n\alpha+1)^2 + 4n(n+1)(x^2+\alpha x)} - n\alpha - 1}{2(n+1)}, \quad x \ge 0,\ n \ge 2.$$

We mention that choosing both in (3.1) and in (3.2) $\alpha = 0$, we reobtain the modified operators in King's sense introduced in [1; §4].

4 Approximation properties, case $J = [0,1]$

Taking in view Theorem 2.1(i) and Lemma 2.2(ii), on the basis of Popoviciu-Bohman-Korovkin theorem, we can state

Theorem 4.1 *Let $L^*_{n,\alpha}$, $n = 2,3,\dots$, be defined by (2.5), where $J = [0,1]$. For any $f \in C([0,1])$ one has*

$$\lim_n \|L^*_{n,\alpha}f - f\|_{C([0,1])} = 0.$$

For exploring the rate of convergence of $L^*_{n,\alpha}$ ($n \ge 2$, $\alpha \in \mathbb{R}_+$) operators, we need the following technical result.

Lemma 4.2 *Let $v_{n,\alpha}$, $n = 2,3,\dots$, be given by (2.4).*

(i) For $\alpha > 0$, one has $x - v_{n,\alpha}(x) \le \dfrac{(a_n-1)x^2 + b_n x}{b_n + \alpha}$.

(ii) For $\alpha = 0$, one has $x - v_{n,\alpha}(x) \le \dfrac{|a_n - 1|}{\sqrt{a_n}}x + \dfrac{b_n}{2a_n}$.

Proof. (i) In the harmony with (2.4), we can write successively

$$x - v_{n,\alpha}(x) = \frac{\left(1 - \dfrac{1}{a_n}\right)x^2 + \dfrac{b_n}{a_n}x}{x + c_{n,\alpha} + \sqrt{c_{n,\alpha}^2 + \dfrac{x^2 + \alpha x}{a_n}}}$$
$$\le \frac{\left(1 - \dfrac{1}{a_n}\right)x^2 + \dfrac{b_n}{a_n}x}{2c_{n,\alpha}} = \frac{(a_n - 1)x^2 + b_n x}{b_n + \alpha}.$$

(ii) For $\alpha = 0$, an upper bound for $e_1 - v_{n,0}$ can be immediately established

$$x - v_{n,0}(x) = x + \frac{b_n}{2a_n} - \sqrt{\frac{x^2}{a_n} + \frac{b_n^2}{4a_n^2}} \le \left(1 - \frac{1}{\sqrt{a_n}}\right)x + \frac{b_n}{2a_n} \le \frac{|a_n - 1|}{\sqrt{a_n}}x + \frac{b_n}{2a_n}. \quad \square$$

For obtaining information regarding the rate of convergence we use the result due to B. Mond [9]. It can be read as follows. For any linear positive functional F on the space $C([0,1])$ we have

$$|F(f) - f(x)| \le |F(e_0) - 1||f(x)| + (F(e_0) + h^{-2}F((e_1 - xe_0)^2))\omega_1(f;h), \tag{4.1}$$

for any $f \in C([0,1])$, $x \in [0,1]$ and $h > 0$.

Theorem 4.3 *Let $L_{n,\alpha}^*$, $n = 2,3,\dots$, be defined by (2.5), where $J = [0,1]$. We assume that the sequence $((a_n - 1)/b_n)_{n\ge2}$ is bounded. Let f belong to $C([0,1])$.*

(i) For $\alpha > 0$, one has

$$|(L_{n,\alpha}^* f)(x) - f(x)| \le \left(1 + (2x + \alpha)x\frac{(a_n - 1)x + b_n}{b_n(b_n + \alpha)}\right)\omega_1(f;\sqrt{b_n}). \tag{4.2}$$

(ii) For $\alpha = 0$, one has

$$|(L_{n,0}^* f)(x) - f(x)| \le \left(1 + \frac{x}{a_n}\left(1 + 2\sqrt{a_n}\frac{|a_n - 1|}{b_n}x\right)\right)\omega_1(f;\sqrt{b_n}). \tag{4.3}$$

Proof. (i) We use the result (4.1) with $h := \sqrt{b_n}$. Taking into account the identity $L_{n,\alpha}^* e_0 = e_0$, Theorem 2.1(ii) and Lemma 4.2(i), the result follows.

(ii) Following a similar route: relation (4.1) with $h := \sqrt{b_n}$, Theorem 2.1(ii) and Lemma 4.2(ii), we obtain (4.3). $\square$

We point out that the additional assumption regarding the boundness of the sequence $((a_n - 1)/b_n)_{n\ge2}$ guarantees that the quantities which appear in front of the modulus $\omega_1(f;\sqrt{b_n})$ are bounded for n tending to infinity. Otherwise, an inequality of the form (4.2) or (4.3) is not useful for evaluating the rate of convergence of the involved operators.

As a matter of fact, applying (4.2) for $B_{n,\alpha}$ operators defined at (1.3), we obtain the following upper bound of the local rate of convergence

$$|(B_{n,\alpha}f)(x) - f(x)| \le \left(1 + \frac{(2x + \alpha)x(1 - x)}{\alpha + 1/n}\right)\omega_1\left(f;\frac{1}{\sqrt{n}}\right),$$

which is similar to [4, Eq. (5)].

Remark 4.4 The requirement regarding the boundness of the sequence $((a_n-1)/b_n)_{n\geq 2}$ is not necessary. We can strike out this claim, but in this case the inequality (4.2) is replaced with a coarser one. For example, for $\alpha>0$, starting from Lemma 4.2(i), we can write

$$x-v_{n,\alpha}(x)\leq \frac{|a_n-1|x^2+b_nx}{b_n+\alpha}\leq \lambda_n(x^2+x),\quad x\in[0,1],$$

where $\lambda_n:=\max\left\{\frac{|a_n-1|}{b_n+\alpha},\frac{b_n}{b_n+\alpha}\right\}$, $n\geq 2$. Relation (2.3) ensures $\lim_n \lambda_n=0$. Choosing $h:=\sqrt{\lambda_n}$ in (4.1), we deduce

$$|(L^*_{n,\alpha}f)(x)-f(x)|\leq (1+x(x+1)(2x+\alpha))\omega_1(f;\sqrt{\lambda_n}),\quad x\in[0,1].$$

As regards inequality (4.3) a similar reasoning can be made.

5 Approximation properties, case $J=\mathbb{R}_+$

Corresponding to the unbounded interval $\mathbb{R}_+$, the function f is allowed to be unbounded, however with some restrictions concerning the growth of f at infinity. In the sequel we discuss functions belonging to the space $C_p(\mathbb{R}_+)$, already presented in *Introduction*. Let us remark that the incorporation of the weight w_p, $p\geq 2$, into our approach is the main point which leads to the global estimate of the rate of convergence.

Lemma 5.1 *Let $L^*_{n,\alpha}$, $b=2,3,\ldots$, be defined by (2.5), where $J=\mathbb{R}_+$.*

(i) For any $p\geq 2$ one has

$$\frac{|(L^*_{n,\alpha}e_1)(x)-x|}{1+x^p}\leq\left|1-\frac{1}{\sqrt{a_n}}\right|+\frac{|b_n(\sqrt{a_n}+1)-\alpha(\sqrt{a_n}-1)|}{\sqrt{a_n}(b_n+\alpha)},\quad x\geq 0; \tag{5.1}$$

(ii) $\lim_n \|L^*_{n,\alpha}e_1-e_1\|_{C_p}=0.$

Proof. (i) Let us fix $n\geq 2$. Since $(L^*_{n,\alpha}e_1)(0)=0$, for $x=0$ the inequality is evident. In what follows, we consider $x>0$. Clearly, for each $x>0$, one has $1+x^p>x$. Taking into account both Lemma 2.2(i) and (2.4), we get

$$\begin{aligned}
\frac{|(L^*_{n,\alpha}e_1)(x)-x|}{1+x^p} &\le \frac{x-v_{n,\alpha}(x)}{x}\\
&= \frac{\left(1-\dfrac{1}{a_n}\right)x+\left(2c_{n,\alpha}-\dfrac{\alpha}{a_n}\right)}{x+c_{n,\alpha}+\sqrt{c^2_{n,\alpha}+\dfrac{x^2+\alpha x}{a_n}}}\\
&\le \frac{\left(1-\dfrac{1}{a_n}\right)x+\dfrac{b_n}{a_n}}{\left(1+\dfrac{1}{\sqrt{a_n}}\right)x+c_{n,\alpha}}\\
&= \frac{\left(1-\dfrac{1}{\sqrt{a_n}}\right)\left[\left(1+\dfrac{1}{\sqrt{a_n}}\right)x+c_{n,\alpha}\right]+\dfrac{b_n}{a_n}-\left(1-\dfrac{1}{\sqrt{a_n}}\right)c_{n,\alpha}}{\left(1+\dfrac{1}{\sqrt{a_n}}\right)x+c_{n,\alpha}}\\
&\le \left|1-\frac{1}{\sqrt{a_n}}\right|+\frac{\left|\dfrac{b_n}{a_n}-\left(1-\dfrac{1}{\sqrt{a_n}}\right)c_{n,\alpha}\right|}{c_{n,\alpha}}\\
&= \left|1-\frac{1}{\sqrt{a_n}}\right|+\frac{|b_n(\sqrt{a_n}+1)-\alpha(\sqrt{a_n}-1)|}{\sqrt{a_n}(b_n+\alpha)}.
\end{aligned}$$

(ii) The claimed identity is implied by (5.1) and the assumption (2.3). □

Further on, we show that the sequence $(L^*_{n,\alpha})_{n\ge 2}$ furnishes a new strong approximation process on the weighed space $C_p(\mathbb{R}_+)$, $p\ge 2$.

Theorem 5.2 *Let $L^*_{n,\alpha}$, $n=2,3,\dots$, be defined by (2.5), where $J=\mathbb{R}_+$. For every $f\in \mathcal{F}(\mathbb{R}_+)\cap C_p(\mathbb{R}_+)$, $p\ge 2$, the following identity*

$$\lim_{n\to\infty}\|L^*_{n,\alpha}f-f\|_{C_p}=0, \tag{5.2}$$

holds.

Proof. It is known that $\{e_0,e_1,e_2\}$ is a Korovkin set in $C_p(\mathbb{R}_+)$, see e.g. [[3], Proposition 4.2.5]. Taking in view both the property $L^*_{n,\alpha}e_0=e_0$ and Lemma 5.1(ii), it remains to prove (5.2) for $f=e_2$. On the basis of Theorem 2.1(i), we get

$$\begin{aligned}
\|L^*_{n,\alpha}e_2-e_2\|_{C_p} &\le \|L^*_{n,\alpha}(e_2+\alpha e_1)-(e_2+\alpha e_1)\|_{C_p}+\alpha\|L^*_{n,\alpha}e_1-e_1\|_{C_p}\\
&= \alpha\|L^*_{n,\alpha}e_1-e_1\|_{C_p}.
\end{aligned}$$

By using again Lemma 5.1(ii), the proof is ended. □

Remark 5.3 The operator $L^*_{n,\alpha}$ is non-expansive on the space $C_p(\mathbb{R}_+)$. Indeed, if $f\in C_p(\mathbb{R}_+)$, then

$$w_p(x)|(L^*_{n,\alpha}f)(x)|\le \sum_{k\in I_n}u_{n,k}(v_{n,\alpha}(x))\|f\|_{C_p}=\|f\|_{C_p},$$

and, consequently, $\|L^*_{n,\alpha}f\|_{C_p} \le \|f\|_{C_p}$ holds.

Also, it is evident that $L^*_{n,\alpha}$ maps $C_B(\mathbb{R}_+)$ into itself. Here $C_B(\mathbb{R}_+)$ stands for the space of all real-valued continuous and bounded functions defined on $\mathbb{R}_+$.

Remark 5.4 In [6], the authors had examined the statistical convergence in Approximation Theory establishing some Korovkin-type theorems. On the basis of [6, Theorem 1], Theorem 2.1(i) leads us to the following result.

If $st - \lim\limits_n \|v_{n,\alpha} - e_1\|_{C(K)} = 0$, then $st - \lim\limits_n \|L^*_{n,\alpha}f - f\|_{C(K)} = 0$, for any function f belonging to $C_B(\mathbb{R}_+)$, where $K \subset \mathbb{R}_+$ is an arbitrary compact.

References

[1] O. Agratini, *Linear operators that preserve some test functions*, International Journal of Mathematics and Mathematical Sciences, Vol. 2006, Article ID94136, pp. 11, DOI 10.1155/IJMMS.

[2] O. Agratini, *On the iterates of a class of summation-type linear positive operators*, Comput. Math. Appl., **55**(2008), 1178-1180.

[3] F. Altomare, M. Campiti, *Korovkin-type Approximation Theory and its Applications*, de Gruyter Studies in Mathematics, Vol. 17, Walter de Gruyter, Berlin, 1994.

[4] D. Cárdenas-Morales, P. Garrancho, F.J. Muñoz-Delgado, *Shape preserving approximation by Bernstein-type operators which fix polynomials*, Applied Mathematics and Computation, **182**(2006), 1615-1622.

[5] O. Duman, C. Orhan, *An abstract version of the Korovkin approximation theorem*, Publ. Math. Debrecen, **69**(2006), f. 1-2, 33-46.

[6] A.D. Gadjiev, C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math., **32**(2002), 129-138.

[7] H. Gonska, P. Piţul, *Remarks on an article of J.P. King*, Schriftenreiche des Fachbereichs Mathematik, Nr. **596**(2005), Universität Duisburg-Essen, pp. 8.

[8] J.P. King, *Positive linear operators which preserve* x^2, Acta Math. Hungar., **99**(2003), no. 3, 203-208.

[9] B. Mond, *On the degree of approximation by linear positive operators*, J. Approx. Theory, **18**(1976), 304-306.