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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

On certain q-analogues of the Bernstein operators

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ABSTRACT. In this note we study the limit of iterates of Lupaş q-analogue of the Bernstein operators. Also, we introduce a new class of q-Bernstein-type operators which fix certain polynomials. Both qualitative and quantitative results are established.

1. INTRODUCTION

Bernstein polynomials hold a special place in Approximation Theory. Abundantly, new papers are published containing their applications and generalizations. During the last decade, the development of q-calculus has led to the discovery of new generalizations of these polynomials involving q-integers. The Phillips' results [10] are the basis of many research papers, and the comprehensive survey due to S. Ostrovska [9] gives a good perspective of these achievements. It is worth mentioning that A. Lupaş [6] was the first introduce a variant of q-Bernstein polynomials but, unfortunately, his work was less known. The merit to bring into light Lupaş' paper is due to S. Ostrovska [8].

The paper is organized in two main sections.

In Section 2 we are dealing with iterates of Lupaş q-analogue of the Bernstein operator. In Section 3, starting from Phillips q-variant, we construct a class of Bernstein-type operators depending on a parameter which fix certain polynomials of second degree. By using the modulus of continuity, the rate of convergence is also obtained.

In connection with q-calculus, for the reader's convenience, we recall the following definitions and notation, see e.g. [11; §8.1].

Let $q > 0$. For any $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, the q-integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad (n \in \mathbb{N}), \quad [0]_q := 0,$$

and the q-factorial $[n]_q!$ as follows

$$[n]_q! := [1]_q [2]_q \dots [n]_q \quad (n \in \mathbb{N}), \quad [0]_q! := 1.$$

Also, for integers $k \in \{0, 1, \dots, n\}$, the q-binomial or the Gaussian coefficients are denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

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Clearly, $[n]_1 = n$, $[n]_1! = n!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_1$ represents $\begin{pmatrix} n \\ k \end{pmatrix}$, the ordinary binomial coefficient.

2. ITERATES OF THE q-BERNSTEIN-LUPAŞ OPERATORS

For each $q > 0$ and $x \in [0, 1]$ setting

$$
\begin{aligned}
(2.1) \qquad w_n(q;x) &= \prod_{k=1}^{n-1}(1 - x + q^k x), \\
a_{n,k}(q;x) &= \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k (1-x)^{n-k}, \\
b_{n,k}(q;x) &= \frac{a_{n,k}(q;x)}{w_n(q;x)},
\end{aligned}
$$

where $0 \le k \le n$, A. Lupaş [6] defined the operators $L_{n,q} : C([0,1]) \to C([0,1])$,

$$
(2.2) \qquad (L_{n,q}f)(x) = \sum_{k=0}^{n} b_{n,k}(q;x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0,1],\ n \in \mathbb{N}.
$$

These are positive linear operators and for $q = 1$ we recover the well-known Bernstein operators. In the case $q \neq 1$, these operators give rational functions rather than polynomials. A. Lupaş proved the following identities

$$
(2.3) \qquad (L_{n,q}e_0)(x) = e_0(x), \quad (L_{n,q}e_1)(x) = e_1(x),
$$

$$
(L_{n,q}e_2)(x) = e_2(x) + \frac{x(1-x)}{[n]_q} - (1-q)\left(1 - \frac{1}{[n]_q}\right)\frac{x^2(1-x)}{1-x+xq}, \ x \in [0,1],
$$

where $e_0(t) = 1$, $e_j(t) = t^j$, $j \in \{1, 2\}$, $t \in [0, 1]$.

By definition the m-th iterate of $L_{n,q}$ is $L^1_{n,q} := L_{n,q}$, $L^m_{n,q} := L_{n,q}(L^{m-1}_{n,q})$, $m = 2, 3, \ldots$. The aim of this section is to study the convergence of the iterates $L^m_{n,q}f$ as m tends to infinity and n is fixed. We mention that for q-Bernstein polynomials introduced by Phillips, a detailed study of their iterates has been done by S. Ostrovska [7]. However, we use a different technique for proving our statement.

Theorem 2.1. ([2]) *Let L_n, $n \in \mathbb{N}$, be defined as follows*

$$
L_n : C([a,b]) \to C([a,b]), \quad (L_n f)(x) = \sum_{k=0}^{n} \psi_{n,k}(x) f(x_{n,k}),
$$

where $0 = x_{n,0} < x_{n,1} < \cdots < x_{n,n} = b$ is a net on $[a, b]$ and for each $0 \le k \le n$ the function $\psi_{n,k}$ belongs to $C([a, b])$ satisfying $\psi_{n,k} \ge 0$. We assume that $L_n e_0 = e_0$ and $L_n e_1 = e_1$, $n \in \mathbb{N}$.

Let us denote $u_n := \min_{x \in [a,b]} (\psi_{n,0}(x) + \psi_{n,n}(x))$. If $u_n > 0$, then the sequence $(L^m_n f)_{m \ge 1}$ verifies

$$
\lim_{m \to \infty} (L^m_n f)(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a), \quad f \in C([a,b]),
$$

uniformly on $[a, b]$.

The proof of this result is based on the contraction principle [2; Theorem 4]. In Theorem 2.1 we also took into account [4; Lemma 3.1] removing from the original statement a superfluous condition.

Choosing $a = 0$, $b = 1$, $x_{n,k} = \frac{[k]_q}{[n]_q}$, $\psi_{n,k} = b_{n,k}(q;\cdot)$, $0 \le k \le n$, and knowing the identities (2.3), all that is left to be proved is

$$\min_{x\in[0,1]} (b_{n,0}(q;x) + b_{n,n}(q;x)) > 0. \tag{2.4}$$

We get

$$\begin{aligned} b_{n,0}(q;x) + b_{n,n}(q;x) &= \frac{(1-x)^n + q^{n(n-1)/2}x^n}{w_n(q;x)} \\ &\ge \frac{\min_{x\in[0,1]}[(1-x)^n + q^{n(n-1)/2}x^n]}{\max_{x\in[0,1]} w_n(q;x)}. \end{aligned} \tag{2.5}$$

By using elements of Calculus, $\min_{x\in[0,1]}[(1-x)^n + q^{n(n-1)/2}x^n]$, $n \ge 2$, is obtained for $x = (1+q^{n/2})^{-1}$ and its value is $(q^{n/2}/(1+q^{n/2}))^{n-1}$. For $n = 1$, the minimum has value 1. On the other hand, by using (2.1), $x \in [0,1]$, we deduce:

(i) for $q \in (0,1]$, $w_n(q;x) \le 1$;

(ii) for $q > 1$, $w_n(q;x) \le (1 - x + q^{n-1}x)^{n-1} = (1 + (q^{n-1}-1)x)^{n-1} \le q^{(n-1)^2}$.

Consequently,

$$\max_{x\in[0,1]} w_n(q;x) \le \max\{1, q^{(n-1)^2}\}, \text{ for any } q > 0.$$

Returning to (2.5) we can write

$$b_{n,0}(q;x) + b_{n,n}(q;x) \ge \left(\frac{q^{n/2}}{1+q^{n/2}}\right)^{n-1} \frac{1}{\max\{1, q^{(n-1)^2}\}},$$

therefore (2.4) holds. On the basis of Theorem 2.1, we can state

Theorem 2.2. *Let $L_{n,q}$, $q > 0$, be defined by (2.2). For any fixed $n \in \mathbb{N}$, one has*

$$\lim_{m\to\infty} (L_{n,q}^m f)(x) = f(0) + (f(1) - f(0))x, \quad x \in [0,1], \ f \in C([0,1]),$$

uniformly on $[0, 1]$.

3. On a family of q-Bernstein-type operators

The q-Bernstein polynomials of $f : [0,1] \to \mathbb{C}$ introduced by G. M. Phillips [10] are defined as follows

$$(P_{n,q}f)(x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1 - q^s x), \quad x \in [0,1], \ n \in \mathbb{N}.$$

From here on, an empty product is taken to be equal to 1. Again, for $q = 1$, the polynomials $P_{n,1}f$, $n \in \mathbb{N}$, are classical Bernstein polynomials.

For $0 < q < 1$, $P_{n,q}$ is a positive linear operator on the space $C([0,1])$. Throughout this section we consider $q \in (0,1)$. These operators satisfy the following properties

$$P_{n,q}e_0 = e_0, \quad P_{n,q}e_1 = e_1, \quad P_{n,q}e_2 = e_2 + \frac{1}{[n]_q}(e_1 - e_2).$$

J. P. King [5] has presented an example of operators of Bernstein type which preserve the test functions e_0 and e_2. A general technique to construct sequences of discrete type operators with the same property as in King's example was indicated in [1] and particular classes such as Szász-Kirakjan, Baskakov, Bernstein-Chlodovsky operators have been modified. Starting with King's result, another modified Bernstein operator was recently achieved [3]. Following the same line, we modify the sequence $(P_{n,q})_n$ into a class depending on a real positive parameter α which preserves the polynomial $e_2 + \alpha e_1$.

Let $\alpha \geq 0$ be fixed. For each $n = 2, 3, \ldots$ and $x \in [0,1]$, setting

$$\beta_{n,q,\alpha} := \frac{1 + [n]_q\alpha}{2([n]_q - 1)},$$

$$v_{n,q,\alpha}(x) := -\beta_{n,q,\alpha} + \sqrt{\beta^2_{n,q,\alpha} + \frac{[n]_q}{[n]_q - 1}(x^2 + \alpha x)}, \tag{3.6}$$

we consider the operators

$$(P^*_{n,q,\alpha}f)(x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q v^k_{n,q,\alpha}(x) \prod_{s=0}^{n-1-k} (1 - q^s v_{n,q,\alpha}(x)), \tag{3.7}$$

where $f \in \mathbb{R}^{[0,1]}$.

By direct computation we can prove

Theorem 3.3. *Let the sequence $(P^*_{n,q,\alpha})_{n\geq 2}$ be defined by (3.7). One has*
(i) $0 \leq v_{n,q,\alpha}(x) \leq x$ for $x \in [0,1]$;
(ii) $P^*_{n,q,\alpha}e_0 = e_0$ and $P^*_{n,q,\alpha}(e_2 + \alpha e_1) = e_2 + \alpha e_1$.

Clearly, $P^*_{n,q,\alpha}$, $n \geq 2$, are linear positive operators preserving both e_0 and the polynomial $e_2 + \alpha e_1$. Also, $P^*_{n,q,\alpha}f$ interpolates f at 0.

For the particular case $q = 1$, $P^*_{n,1,\alpha}$ turns into the operator introduced in [3]. Moreover, $P^*_{n,1,0}$ represents King's example [5].

Since $P^*_{n,q,\alpha}e_1 = v_{n,q,\alpha}$, it is obvious that our sequence does not form an approximation process. In order to satisfy this property, for each $n \geq 2$, the constant $q \in (0,1)$ will be replaced by a number $q_n \in (0,1)$.

Theorem 3.4. *Let $(q_n)_{n\geq 2}$, $0 < q_n < 1$, be a sequence such that $\lim_n q_n = 1$ and $\lim_n q_n^n$ exists. Let $P^*_{n,q_n,\alpha}$, $n \geq 2$, be defined as in (3.7). For any $f \in C([0,1])$ one has*

$$\lim_n (P^*_{n,q_n,\alpha}f)(x) = f(x), \text{ uniformly in } x \in [0,1].$$

Proof. The assumptions made upon the sequence $(q_n)_{n\geq 2}$ guarantee that $\lim_n [n]_{q_n} = \infty$. This implies

$$\lim_n v_{n,q_n,\alpha}(x) = x, \text{ uniformly in } x \in [0,1],$$

and, consequently, $\lim\limits_{n} P^*_{n,q_n,\alpha} e_1 = e_1$.

Further on, Theorem 3.3 (ii) ensures that the other two hypotheses of Bohman-Korovkin theorem are satisfied. The conclusion follows. □

Based on a classical result due to Shisha and Mond [12], for any linear positive operator $L : C([0,1]) \to B([0,1])$, the following inequality involving the modulus of continuity $\omega(f;\cdot)$ takes place

$$|(Lf)(x) - f(x)| \tag{3.8}$$

$$\leq |f(x)||(Le_0)(x) - 1| + \left((Le_0)(x) + \frac{1}{\delta}\sqrt{(L\psi_x^2)(x)}\sqrt{(Le_0)(x)} \right) \omega(f;\delta),$$

$x \in [0,1]$, $\delta > 0$, $f \in C([0,1])$, where $\psi_x(t) = t - x$, $t \in [0,1]$.

In what follows, we study the general case $\alpha > 0$. By a straight computation, we get

$$(P^*_{n,q_n,\alpha}\psi_x^2)(x) = (2x+\alpha)(x - v_{n,q_n,\alpha}(x)), \quad x \in [0,1].$$

Also, by using (3.6), for each $n = 2,3,\dots$ and $x \in [0,1]$ we can write

$$x - v_{n,q_n,\alpha}(x) \tag{3.9}$$

$$\begin{aligned} &\leq \frac{1}{2\beta_{n,q_n,\alpha}} \left((x + \beta_{n,q_n,\alpha})^2 - \left(\beta^2_{n,q_n,\alpha} + \frac{[n]_{q_n}}{[n]_{q_n} - 1}(x^2 + \alpha x) \right) \right) \\ &= \frac{x(1-x)}{1 + [n]_{q_n}\alpha}. \end{aligned}$$

We point out that the above upper bound is useful for our purposes only for $\alpha > 0$.

Returning to (3.8) and choosing $\delta := \dfrac{1}{\sqrt{[n]_{q_n}}}$, we can state

Theorem 3.5. *Let $(q_n)_{n\geq 2}$, $0 < q_n < 1$, be a sequence and $P^*_{n,q_n,\alpha}$, $n \geq 2$, be defined as in (3.7) with $\alpha > 0$. For any $f \in C([0,1])$ one has*

$$|(P^*_{n,q_n,\alpha}f)(x) - f(x)| \leq \left(1 + \sqrt{\frac{[n]_{q_n}}{1 + [n]_{q_n}\alpha}(2x+\alpha)x(1-x)} \right) \omega\left(f; \frac{1}{\sqrt{[n]_{q_n}}} \right).$$

Under the assumptions of Theorem 3.5, the following global estimate takes place

$$\|P^*_{n,q_n,\alpha}f - f\| \leq \left(1 + \sqrt{\frac{1}{2\alpha} + \frac{1}{4}} \right) \omega\left(f; \frac{1}{\sqrt{[n]_{q_n}}} \right),$$

where $\|\cdot\|$ stands for sup-norm of the space $C([0,1])$.

Remark 3.1. For the particular case $\alpha = 0$, instead of (3.9), we easily find another upper bound of the form $\mathcal{O}\left(\dfrac{1}{[n]_{q_n}} \right)$. For example, one has

$$(P^*_{n,q_n,0}\psi_x^2)(x) \leq \frac{2x^2(1-x)}{([n]_{q_n} - 1)x + 1}, \quad x \in [0,1],\ n = 2,3,\dots$$

Consequently, by using (3.8) with $\delta := \dfrac{1}{\sqrt{[n]_{q_n}}}$, we get

$$|(P^*_{n,q_n,0}f)(x) - f(x)| \le \left(1 + \sqrt{\frac{2x^2(1-x)[n]_{q_n}}{([n]_{q_n}-1)x+1}}\right)\omega\left(f;\frac{1}{\sqrt{[n]_{q_n}}}\right),$$

$x \in [0,1]$, $n = 2, 3, \ldots$

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