International Journal of Pure and Applied Mathematics

Volume 47 No. 2 2008, 199-206

EXISTENCE RESULTS FOR SYSTEMS OF NONLINEAR EVOLUTION EQUATIONS

Radu Precup[1 §], Adrian Viorel[2]

[1,2]Department of Applied Mathematics
Faculty of Mathematics and Computer Sciences
Babeş–Bolyai University
Cluj-Napoca, 400084, ROMANIA
[1]e-mail: r.precup@math.ubbcluj.ro
[2]e-mail: aviorel@gmail.com

Abstract: Existence results for semilinear systems of abstract evolution equations are established by means of Perov, Schauder and Leray-Schauder Fixed Point Theorems and a new technique for the treatment of systems based on vector-valued metrics and convergent to zero matrices.

AMS Subject Classification: 35K90, 35K45

Key Words: abstract parabolic evolution equation, parabolic system, initial value problem, Fixed Point Theorem, vector-valued norm

1. Introduction and Preliminaries

In the forthcoming paper [3] the first author developed a technique for the investigation of systems of nonlinear operator equations which is based on vector-valued metrics and convergent to zero matrices together with fundamental principles of nonlinear functional analysis. It is shown in [3] that the use of vector-valued metrics is more appropriate when treating systems of equations. In this paper we are concerned with the existence (and the uniqueness) of solutions for the Cauchy problem associated to a semilinear system of abstract evolution equations:

Received: June 11, 2008

§Correspondence author

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1 u_1(t) = F_1(t, u_1(t), u_2(t)) \,, \\ \dfrac{du_2}{dt}(t) + A_2 u_2(t) = F_2(t, u_1(t), u_2(t)) \,, \\ u_1(0) = u_1^0 \,, \\ u_2(0) = u_2^0 \,. \end{cases} \tag{1.1}$$

Here the linear operator $A_i : D(A_i) \subseteq X_i \to X_i$ is densely defined on the real Banach space X_i and generates the strongly continuous semigroup of contractions $\{S_i(t), t \geq 0\}$, for $i = 1, 2$. The Hille-Yosida Theorem (see [1], [2] , [6] and [7]) gives the following necessary and sufficient codition for A_i to generate a semigroup: for any $x, y \in D(A_i)$ and any $\lambda > 0$, $\|x - y\| \leq \|x - y + \lambda(A_i x - A_i y)\|$, and $I - A_i$ is surjective. If $(X_i, \langle \,.\,, .\, \rangle_{X_i})$ is a Hilbert space the necessary and sufficient condition is that $I - A_i$ is surjective and $\langle A_i x, x \rangle_{X_i} \geq 0$ for all $x \in D(A_i)$.

We shall look for *global mild solutions* on the interval $[0, T]$, i.e., $(u_1, u_2) \in C([0, T], X_1) \times C([0, T], X_1)$ satisfying

$$u_i(t) = S_i(t) u_i^0 + \int_0^t S_i(t - \tau) F_i(\tau, u_1(\tau), u_2(\tau)) d\tau \ \text{ for all } t \in [0, T], \quad i = 1, 2. \tag{1.2}$$

The nonlinear operator defined by the right hand side of (1.2) will be denoted by $N_i(u)$, where $u = (u_1, u_2) \in C([0, T], X_1) \times C([0, T], X_2)$.

In the next section three different Fixed Point Principles will be used in order to prove the existence of solutions for the semilinear problem, namely the Fixed Point Theorems of Perov, Schauder and Leray-Schauder (see [4]). In all three cases a key role will be played by the so called convergent to zero matrices. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k \to 0 \quad \text{as} \quad k \to \infty.$$

It is known that the property of being convergent to zero is equivalent to each of the following three conditions (for details see [3], [4] and [5]):

(a) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + \ldots$ (where I stands for the unit matrix of the same order as M);

(b) the eigenvalues of M are located inside the unit disc of the complex plane;

(c) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

We finish this introductory section by recalling three fundamental results which will be used in Section 2. Let X be a nonempty set. By a *vector-valued metric* on X we mean a mapping $d : X \times X \to \mathbf{R}_+^n$ such that:

(i) $d(u,v) \geq 0$ for all $u, v \in X$ and if $d(u,v) = 0$ then $u = v$;

(ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbf{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \ldots, n$. We call the pair (X, d) a *generalized metric space*. For such a space convergence and completness are similar to those in usual metric spaces.

An operator $N : X \to X$ is said to be *contractive* (with respect to the vector-valued metric d on X) if there exists a convergent to zero matrix M such that

$$d(N(u), N(v)) \leq M d(u,v) \quad \text{for all } u, v \in X.$$

Theorem 1. (Perov) *Let (X, d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point u^* and for each $u_0 \in X$ we have*

$$d(N^k(u_0), u^*) \leq M^k (I - M)^{-1} d(u_0, N(u_0)) \ \textit{ for all } k \in \mathbf{N}.$$

Theorem 2. (Schauder) *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $N : D \to D$ a completely continuous operator (i.e., N is continuous and $N(D)$ is relatively compact). Then N has at least one fixed point.*

Theorem 3. (Leray-Schauder) *Let $(X, ||\,.\,||)$ be a Banach space, $R > 0$ and $N : \overline{B}_R(0; X) \to X$ a completely continuous operator. If $||u|| < R$ for every solution u of the equation $u = \lambda N(u)$ and any $\lambda \in (0,1)$, then N has at least one fixed point.*

2. Main Results

Our first result is an existence and uniqueness theorem for the case of nonlinearities which satisfy a Lipschitz condition. Under the basic assumptions on X_i and A_i from Section 1, we have:

Theorem 4. *Suppose that $F_i : [0,T] \times X_1 \times X_2 \to X_i$ satisfies the Lipschitz condition*

$$||F_i(t,u) - F_i(t,v)||_{X_i} \leq a_{i1}(t) ||u_1 - v_1||_{X_1} + a_{i2}(t) ||u_2 - v_2||_{X_2} \qquad (2.1)$$

for all $u = (u_1, u_2)$, $v = (v_1, v_2) \in X_1 \times X_2$, $t \in [0, T]$ *and* $i = 1, 2$, *where* $a_{ij} \in L^p([0, T], \mathbf{R}_+)$ *for* $i, j = 1, 2$. *Then for any* $(u_1^0, u_2^0) \in X_1 \times X_2$ *the Cauchy problem* (1.1) *has a unique global mild solution.*

Proof. Let $E_i := C([0, T], X_i)$ be endowed with the Bielecki norm

$$||u||_{E_i} := \sup_{t \in [0,T]} e^{-kt} ||u(t)||_{X_i}$$

where $k > 0$ will be chosen later, and let $E := E_1 \times E_2$ be endowed with the vector-valued metric

$$d : E \to \mathbf{R}_+^2, \quad d(u, v) = \begin{pmatrix} ||u_1 - v_1||_{E_1} \\ ||u_2 - v_2||_{E_2} \end{pmatrix}.$$

Clearly (E, d) is a complete generalized metric space. Finding a mild solution to the Cauchy problem (1.1) comes back to finding a fixed point u for the nonlinear operator

$$N(u_1, u_2) := (N_1(u_1, u_2),\ N_2(u_1, u_2))$$

defined by (1.2) (i.e., a pair $u = (u_1, u_2) \in E$ with $N(u) = u$). In order to apply Perov's Fixed Point Theorem we need to show that N maps E into itself and that N is contractive.

Indeed, $N_i(u_1, u_2)$ is a continuous X_i-valued function as a consequence of the continuity properties of the semigroup $\{S_i(t), t \geq 0\}$ and of the integral operator. Notice that Hölder's inequality guarantees that

$$\int_0^t a_{ij}(\tau) e^{k\tau} d\tau \leq ||a_{ij}||_{L^p} \left(\int_0^t e^{qk\tau} d\tau \right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $u, v \in E$, using (2.1), we obtain

$$\begin{aligned}
& ||N_i(u)(t) - N_i(v)(t)||_{X_i} \\
= & \left\| \int_0^t S_i(t - \tau) \left(F_i(\tau, u(\tau)) - F_i(\tau, v(\tau)) \right) d\tau \right\|_{X_i} \\
\leq & \int_0^t ||S_i(t - \tau)|| ||F_i(\tau, u(\tau)) - F_i(\tau, v(\tau))||_{X_i} d\tau \\
\leq & \int_0^t ||F_i(\tau, u(\tau)) - F_i(\tau, v(\tau))||_{X_i} d\tau \\
\leq & \int_0^t \left(a_{i1}(\tau) ||u_1(\tau) - v_1(\tau)||_{X_1} + a_{i2}(\tau) ||u_2(\tau) - v_2(\tau)||_{X_2} \right) d\tau \\
\leq & ||u_1 - v_1||_{E_1} \int_0^t a_{i1}(\tau) e^{k\tau} d\tau + ||u_2 - v_2||_{E_2} \int_0^t a_{i2}(\tau) e^{k\tau} d\tau
\end{aligned}$$

$$\leq \quad \frac{||a_{i1}||_{L^p}}{(qk)^{1/q}}||u_1 - v_1||_{E_1}e^{kt} + \frac{||a_{i2}||_{L^p}}{(qk)^{1/q}}||u_2 - v_2||_{E_2}e^{kt}.$$

It follows that

$$||N_i(u) - N_i(v)||_{E_i} \leq \frac{||a_{i1}||_{L^p}}{(qk)^{1/q}}||u_1 - v_1||_{E_1} + \frac{||a_{i2}||_{L^p}}{(qk)^{1/q}}||u_2 - v_2||_{E_2}$$

for $i = 1, 2$. These inequalities can be written in the matrix form

$$d(N(u), N(v)) \leq M d(u, v)\,,$$

where

$$M = \left(\frac{||a_{ij}||_{L^p}}{(qk)^{1/q}}\right)_{i,j=1,2}.$$

Clearly, M converges to zero for k large enough, and so Perov's theorem can be applied. □

Assuming that the operator N is completely continuous we can weaken condition (2.1) to a at most linear growth condition. But now Schauder's Fixed Point Theorem that we apply will only guarantee the existence not also the uniqueness of the solution.

Theorem 5. *If the operator N is completely continuous and F_i satisfies*

$$||F_i(t, u)||_{X_i} \leq a_{i1}(t)||u_1||_{X_1} + a_{i2}(t)||u_2||_{X_2} + b_i(t) \tag{2.2}$$

for all $u = (u_1, u_2) \in X_1 \times X_2$, where $a_{ij} \in L^p([0, T], \mathbf{R}_+)$ and $b_i \in L^1([0, T], \mathbf{R}_+)$, for $i, j = 1, 2$, then problem (1.1) has at least one global mild solution.

Proof. In order to apply Schauder's Fixed Point Principle we need to find a nonempty closed bounded convex set $D \subset E$ such that

$$N(D) \subseteq D. \tag{2.3}$$

Let us consider the set $D := \overline{B}_{R_1}(0; E_1) \times \overline{B}_{R_2}(0; E_2)$, where $\overline{B}_{R_i}(0; E_i)$ is the closed ball of radius R_i in E_i. We try to find $R_1, R_2 \geq 0$ such that (2.3) holds. Using (2.2) and the Hölder inequality we deduce that

$$\begin{aligned} ||N_i(u)(t)||_{X_i} &\leq ||S_i(t)u_i^0||_{X_i} + \left\|\int_0^t S_i(t - \tau)\left(F_i(\tau, u(\tau))\right) d\tau\right\|_{X_i} \\ &\leq ||u_i^0||_{X_i} + \int_0^t \left(a_{i1}(\tau)||u_1(\tau))||_{X_1} + a_{i2}(\tau)||u_2(\tau)||_{X_2} + b_i(\tau)\right) d\tau \\ &\leq \frac{||a_{i1}||_{L^p}}{(qk)^{1/q}}||u_1||_{E_1}e^{kt} + \frac{||a_{i2}||_{L^p}}{(qk)^{1/q}}||u_2||_{E_2}e^{kt} + ||u_i^0||_{X_i} + ||b_i||_{L^1}. \end{aligned}$$

This means that for $u \in D$, i.e., $||u_i||_{E_i} \leq R_i$ for $i = 1, 2$, we have

$$||N_i(u)||_{E_i} \leq \tilde{a}_{i1}R_1 + \tilde{a}_{i2}R_2 + \tilde{b}_i\,,$$

where

$$\tilde{a}_{ij} = \frac{||a_{ij}||_{L^p}}{(qk)^{1/q}} \text{ and } \tilde{b}_i = ||b_i||_{L^1} + ||u_i^0||_{X_i}.$$

Thus, for any nonnegative solutions $R_1, R_2 \geq 0$ of the algebraic system

$$(I - M) \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} \tag{2.4}$$

where

$$M = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix},$$

we will have $||N_i(u)||_{E_i} \leq R_i$ for $i = 1, 2$, i.e., $N(D) \subseteq D$. For k large enough, matrix M is convergent to zero. Therefore, according to (c), (2.4) has a unique nonnegative solution, as we wished. □

Now in the case of Hilbert spaces, and if all mild solutions are classical solutions (i.e., they are in $C([0,T], D(A_i)) \cap C^1([0,T], X_i)$ and satisfy (1.1)), we have the following result based on the Leray-Schauder Fixed Point Theorem.

Theorem 6. *Let $(X_i, \langle .,. \rangle_{X_i})$, $i = 1, 2$ be real Hilbert spaces. If all mild solutions of the equations $u_i = \lambda N_i(u)$, $\lambda \in (0,1)$, are classical solutions, the nonlinear operator N is completely continuous and F_i satisfies*

$$\langle F_i(t,u), u_i \rangle_{X_i} \leq a_{i1}(t)||u_1||_{X_1}^2 + a_{i2}(t)||u_2||_{X_2}^2 + b_i(t) \tag{2.5}$$

for all $u \in X_1 \times X_2$, where $a_{ij} \in L^p([0,T], \mathbf{R}_+)$ and $b_i \in L^1([0,T], \mathbf{R}_+)$ for $i, j = 1, 2$, then problem (1.1) has at least one solution.

Proof. For each solution to the equation $u_i = \lambda N_i(u)$, $\lambda \in (0,1)$, we can write

$$\begin{aligned} u_i(t) &= \lambda S_i(t)u_i^0 + \lambda \int_0^t S_i(t-\tau)F_i(\tau, u_1(\tau), u_2(\tau))d\tau \\ &= S_i(t)(\lambda u_i^0) + \int_0^t S_i(t-\tau)(\lambda F_i(\tau, u_1(\tau), u_2(\tau)))d\tau \end{aligned}$$

for all $t \in [0,T]$, $i = 1, 2$. Since by our assumption, u_i is a classical solution, we have

$$\begin{cases} \dfrac{du_i}{dt}(t) + A_i u_i(t) = \lambda F_i(t, u_1(t), u_2(t)), \\ u_i(0) = \lambda u_i^0, \end{cases} \tag{2.6}$$

for all $t \in [0,T]$, $i = 1, 2$. Now taking the inner product in X_i with $u_i(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}||u_i(t)||_{X_i}^2 + \langle A_i u_i(t), u_i(t) \rangle_{X_i} = \lambda \langle F_i(t, u_1(t), u_2(t)), u_i(t) \rangle_{X_i}.$$

Then using $\langle A_i x, x\rangle_{X_i} \geq 0$ for all $x \in D(A_i)$ and (2.5), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u_i(t)||^2_{X_i} &\leq \lambda \langle F_i(t, u_1(t), u_2(t)), u_i(t)\rangle_{X_i} \\ &\leq a_{i1}(t)||u_1(t)||^2_{X_1} + a_{i2}(t)||u_2(t)||^2_{X_2} + b_i(t).\end{aligned}$$

Integrating with respect to t, we deduce that

$$||u_i(t)||^2_{X_i} \leq ||u_i^0||^2_{X_i} + 2\int_0^t (a_{i1}(\tau)||u_1(\tau))||^2_{X_1} + a_{i2}(\tau)||u_2(\tau)||^2_{X_2} + b_i(\tau))d\tau$$

for all $t \in [0, T]$, $i = 1, 2$. From this inequality, by using the same technique based on the Bielecki norm, as in the proof of Theorem 5, we obtain that

$$||u_i||^2_{E_i} \leq \tilde{a}_{i1}||u_1||^2_{E_1} + \tilde{a}_{i2}||u_2||^2_{E_2} + \tilde{b}_i \text{ for } i = 1, 2, \tag{2.7}$$

where

$$\tilde{a}_{ij} = 2\frac{||a_{ij}||_{L^p}}{(2qk)^{1/q}} \text{ and } \tilde{b}_i = 2||b_i||_{L^1} + ||u_i^0||^2_{X_i}.$$

Thus using the notation $M = (\tilde{a}_{ij})_{i,j=1,2}$ system (2.7) can be written in the matrix form as follows

$$(I - M)\begin{pmatrix} ||u_1||^2_{E_1} \\ ||u_2||^2_{E_2}\end{pmatrix} \leq \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2\end{pmatrix}. \tag{2.8}$$

For a sufficiently large k, matrix M is convergent to zero. Hence, $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements. Thus (2.8) becomes

$$\begin{pmatrix} ||u_1||^2_{E_1} \\ ||u_2||^2_{E_2}\end{pmatrix} \leq (I - M)^{-1}\begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2\end{pmatrix}.$$

This guarantees the a priori boundedness of the solutions $u = (u_1, u_2)$ of the equations $u = \lambda N(u)$, for all $\lambda \in (0, 1)$. Thus we may apply the Leray-Schauder Fixed Point Theorem. □

Notice that sufficient conditions for the complete continuity of operator N, as well as for that mild solutions be classical solutions can be found in the literature, see for example [1], [6] and [7].

References

[1] T. Cazenave, A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford University Press, New York (1998).

[2] J.A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, New York (1985).

[3] R. Precup, The role of convergent to zero matrices in the study of semilinear operator systems, *Mathematical and Computer Modelling* (2008).

[4] R. Precup, *Methods in Nonlinear Integral Equations*, Kluwer, Dordrecht (2002).

[5] I.A. Rus, *Principles ans Applications of the Fixed Point Theory*, Romanian, Dacia, Cluj-Napoca (1979), In Romanian.

[6] I. Vrabie, C_0*-Semigroups and Applications*, Elsevier, Amsterdam (2003).

[7] S. Zheng, *Nonlinear Evolution Equations*, Chapman and Hall, CRC (2004).