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Inequalities and Approximation Theory

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Abstract. The purpose of this paper is twofold. Firstly, we present an equivalence property involving isotonic linear functionals. Secondly, by using the contraction principle, we give a method for obtaining the limit of iterates of some classes of linear positive operators.

1. Introduction

In Approximation Theory a tool with rich mathematical content and great potential for applications is given by linear methods of approximation generated by sequences of linear operators, the essential ingredient being that of positivity.

The main objective of this survey paper is to present results which spring from standard inequalities enriching the mentioned research field.

In this respect, the paper is organized in two main sections.

Taking into account that the class of convex functions is characterized by the well-known inequality of Jensen, the following question arises in a natural way: what are the connections between the Jensen's inequality on $C([a, b])$, the existence of a sequence of approximating and convexity-preserving positive linear polynomial operators which reproduce the affine functions and Bohman-Korovkin's theorem? The aim of Section 2 is to show that the three above mentioned basic results together with a certain generalization of Jensen's inequality due to B. Jessen are equivalent. This equivalence property emphasizes the role of convexity and convexity-preserving operators in the approximation of functions by positive linear operators. Once again, the powerful criterion due to T. Popoviciu, H. Bohman and P.P. Korovkin is pointed out. It helps us to decide if a sequence of positive

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linear operators tends to the identity operator with respect to the uniform norm of the Banach space $C([a,b])$.

Further on, the chief aim of Section 3 is to study the convergence of the iterates associated to any operator belonging to certain classes of summation-type linear operators. These sequences preserve some test functions comprised by Popoviciu-Bohman-Korovkin theorem. For our purpose, we use a general technique based on the contraction principle and on the inequalities intrinsically entailed by this classic result. Some examples are also delivered.

2. Inequalities of Jensen type and approximation processes

The aim of this section is to bring into light an equivalence property involving isotonic linear functionals. The result emphasizes the role of convexity and of convexity-preserving operators in the approximation of functions by linear positive operators.

Let $C([a,b])$ be the Banach space of all real-valued and continuous functions defined on $[a,b]$, equipped with the norm $\|\cdot\|$ of the uniform convergence. We denote by e_n the monomials given by $e_n(x)=x^n$, $x\in[a,b]$ and $n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}$.

Bohman-Korovkin's theorem states: if $(L_n)_{n\geq 1}$ is a sequence of positive linear operators mapping $C([a,b])$ into itself such that $\lim\limits_{n\to\infty}\|L_ne_i-e_i\|=0$ for $i=0,1,2$, then one has $\lim\limits_{n\to\infty}\|L_nf-f\|=0$ for every $f\in C([a,b])$.

An abstract version of this result was proved in [An2].

In fact, this criterion should be called Popoviciu-Bohman-Korovkin. The research of the Romanian mathematician Tiberiu Popoviciu (1906-1975) was published in 1951 in Romanian language [Po2] and thus his contribution remained unknown for many mathematicians. In his proof, Popoviciu considered that the operators L_n, $n\in\mathbb{N}$, reproduce the constant functions.

Jensen's inequality in $C([a,b])$ says: if the function $f\in C([a,b])$ is convex on $[a,b]$, then for each $m\in\mathbb{N}$, $x_k\in[a,b]$ and $p_k\geq 0$, $k=1,2,\ldots,m$, such that $\sum\limits_{k=1}^{m}p_k=1$,one has

$$f\left(\sum_{k=1}^{m}p_kx_k\right)\leq\sum_{k=1}^{m}p_kf(x_k).$$

We recall: if $\mathcal{L}$ is a linear class of real-valued functions, then an isotonic linear functional $A:\mathcal{L}\to\mathbb{R}$ is a functional satisfying the conditions

(c_1) $A(\alpha f+\beta g)=\alpha A(f)+\beta A(g)$ for all $f,g\in\mathcal{L}$ and $\alpha,\beta\in\mathbb{R}$,

(c_2) if $f\in\mathcal{L}$ and $f\geq 0$, then $A(f)\geq 0$.

The above conditions guarantee that A is monotone: for each $f,g\in\mathcal{L}$ such that $f\leq g$, one has $A(f)\leq A(g)$.

The mapping A is said to be normalized if $A(1)=1$.

A generalization of Jensen's inequality involving isotonic linear functionals is due to B. Jessen [Je]. A short proof of this generalization and other related results can be found in P. Beesack and J. Pečarić [BePe].

A particular form of Jessen's inequality states: if $A : C([a,b]) \to \mathbb{R}$ is an isotonic normalized linear functional, then for every convex function $f \in C([a,b])$ the following inequality

$$f(A(e_1)) \leq A(f) \tag{2.1}$$

holds.

It is worth mentioning that this inequality and other similar inequalities appear in the works of M.L. Slater [Sl], J. Pečarić [Pe], J. Pečarić and D. Andrica [PeAn].

In what follows by Jessen's inequality we shall understand the inequality (2.1).

The following result related to these fundamental inequalities and the approximation by positive linear operators was proved by D. Andrica and C. Badea [AnBa].

Theorem 2.1. *The following statements are equivalent:*

(i) Jensen's inequality for convex functions in $C([a,b])$;

(ii) there is a sequence of approximating and convex-preserving positive linear polynomial operators which reproduce the affine functions;

(iii) Popoviciu-Bohman-Korovkin's theorem in the space $C([a,b])$;

(iv) Jessen's inequality for positive linear functionals on $C([a,b])$.

The technique of the proof of the above equivalences consists in the following five steps: $(i) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (i)$.

Proof $(i) \Rightarrow (iv)$.

We consider the convex function $f \in C([a,b])$. It is well-known that $f'_+(y)$ exists for every $y \in (a,b)$ and, for every $x \in [a,b]$, we have

$$f(x) \geq f(y) + f'_+(y)(x-y), \tag{2.2}$$

see e.g. A.W. Roberts and D.E. Varberg [RoVa; *p. 12*].

In (2.2) we consider $y = A(e_1) \in (a,b)$ and we apply the functional A with respect to x. Since A is monotone, it results $A(f) \geq f(A(e_1))$. Consequently (iv) holds.

Proof $(iv) \Rightarrow (iii)$.

Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators on $C([a,b])$ with $\lim\limits_{n\to\infty} \|L_n e_i - e_i\| = 0$, $i = 0,1,2$. Because $\lim\limits_{n\to\infty} \|L_n e_0 - e_0\| = 0$ we can assume that $L_n(e_0;x) > 0$ and $L_n(e_0,x) < K$, K constant, for every $x \in [a,b]$ and all positive integers n.

For a fixed $x \in [a,b]$ we consider the functionals $A_n : C([a,b]) \to \mathbb{R}$, $A_n(f) = L_n(f;x)/L_n(e_0;x)$. It is obvious that $A_n(e_0) = 1$.

If $f \in C^2([a,b])$ let us denote by

$$m = \min_{t\in[a,b]} f''(t), \quad M = \max_{t\in[a,b]} f''(t).$$

Now we can apply (iv) for the above-defined functionals A_n and for the convex functions $f_1 = f - \frac{m}{2}e_2$, $f_2 = \frac{M}{2}e_2 - f$. We get

$$\begin{aligned}
&\frac{1}{2} \cdot \frac{m}{L_n(e_0;x)}[L_n(e_0;x)L_n(e_2;x) - L_n^2(e_1;x)] \\
&\leq L_n(f;x) - L_n(e_0;x) \cdot f\left(\frac{L_n(e_1;x)}{L_n(e_0;x)}\right) \\
&\leq \frac{1}{2} \cdot \frac{M}{L_n(e_0;x)}[L_n(e_0;x)L_n(e_2;x) - L_n^2(e_1;x)]
\end{aligned}$$

for every $x \in [a, b]$ and $n = 1, 2, \ldots$ Therefore

$$\begin{aligned}
&|L_n(f;x) - L_n(e_0;x)f(L_n(e_1;x)/L_n(e_0;x))| \\
&\leq \frac{\|f''\|}{2L_n(e_0;x)}[L_n(e_0;x)L_n(e_2;x) - L_n^2(e_1;x)].
\end{aligned} \tag{2.3}$$

Using the following inequality

$$\begin{aligned}
\Bigg| L_n(f;x) - f(x)| \leq |f(x) - L_n(e_0;x)f\left(\frac{L_n(e_1;x)}{L_n(e_0;x)}\right)\Bigg| \\
+ \left|L_n(f;x) - L_n(e_0;x)f\left(\frac{L_n(e_1;x)}{L_n(e_0;x)}\right)\right|
\end{aligned}$$

and relation (2.3) we conclude that

$$\begin{aligned}
|L_n(f;x) - f(x)| \leq \left|f(x) - f\left(\frac{L_n(e_1;x)}{L_n(e_0;x)}\right)\right| + \|f\||L_n(e_0;x) - 1| \\
+ \frac{\|f''\|}{2L_n(e_0;x)}[L_n(e_0;x)L_n(e_2;x) - L_n^2(e_1;x)].
\end{aligned} \tag{2.4}$$

Since f is continuous on $[a, b]$, it is also uniformly continuous on $[a, b]$ and we obtain

$$|f(x) - f(L_n(e_1;x)/L_n(e_0;x))| \to 0$$

uniformly as $n \to \infty$. On the other hand, using the fact that $\{e_0, e_1, e_2\}$ is a set of test functions, we have

$$[L_n(e_0;x)L_n(e_2;x) - L_n^2(e_1;x)]/L_n(e_0;x) \to 0$$

uniformly as $n \to \infty$. From these remarks and relation (2.4) we deduce

$$\lim_{n\to\infty} \|L_n f - f\| = 0,$$

for every $f \in C^2([a, b])$.

On the other hand, for every $f \in C([a,b])$ we have

$$|L_n(f;x)| \leq \|f\| L_n(e_0;x) \leq K\|f\|,$$

consequently $\|L_n f\| \leq K\|f\|$ and one obtains $\|L_n\| \leq K, n = 0, 1, \dots$

Because $C^2([a,b])$ is a dense subspace in $C([a,b])$ the proof is complete via Banach-Steinhaus theorem.

We point out that the construction of the functions f_1 and f_2 has been used by A. Lupaş [Lu] in order to obtain the improvements of some inequalities due to D. Andrica and I. Raşa [AnRa]. The same idea has appeared in D. Andrica, I. Raşa and Gh. Toader [AnRaTo].

By using the same technique one could find a more general class than the class of approximating positive linear interpolation operators. This can be done by using Jensen-Steffensen' inequality (see D.S. Mitrinović [Mi; *p. 109*]) instead of Jensen's inequality.

Proof that $(iii) \Rightarrow (ii)$.

Let B_n be the n-th classical Bernstein operator on $[a,b]$, i.e.,

$$B_n(f;x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} (x-a)^k (b-x)^{n-k} f\left(a + k\frac{b-a}{n}\right). \quad (2.5)$$

Because $B_n e_k = e_k$, $k = 0, 1$ and $B_n e_2 = e_2 + \dfrac{e_1 - e_2}{n}$ we get by Popoviciu-Bohman-Korovkin's theorem that Bernstein operators are approximating operators. At the same time B_n is a convexity-preserving polynomial operator (see T. Popoviciu [Po1]), which reproduces the affine functions.

Proof $(ii) \Rightarrow (iv)$.

Let $(L_n)_{n\geq 1}$ be a sequence of approximating and convexity-preserving positive linear polynomial operators which preserve the affine functions. The existence of a such sequence is guaranteed by (ii). Let f be a convex function of $C^2([a,b])$. Using the Taylor's formula we get

$$f(x) \geq f(t) + (x-t)f'(t),$$

for every $x, t \in [a,b]$.

Applying the functional A with respect to x (for $t = A(e_1)$), it results (2.1) for every convex function $f \in C^2([a,b])$. For these functions we similarly have $L_n(f;x) \geq f(x)$. However, this inequality holds for an arbitrary convex function $f \in C([a,b])$. Indeed, if f is convex on $[a,b]$, then $L_m f \in C^2([a,b])$ is also a convex function and this implies $L_n(L_m f;x) \geq f(x)$. Letting m tends to infinity we get $L_n(f;x) \geq f(x)$ for every convex function $f \in C([a,b])$.

Finally, we complete the proof of Jessen's inequality (2.1) by using an idea of D. Andrica [An1].

Let $f \in C([a,b])$ be a convex function. Using the last inequality and the fact that the operators L_n form an approximation process, we deduce: for every $\varepsilon > 0$

there is a positive integer $N = N(\varepsilon)$ such that for all $n \geq N$ one has

$$0 \leq L_n(f;x) - f(x) < \varepsilon, \quad x \in [a,b].$$

Thus

$$A(L_n f) \leq A(f) + \varepsilon, \quad n = 1, 2, \dots \tag{2.6}$$

Because $L_n f \in C^2([a,b])$ is also a convex function we find, see (2.1), that

$$L_n(A(e_1)) \leq A(L_n f), \quad n = 1, 2, \dots \tag{2.7}$$

Hence, from (2.6) and (2.7) we have

$$L_n(A(e_1)) \leq A(f) + \varepsilon, \quad n = 1, 2, \dots$$

Because the operators L_n are approximating, we get

$$f(A(e_1)) \leq A(f) + \varepsilon$$

for every $\varepsilon > 0$. Consequently, Jessen's inequality is proved and the implication $(ii) \Rightarrow (iv)$ is true.

Proof $(iv) \Rightarrow (i)$. Examining Jessen's inequality, this assertion is evident.

The proof of Theorem 2.1 is complete. □

Remark 2.2. Taking into account the above proof we deduce that (ii) can be replaced by the following weaker assertion:

(ii)' there is a sequence of approximating and convex-preserving positive linear operators (L_n) which reproduce the affine functions and verify the condition $L_n f \in C^2([a,b])$ for every convex function f.

3. Contractive mappings and iterates of certain classes of linear positive operators

The Banach fixed-point theorem, also called contractive mapping theorem, furnishes an elegant method for obtaining the limit of over-iteration of a given linear positive operator. Over-iteration means that for a fixed operator its m-th powers are investigated when m goes to infinity. The technique described by O. Agratini and I.A. Rus was already the topic of different papers, see e.g. [Ru], [AgRu1], [AgRu2], [Ag] containing the study of several classes of linear positive operators on some spaces of functions defined on subsets of $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$. As usual, we set the powers of the operator A by $A^0 = I_X$, $A^1 = A$, $A^{m+1} = A \circ A^m$, $m \in \mathbb{N}$, where I_X indicates the identity operator on the space X.

At first, as in [AgRu1], we introduce the following general discrete type sequence of operators acting on the space $C([a,b])$

$$L_n : C([a,b]) \to C([a,b]), \quad (L_n f)(x) = \sum_{k=0}^{n} \psi_{n,k}(x) f(x_{n,k}), \tag{3.1}$$

where Δ_n is a given net on $[a, b]$, $\Delta_n(a = x_{n,0} < x_{n,1} < \cdots < x_{n,n} = b)$, and every function $\psi_{n,k}$ $(0 \le k \le n)$ belongs to $C([a, b])$ such that

$$\psi_{n,k} \ge 0 \; (0 \le k \le n), \quad \sum_{k=0}^{n} \psi_{n,k} = e_0, \quad \psi_{n,0}(a) = \psi_{n,n}(b) = 1. \tag{3.2}$$

Let us denote

$$u_n := \min_{x \in [a,b]} (\psi_{n,0}(x) + \psi_{n,n}(x)). \tag{3.3}$$

According to Bohman-Korovkin theorem, the necessary and sufficient conditions which offer to $(L_n)_{n \ge 1}$ the attribute of approximation process are the following

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_{n,k}^i \psi_{n,k} = e_i \text{ uniformly on } [a, b], \quad i \in \{1, 2\}.$$

In what follows we study the convergence for over-iteration of the L_n operator in two distinct variants, see [AgRu1], [Ag] respectively.

(i) The operators L_n, $n \in \mathbb{N}$, have degree of exactness one, this meaning

$$\sum_{k=0}^{n} x_{n,k} \psi_{n,k} = e_1. \tag{3.4}$$

(ii) The operators L_n, $n \in \mathbb{N}$, preserve the quadratic test function, this meaning

$$\sum_{k=0}^{n} x_{n,k}^2 \psi_{n,k} = e_2. \tag{3.5}$$

We mention that both results take into account only the iterates of a fixed operator.

Theorem 3.1. *Let the operators L_n, $n \in \mathbb{N}$, be defined by (3.1)-(3.2) such that (3.4) holds.*

If u_n defined by (3.3) satisfies $u_n > 0$, then one has

$$\lim_{m\to\infty} (L_n^m f)(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a), \quad f \in C([a, b]), \tag{3.6}$$

uniformly on $[a, b]$.

Proof. We define the sets

$$X_{\alpha,\beta} := \{f \in C([a, b]) \mid f(a) = \alpha, \; f(b) = \beta\}, \quad (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}.$$

Clearly, every $X_{\alpha,\beta}$ is a closed subset of $C([a, b])$ and the system $X_{\alpha,\beta}$, $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$, makes up a partition of this space. Since the non-negative functions $\psi_{n,k}$, $0 \le k \le n$, verify $\psi_{n,0}(a) = \psi_{n,n}(b) = 1$ and $\sum_{k=0}^{n} \psi_{n,k}(x) = 1$, we get

$(L_n f)(a) = f(a)$ and $(L_n f)(b) = f(b)$. In other words, for all $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ and $n \in \mathbb{N}$, $X_{\alpha,\beta}$ is an invariant subset of L_n.

Further on, we prove that $L_n|_{X_{\alpha,\beta}} : X_{\alpha,\beta} \to X_{\alpha,\beta}$ is a contractive operator with contractivity constant $1 - u_n \in [0, 1)$. Indeed, if f and g belong to $X_{\alpha,\beta}$ then, for every $x \in [a, b]$, we can write

$$\begin{aligned}
|(L_n f)(x) - (L_n g)(x)| &= \left|\sum_{k=1}^{n-1} \psi_{n,k}(x)(f-g)(x_{n,k})\right| \\
&\le \sum_{k=1}^{n-1} \psi_{n,k}(x)\|f-g\| \\
&= (1 - \psi_{n,0}(x) - \psi_{n,n}(x))\|f-g\| \\
&\le (1-u_n)\|f-g\|,
\end{aligned}$$

and consequently, $\|L_n f - L_n g\| \le (1 - u_n)\|f - g\|$. The assumption $u_n > 0$ guarantees our statement.

On the other hand, the function $p^*_{\alpha,\beta} := \dfrac{\alpha b - \beta a}{b-a} e_0 + \dfrac{\beta - \alpha}{b-a} e_1$ belongs to $X_{\alpha,\beta}$. Since L_n reproduces the affine functions, $p^*_{\alpha,\beta}$ is a fixed point of L_n.

For any $f \in C([a, b])$ one has $f \in X_{f(a),f(b)}$ and, by using the contraction principle, we get $\lim\limits_{m\to\infty} L_n^m f = p^*_{f(a),f(b)}$. The desired result (3.6) is obtained. □

In a similar manner we can prove the following result. In this new case the fixed point element is the function

$$p^*_{\alpha,\beta} = \frac{\alpha b^2 - \beta a^2}{b^2 - a^2} e_0 + \frac{\beta - \alpha}{b^2 - a^2} e_2 \in X_{\alpha,\beta}.$$

Theorem 3.2. *Let the operators L_n, $n \in \mathbb{N}$, be defined by (3.1)-(3.2) such that $b \ne -a$ and (3.5) holds.*

If u_n defined by (3.3) satisfies $u_n > 0$, then one has

$$\lim_{m\to\infty} (L_n^m f)(x) = \frac{1}{b^2 - a^2}(f(a)b^2 - f(b)a^2 + (f(b) - f(a))x^2), \qquad (3.7)$$

$f \in C([a, b])$, uniformly on $[a, b]$.

Remark 3.3. In Theorem 3.1 the conditions $\psi_{n,0}(a) = \psi_{n,n}(b) = 1$ are automatically satisfied. $L_n e_i = e_i$ for $i = 0$ and $i = 1$ imply interpolation at the endpoints of the function. This fact was bring to the first author by H. Gonska, D. Kacsó and P. Piţul [GoKaPi, *Lemma 3.1*] accompanied by a nice proof.

Remark 3.4. If $a < b \le 0$ or $0 \le a < b$, then the identities $\psi_{n,0}(a) = \psi_{n,n}(b) = 1$ claimed in (3.2) are implied by the identities $L_n e_j = e_j$, $j \in \{0, 2\}$. Let analyse the case $0 \le a < b$. Choosing in the previous identities $x = a$, we deduce $\sum\limits_{k=1}^{n} \psi_{n,k}(a)(a^2 - x^2_{n,k}) = 0$. Since $\psi_{n,k}(a) \ge 0$ and $a^2 - x^2_{n,k} < 0$ for all $k = \overline{1,n}$, clearly $\psi_{n,k}(a) = 0$ for each $k = \overline{1,n}$. Since $L_n e_0 = e_0$, we get $\psi_{n,0}(a) = 1$. In a similar manner we prove $\psi_{n,n}(b) = 1$.

Moreover, if $a < 0 < b$, then $0 \in \{x_{n,1}, \ldots, x_{n,n-1}\}$. Indeed, assuming contrary, this means $x_{n,k} \neq 0$ for all $k = \overline{1, n-1}$, and choosing $x = 0$ in (3.5) we obtain $\psi_{n,k}(0) = 0$ for all $= \overline{0, n}$. This fact is in contradiction with the identity $L_n e_0 = e_0$.

Examples 3.5.

1° Bernstein operators defined by (2.5) satisfy the hypothesis of Theorem 3.1, consequently (3.6) takes place. We reobtain the result established long time ago by R.P. Kelinsky and T.J. Rivlin [KeRi].

2° In (3.1) choosing $a = 0$, $b = 1$, $x_{n,k} = k/n$,

$$\psi_{n,k}(x) = \binom{n}{k} r_n^k(x)(1 - r_n(x))^{n-k},$$

where $r_1(x) = x^2$ and

$$r_n(x) = -\frac{1}{2(n-1)} + \left(\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}\right)^{1/2}, \text{ for } n \geq 2,$$

our operators become the King operators, see [Ki].

This is an unexpected example of non polynomials operators which preserve the monomials e_0 and e_2. Applying Theorem 3.2, King operators L_n, $n \in \mathbb{N}$, enjoy the following property

$$\lim_{m \to \infty} (L_n^m f)(x) = f(0) + (f(1) - f(0))x^2, \quad f \in C([0,1]),$$

uniformly on $[0, 1]$.

It is worth to point out the following more general result as regards the limit behavior of the k_n-th iterates of positive linear approximation operators L_n, $n \in \mathbb{N}$, obtained by S. Karlin and Z. Ziegler [KaZi, *Theorem 1*].

Theorem 3.6. *Let $(L_n)_{n \geq 1}$ be a sequence of positive linear operators on $C([a,b])$ satisfying $L_n e_j = e_j$, $j \in \{0, 1\}$, for all $n \in \mathbb{N}$.*

(a) A necessary and sufficient condition for

$$\lim_{n \to \infty} \|L_n^{k_n} f - Pf\| = 0,$$

where P is the projection operator $(Pf)(x) = f(a) + \dfrac{x-a}{b-a}(f(b) - f(a))$, is that

$$\lim_{n \to \infty} \|L_n^{k_n} e_2 - (a+b)e_1 - abe_0\| = 0.$$

(b) A necessary and sufficient condition for

$$\lim_{n \to \infty} \|L_n^{k_n} f - f\| = 0$$

is that

$$\lim_{n \to \infty} \|L_n^{k_n} e_e - e_2\| = 0.$$

Returning to Bernstein operators, the above theorem yields the following conclusions [KaZi, *p. 323*]:

$$\lim_{n\to\infty} \|B_n^{k_n} f - f\| = 0 \text{ iff } \lim_{n\to\infty} \frac{k_n}{n} = 0,$$

$$\lim_{n\to\infty} \|B_n^{k_n} f - Pf\| = 0 \text{ iff } \lim_{n\to\infty} \frac{k_n}{n} = \infty.$$

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