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A compression type mountain pass theorem in conical shells

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Abstract

In this paper we present a compression type version of the mountain pass lemma in a conical shell with respect to two norms. An application to second-order ordinary differential equations is included.

Keywords: Critical point; Mountain pass lemma; Compression; Cone; Positive solution

1. Introduction

The so-called mountain pass theorem of Ambrosetti and Rabinowitz [1] is one of the most used tools in studying nonlinear equations having a variational form (see [2,6,10,14,15,20,24] and [25]). It concerns a real-valued C^1 functional $E(u)$ defined on a real Banach space X, for which one desires to find a critical point, i.e., a point u, where $E'(u) = 0$. Such a point is obtained by considering an optimal path in the set of all continuous paths connecting two given points separated by a "mountain range." A number of authors have been interested to restrict the competing paths to a bounded region in order to locate a critical point. For example, in [9] the authors gave a variant of the mountain pass theorem in a convex set M of the Hilbert space X (identified to its dual), using the Schauder invariance condition $(I - E')(M) \subset M$, while in [21] (see also [22,23] and [18]) a critical point is located in a ball M of X under the Leray–Schauder boundary condition (see [16]) for $I - E'$. Here I stands for the identity map of X. The Schauder and the Leray–Schauder conditions are used to solve the difficult problem of constructing paths which do not leave region M. Such a construction suggested in [12] to introduce the notion of an invariant set of descending flow of E with respect to a pseudogradient of E. Thus the difficult problem is reduced to prove that for a given set M, there exists a pseudogradient with respect to which M is an invariant set of descending flow (a difficult problem as well). Related topics can be found in [5,8,13,17] and [19].

In this paper we shall not identify X to its dual X'. More exactly we consider two real Hilbert spaces, X with inner product and norm $(.,.)$, $|.|$, and H with inner product and norm $\langle .,. \rangle$, $\|.\|$, and we assume that $X \subset H$, X is dense in H, the injection being continuous. We shall denote by c_0 the imbedding constant with

$$\|u\| \leqslant c_0 |u| \quad \text{for all } u \in X.$$

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We identify H to its dual H', thanks to the Riesz representation theorem and we obtain

$$X \subset H \equiv H' \subset X',$$

where each space is dense in the following one, the injections being continuous. By $\langle .,. \rangle$ we also denote a natural duality between X and X', that is $\langle x^*, x\rangle = x^*(x)$ for $x \in X$ and $x^* \in X'$. When $x^* \in H$, one has that $\langle x^*, x\rangle$ is exactly the scalar product in H of x and x^*. Let L be the linear continuous operator from X to X' (the canonical isomorphism of X onto X'), given by

$$(u, v) = \langle Lu, v\rangle \quad \text{for all } u, v \in X,$$

and let J from X' into X be the inverse of L. Then

$$(Ju, v) = \langle u, v\rangle \quad \text{for all } u \in X',\ v \in X.$$

This for $u \in H$ implies

$$|Ju|^2 = \langle u, Ju\rangle \leqslant \|u\| \|Ju\| \leqslant c_0 \|u\| |Ju|.$$

Hence

$$|Ju| \leqslant c_0 \|u\|. \tag{1.1}$$

We consider a C^1 real functional E defined on X and we are interested in the equation $E'(u) = 0$.

By a wedge of X we shall understand a convex closed nonempty set $K \subset X$, $K \neq \{0\}$, with $\lambda u \in K$ for every $u \in K$ and $\lambda \geqslant 0$. Thus K has not necessarily be a cone (when $K \cap (-K) = \{0\}$) and, in particular, K might be the whole space X.

In what follows we shall assume that J is "positive" with respect to K, i.e.,

$$Ju \in K \quad \text{for every } u \in K.$$

Let R_0, R_1 be such that $0 < R_0 < c_0 R_1$ and let $K_{R_0 R_1}$ be the conical shell

$$K_{R_0 R_1} = \big\{u \in K\colon\ \|u\| \geqslant R_0,\ |u| \leqslant R_1\big\}.$$

In applications, $|.|$ is the specific energy norm, while $\|.\|$ is an L^p-norm which can be used instead of $|.|$ because of its monotonicity property with respect to the order relation.

Notice that there exists a number R with $R \leqslant R_1$ and

$$|Ju| \geqslant R > 0 \quad \text{for all } u \in K_{R_0 R_1}. \tag{1.2}$$

Indeed, otherwise, there would be a sequence (u_k) of elements in $K_{R_0 R_1}$ with $|Ju_k| \to 0$ as $k \to \infty$. Now, from

$$R_0^2 \leqslant \|u_k\|^2 = \langle u_k, u_k\rangle = (Ju_k, u_k) \leqslant |Ju_k||u_k| \leqslant R_1 |Ju_k|$$

letting $k \to \infty$, we derive the contradiction $R_0^2 \leqslant 0$.

In this paper, starting from the results in [21,22], we present a variant of the mountain pass theorem in the conical shell $K_{R_0 R_1}$ assuming that the operator $I - JE'$ satisfies a compression boundary condition like that in the corresponding fixed point theorem of Krasnoselskii [11]. The localization result immediately yields multiplicity results for functionals with a "wavily relief." A simple application to nonlinear boundary value problems is presented to illustrate the theory.

We finish this introductory section by a technical result about differential equations in closed convex sets (see [3]).

Lemma 1.1. *Let X be a Banach space, D a closed convex set in X. Assume that $W : D \to X$ is a locally Lipschitz map which satisfies*

$$\big|W(u)\big| \leqslant C, \qquad \varliminf_{\lambda \to 0^+} \frac{1}{\lambda} d\big(u + \lambda W(u), D\big) = 0 \tag{1.3}$$

for all $u \in D$. Then, for any $u \in D$, the initial value problem in Banach space

$$\frac{d\sigma}{dt} = W(\sigma), \quad \sigma(0) = u$$

has a unique solution $\sigma(u,t)$ *on* $\mathbf{R}_+$, *and* $\sigma(u,t) \in D$ *for every* $t \in \mathbf{R}_+$.

Notice that the lim condition in (1.3) holds in particular, if for any $u \in D$, there exists $\lambda_u > 0$ with $u + \lambda_u W(u) \in D$. Indeed, if such a λ_u exists, then for every $\lambda \in (0, \lambda_u]$, one has

$$u + \lambda W(u) = \left(1 - \lambda\lambda_u^{-1}\right)u + \lambda\lambda_u^{-1}\left(u + \lambda_u W(u)\right) \in D$$

since D is convex.

2. Main results

Theorem 2.1. *Assume that there exist* $u_0, u_1 \in K_{R_0R_1}$ *and* $\nu_0, r > 0$, $|u_0| < r < |u_1|$, *such that the following conditions are satisfied*:

$$u - JE'(u) \in K \quad \text{for all } u \in K; \tag{2.1}$$

$$\left(JE'(u), Ju\right) \leqslant \nu_0 \quad \text{for all } u \in K_{R_0R_1} \text{ with } \|u\| = R_0; \tag{2.2}$$

$$\left(JE'(u), u\right) \geqslant -\nu_0 \quad \text{for all } u \in K_{R_0R_1} \text{ with } |u| = R_1; \tag{2.3}$$

$$\max\left\{E(u_0), E(u_1)\right\} < \inf_{\substack{u \in K_{R_0R_1} \\ |u| = r}} E(u). \tag{2.4}$$

Let

$$\Gamma = \left\{\gamma \in C\left([0,1]; K_{R_0R_1}\right):\ \gamma(0) = u_0,\ \gamma(1) = u_1\right\}$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E\left(\gamma(t)\right).$$

Then there exists a sequence (u_k) *with* $u_k \in K_{R_0R_1}$ *such that*

$$E(u_k) \to c \quad \text{as } k \to \infty \tag{2.5}$$

and one of the following three properties holds:

$$E'(u_k) \to 0 \quad \text{as } k \to \infty; \tag{2.6}$$

$$\begin{cases} \|u_k\| = R_0, \quad \left(JE'(u_k), Ju_k\right) \geqslant 0 \quad \text{and} \\ JE'(u_k) - \dfrac{(JE'(u_k), Ju_k)}{|Ju_k|^2} Ju_k \to 0 \quad (\text{in } X) \quad \text{as } k \to \infty; \end{cases} \tag{2.7}$$

$$\begin{cases} |u_k| = R_1, \quad \left(JE'(u_k), u_k\right) \leqslant 0 \quad \text{and} \\ JE'(u_k) - \dfrac{(JE'(u_k), u_k)}{R_1^2} u_k \to 0 \quad (\text{in } X) \quad \text{as } k \to \infty. \end{cases} \tag{2.8}$$

If in addition, any sequence (u_k) *as above has a convergent* (*in* X) *subsequence and* E *satisfies the boundary conditions*

$$JE'(u) - \lambda Ju \neq 0 \quad \text{for } u \in K_{R_0R_1},\ \|u\| = R_0,\ \lambda > 0, \tag{2.9}$$

$$JE'(u) + \lambda u \neq 0 \quad \text{for } u \in K_{R_0R_1},\ |u| = R_1,\ \lambda > 0, \tag{2.10}$$

then there exists $u \in K_{R_0R_1}$ *with*

$$E'(u) = 0 \quad \text{and} \quad E(u) = c.$$

Remark 2.1. Let $N(u) := u - JE'(u)$. Conditions (2.9), (2.10) can be written under the form

$$N(u) + \lambda Ju \neq u \quad \text{for } \|u\| = R_0,\ \lambda > 0, \tag{2.11}$$

$$N(u) \neq (1+\lambda)u \quad \text{for } |u| = R_1,\ \lambda > 0. \tag{2.12}$$

The proof of Theorem 2.1 needs some lemmas.

Lemma 2.1.

(1^0) *Let $w, v \in X \setminus \{0\}$ and $\alpha, \theta \in \mathbf{R}_+$ such that $0 < \alpha < 1 - \theta$ and $-(w, v) \leqslant \theta|w||v|$. Then there exists an element $h \in X$ with*

$$|h| = 1, \quad (w, h) \leqslant -\alpha|w| \quad \textit{and} \quad (v, h) < 0. \tag{2.13}$$

Moreover, if $v \in K$ and $v - w \in K$, then there exists $\lambda^* > 0$ with

$$v + \mu h \in K \quad \textit{for all } \mu \in [0, \lambda^*]. \tag{2.14}$$

(2^0) *Let $w, v \in X \setminus \{0\}$ and $\alpha, \theta \in \mathbf{R}_+$ such that $0 < \alpha < 1 - \theta$ and $(w, Jv) \leqslant \theta|w||Jv|$. Then there exists an element $h \in X$ with*

$$|h| = 1, \quad (w, h) \leqslant -\alpha|w| \quad \textit{and} \quad (Jv, h) \geqslant (1 - \theta - \alpha)|Jv| > 0. \tag{2.15}$$

Moreover, if $v \in K$ and $v - w \in K$, then there exists $\lambda^* > 0$ with

$$v + \mu h \in K \quad \textit{for all } \mu \in [0, \lambda^*]. \tag{2.16}$$

In case that $1 - \theta < 2\alpha$, and $|v| \geqslant \rho > 0$, $|w| \geqslant \omega > 0$, then λ^* in (2.14) *and* (2.16) *only depends on ρ and ω* (*being independent of v and w*).

Proof. (1^0) Let $h_0 = -\frac{w}{|w|} - \beta\frac{v}{|v|}$, with $\beta = \frac{1-\alpha}{\theta+\alpha}$. One has $\beta > \theta$ since $\alpha < 1 - \theta$, and $0 < |h_0| \leqslant 1 + \beta$. Also

$$(w, h_0) = -|w| - \frac{\beta}{|v|}(w, v) \leqslant -|w| + \beta\theta|w| = -\alpha(1+\beta)|w| \leqslant -\alpha|h_0||w|$$

and

$$(v, h_0) = -\left(v, \frac{w}{|w|}\right) - \beta|v| \leqslant \theta|v| - \beta|v| < 0.$$

Obviously, for any $\lambda > 0$, $h := \lambda h_0$ also satisfies

$$(w, h) \leqslant -\alpha|h||w| \quad \text{and} \quad (v, h) < 0.$$

Let $h := \frac{h_0}{|h_0|}$. Then (2.13) holds.

Assume now that $v \in K$ and $v - w \in K$. Then

$$v + \mu h = v - \frac{\mu}{|h_0|}\frac{w}{|w|} - \frac{\mu}{|h_0|}\beta\frac{v}{|v|} = \frac{\mu}{|h_0||w|}(v - w) + \left(1 - \frac{\mu}{|h_0||w|} - \frac{\mu\beta}{|h_0||v|}\right)v \in K$$

for $\mu > 0$ small enough that

$$1 - \frac{\mu}{|h_0||w|} - \frac{\mu\beta}{|h_0||v|} \geqslant 0, \tag{2.17}$$

since both v, $v - w$ belong to K.

(2^0) Let $h_0 = -\frac{w}{|w|} + \beta\frac{Jv}{|Jv|}$, with $\beta = \frac{1-\alpha}{\theta+\alpha}$. Here again $\beta > \theta$ and $0 < |h_0| \leqslant 1 + \beta$. Also

$$(w, h_0) = -|w| + \frac{\beta}{|Jv|}(w, Jv) \leqslant -|w| + \beta\theta|w| = -\alpha(1+\beta)|w| \leqslant -\alpha|h_0||w|$$

and

$$(Jv, h_0) = -\left(Jv, \frac{w}{|w|}\right) + \beta|Jv| \geqslant -\theta|Jv| + \beta|Jv| = \frac{(\theta+1)(1-\theta-\alpha)}{\theta+\alpha}|Jv|.$$

Let $h := \frac{h_0}{|h_0|}$. Then $(w, h) \leqslant -\alpha|w|$ and since $|h_0| \leqslant 1 + \beta = \frac{\theta+1}{\theta+\alpha}$,

$$(Jv, h) \geqslant (1-\theta-\alpha)|Jv|.$$

In case that $v \in K$ and $v - w \in K$, we have that $Jv \in K$ and consequently

$$\begin{aligned} v + \mu h &= v - \frac{\mu}{|h_0|}\frac{w}{|w|} + \frac{\mu}{|h_0|}\beta\frac{Jv}{|Jv|} \\ &= \left(1 - \frac{\mu}{|h_0||w|}\right)v + \frac{\mu}{|h_0||w|}(v-w) + \frac{\mu}{|h_0|}\beta\frac{Jv}{|Jv|} \in K \end{aligned}$$

for

$$1 - \frac{\mu}{|h_0||w|} \geqslant 0. \tag{2.18}$$

Finally, if $1-\theta < 2\alpha$, then $\beta < 1$, so $|h_0| \geqslant 1-\beta > 0$. Consequently (2.17) and (2.18) hold for $\mu \leqslant \frac{1-\beta}{\frac{1}{\omega}+\frac{1}{\rho}}$. □

Lemma 2.2. *Let $a > 0$, $G : K_{R_0R_1} \to X$ a continuous map, $\widehat{D} = \{u \in K_{R_0R_1}: |G(u)| \geqslant a\}$, and $D_0 \subset \{u \in \widehat{D}: \|u\| = R_0\}$, $D_1 \subset \{u \in \widehat{D}: |u| = R_1\}$ closed sets. Assume that*

$$u - G(u) \in K \quad \textit{for all } u \in K_{R_0R_1},$$

and there exists $\theta \in [0, 1)$ such that

$$\begin{aligned} &\big(Ju, G(u)\big) \leqslant \theta|Ju|\big|G(u)\big| \quad \textit{for all } u \in D_0, \\ &-\big(u, G(u)\big) \leqslant \theta|u|\big|G(u)\big| \quad \textit{for all } u \in D_1. \end{aligned}$$

Then there exists $\alpha > 0$ and a locally Lipschitz map $H : \widehat{D} \to X$ such that

$$\big|H(u)\big| \leqslant 1, \quad u + H(u) \in K, \quad \big(G(u), H(u)\big) \leqslant -\alpha\big|G(u)\big| \quad \textit{for } u \in \widehat{D},$$

and

$$\begin{aligned} &\big(Ju, H(u)\big) > 0 \quad \textit{for } u \in D_0, \\ &\big(u, H(u)\big) < 0 \quad \textit{for } u \in D_1. \end{aligned}$$

Proof. Let α' be such that $0 < \alpha' < 1-\theta < 2\alpha'$. Using Lemma 2.1, applied to $w := G(u)$ and $v := u$ if $u \in D_0 \cup D_1$, we may define a map $h : \widehat{D} \to X$ with $|h(u)| = 1$ for all $u \in \widehat{D}$ and with the following properties:

$$\begin{aligned} &h(u) = -\big|G(u)\big|^{-1}G(u) \quad \text{for } u \in \widehat{D} \setminus (D_0 \cup D_1); \\ &\big(G(u), h(u)\big) \leqslant -\alpha'\big|G(u)\big| \quad \text{and} \quad \big(Ju, h(u)\big) \geqslant (1-\theta-\alpha')R > 0 \quad \text{in } D_0; \\ &\big(G(u), h(u)\big) \leqslant -\alpha'\big|G(u)\big| \quad \text{and} \quad \big(u, h(u)\big) < 0 \quad \text{in } D_1; \\ &u + \mu h(u) \in K \quad \text{for all } \mu \in \big[0, \lambda^*\big],\ u \in \widehat{D}, \end{aligned}$$

and some $\lambda^* = \lambda^*(\theta, \alpha', a) > 0$. Notice that for $u \in \widehat{D} \setminus (D_0 \cup D_1)$, we have

$$u + \mu h(u) = \left(1 - \frac{\mu}{|G(u)|}\right)u + \frac{\mu}{|G(u)|}\big(u - G(u)\big) \in K$$

for $0 \leqslant \mu \leqslant a$, since $u, u - G(u) \in K$ and $\frac{\mu}{|G(u)|} \in [0, 1]$.

Clearly we may assume without loss of generality that $\lambda^* < 1$.

We have

$$\big(G(u), h(u)\big) \leqslant -\alpha'\big|G(u)\big| \quad \text{for all } u \in \widehat{D}.$$

Fix any number $\alpha'' \in (0, \alpha')$. Since G is continuous, for each $u \in \widehat{D}$, there is in $\widehat{D}$ an open neighborhood $V(u)$ of u such that

$$\big(G(v), h(u)\big) \leqslant -\alpha''\big|G(v)\big| \quad \text{for all } v \in V(u).$$

For $u \in D_0$, we may also assume that

$$\big(Jv, h(u)\big) \geqslant \frac{1}{2}(1-\theta-\alpha')R > 0 \quad \text{for all } v \in V(u),$$

while for $u \in D_1$, that

$$\big(v, h(u)\big) < 0 \quad \text{for all } v \in V(u).$$

For $u \in \widehat{D} \setminus (D_0 \cup D_1)$, take $V(u)$ so small that $V(u) \cap (D_0 \cup D_1) = \emptyset$. We may also assume that $\operatorname{diam} V(u) \leqslant r$ for every $u \in \widehat{D}$, where $r > 0$ is small and will be chosen later. The collection $\{V(u)\colon\ u \in \widehat{D}\}$ is an open covering of $\widehat{D}$. Since $\widehat{D}$ is paracompact, the covering has a locally finite refinement $\{V_\tau\}$. Let $\{\psi_\tau\}$ be a locally Lipschitz partition of unity subordinated to this refinement, and for each τ, let $u_\tau \in \widehat{D}$ be an element for which $V_\tau \subset V(u_\tau)$.

Let $b(u_\tau) := u_\tau + \lambda^* h(u_\tau)$. Clearly, $b(u_\tau) \in K$ for every τ. Now let $H : \widehat{D} \to X$ be given by

$$H(v) = -v + \sum_\tau \psi_\tau(v) b(u_\tau).$$

Clearly H is locally Lipschitz. For every $v \in \widehat{D}$, we have

$$\begin{aligned} \big|H(v)\big| &= \left|\sum_\tau \psi_\tau(v)\big(b(u_\tau) - u_\tau\big) + \sum_\tau \psi_\tau(v)(u_\tau - v)\right| \\ &\leqslant \lambda^* + r \leqslant 1 \end{aligned}$$

if $r \leqslant 1 - \lambda^*$ (recall that we have assumed $\lambda^* < 1$). Also

$$\begin{aligned} \big(G(v), H(v)\big) &= -\big(G(v), v\big) + \sum_\tau \psi_\tau(v)\big(G(v), b(u_\tau)\big) \\ &= \lambda^* \sum_\tau \psi_\tau(v)\big(G(v), h(u_\tau)\big) + \sum_\tau \psi_\tau(v)\big(G(v), u_\tau - v\big) \\ &\leqslant -\lambda^*\alpha''\big|G(v)\big| + r\big|G(v)\big| \\ &= -\big(\lambda^*\alpha'' - r\big)\big|G(v)\big|. \end{aligned}$$

Hence

$$\big(G(v), H(v)\big) \leqslant -\alpha\big|G(v)\big|,$$

where $\alpha := \lambda^*\alpha'' - r > 0$ and $r < \lambda^*\alpha''$. Also, if $v \in D_1$, then

$$\big(v, H(v)\big) = \lambda^* \sum_\tau \psi_\tau(v)\big(v, h(u_\tau)\big) + \sum_\tau \psi_\tau(v)(v, u_\tau - v). \tag{2.19}$$

We have $(v, h(u_\tau)) < 0$ whenever $v \in V(u_\tau)$. Hence the first sum in (2.19) is < 0. In addition

$$(v, u_\tau - v) = (v, u_\tau) - |v|^2 \leqslant |v||u_\tau| - |v|^2 = 0$$

if $v \in V(u_\tau)$, since both $v, u_\tau \in D_1$. Thus the second sum in (2.19) is $\leqslant 0$. Therefore $(v, H(v)) < 0$ on D_1.

Next assume that $v \in D_0$. Then

$$\big(Jv, H(v)\big) = \lambda^* \sum_\tau \psi_\tau(v)\big(Jv, h(u_\tau)\big) + \sum_\tau \psi_\tau(v)(Jv, u_\tau - v). \tag{2.20}$$

Here

$$\lambda^* \sum_\tau \psi_\tau(v)\big(Jv, h(u_\tau)\big) \geqslant \lambda^* \frac{1}{2}(1-\theta-\alpha')R > 0, \tag{2.21}$$

while

$$\left|\sum_\tau \psi_\tau(v)(Jv, u_\tau - v)\right| \leqslant \sum_\tau \psi_\tau(v)\big|(Jv, u_\tau - v)\big| \leqslant r|Jv| \leqslant rc_0R_0 \tag{2.22}$$

as follows from (1.1). Now if r is chosen such that $\lambda^*\frac{1}{2}(1-\theta-\alpha')R - rc_0R_0 > 0$, then (2.20)–(2.21) guarantee that $(Jv, H(v)) > 0$ on D_0. Therefore $r > 0$ has to satisfy the following conditions:

$$r \leqslant 1-\lambda^*, \quad r < \lambda^*\alpha'' \quad \text{and} \quad \lambda^*\frac{1}{2}(1-\theta-\alpha')R - rc_0R_0 > 0.$$

Finally $v + H(v) = \sum_\tau \psi_\tau(v)b(u_\tau) \in K$ for all $v \in \widehat{D}$. $\square$

Lemma 2.3. *Assume all the assumptions of Theorem* 2.1 *hold. In addition assume that there are constants* $\delta > 0$ *and* $\theta \in [0,1)$ *such that for* $u \in K_{R_0R_1}$ *satisfying* $\|u\| = R_0$ *and* $|E(u) - c| \leqslant \delta$, *one has*

$$\big(JE'(u), Ju\big) \leqslant \theta|Ju|\big|JE'(u)\big|, \tag{2.23}$$

and for $u \in K_{R_0R_1}$ *satisfying* $|u| = R_1$ *and* $|E(u) - c| \leqslant \delta$, *one has*

$$-\big(JE'(u), u\big) \leqslant \theta|u|\big|JE'(u)\big|. \tag{2.24}$$

Then there exists a sequence of elements $u_k \in K_{R_0R_1}$ *with*

$$E(u_k) \to c \quad \textit{and} \quad E'(u_k) \to 0 \quad \textit{as } k \to \infty. \tag{2.25}$$

Proof. Assume there are no sequences satisfying (2.25). Then there would be constants $\delta, a > 0$ such that

$$\big|JE'(u)\big| \geqslant a$$

for all u in

$$Q = \big\{u \in K_{R_0R_1}\colon \big|E(u) - c\big| \leqslant 3\delta\big\}.$$

Clearly, we may assume that $3\delta < c - \max\{E(u_0), E(u_1)\}$ and that (2.23), (2.24) hold in $\widetilde{Q_0} = \{u \in Q\colon \|u\| = R_0\}$ and $\widetilde{Q_1} = \{u \in Q\colon |u| = R_1\}$, respectively. Denote

$$\begin{aligned}
&Q_0 = \big\{u \in K_{R_0R_1}\colon \big|E(u) - c\big| \leqslant 2\delta\big\},\\
&Q_1 = \big\{u \in K_{R_0R_1}\colon \big|E(u) - c\big| \leqslant \delta\big\},\\
&Q_2 = K_{R_0R_1} \setminus Q_0,\\
&\eta(u) = \frac{d(u, Q_2)}{d(u, Q_1) + d(u, Q_2)}.
\end{aligned}$$

We have

$$\eta(u) = 1 \quad \text{in } \overline{Q_1}, \qquad \eta(u) = 0 \quad \text{in } \overline{Q_2}, \qquad 0 < \eta(u) < 1 \quad \text{for } u \in K \setminus (\overline{Q_1} \cup \overline{Q_2}).$$

We now apply Lemma 2.2 to $G(u) = JE'(u)$, $D_0 = \widetilde{Q_0}$ and $D_1 = \widetilde{Q_1}$. It follows that there exists $\alpha > 0$ and a locally Lipschitz map $H : \widehat{D} \to X$ (here $\widehat{D}$ means the set $\{u \in K_{R_0R_1}\colon |JE'(u)| \geqslant a\}$) such that

$$\begin{aligned}
&\big|H(u)\big| \leqslant 1, \quad \big(JE'(u), H(u)\big) \leqslant -\alpha\big|JE'(u)\big| \quad \text{for } u \in \widehat{D},\\
&\big(Ju, H(u)\big) > 0 \quad \text{for } u \in \widetilde{Q_0},\\
&\big(u, H(u)\big) < 0 \quad \text{for } u \in \widetilde{Q_1},
\end{aligned} \tag{2.26}$$

and

$$u + H(u) \in K \quad \text{for all } u \in \widehat{D}. \tag{2.27}$$

Define $W : K_{R_0R_1} \to X$ by

$$W(u) = \begin{cases} \eta(u)H(u) & \text{for } u \in \widehat{D}, \\ 0 & \text{for } u \in K_{R_0R_1} \setminus \widehat{D}. \end{cases}$$

This map is locally Lipschitz and can be extended to a locally Lipschitz map on the whole K. Indeed, let

$$W_0(u) = \eta(u)u + W(u) \quad \text{for } u \in K_{R_0R_1}.$$

Then W_0 is locally Lipschitz on $K_{R_0R_1}$ and

$$W_0(u) = \begin{cases} \eta(u)\sum_\tau \psi_\tau(u)b(u_\tau) & \text{for } u \in \widehat{D}, \\ \eta(u)u & \text{for } u \in K_{R_0R_1} \setminus \widehat{D} \end{cases}$$

which shows that $W_0(u) \in K$ for all $u \in K_{R_0R_1}$. Let $\widetilde{W_0}$ be the locally Lipschitz extension of W_0 to the whole K, as in the proof of Dugundji's extension theorem (see [4, p. 44]). Then $\widetilde{W_0}(u) \in K$ for all $u \in K$. Now we define the extension of W to K, by

$$W(u) = -\eta(u)u + \widetilde{W_0}(u), \quad u \in K.$$

Let σ be the semiflow generated by W as shows Lemma 1.1. Note $\sigma(u, .)$ does not exit K since for each $v \in K$, there is $\lambda > 0$ with $v + \lambda W(v) \in K$. Indeed, this is true for every $\lambda > 0$ if $\eta(v) = 0$, and for $\lambda = \frac{1}{\eta(v)}$ in case that $\eta(v) > 0$. We claim that $\sigma(u, .)$ does not exit $K_{R_0R_1}$ for $t \in \mathbf{R}_+$ if $u \in K_{R_0R_1}$. To prove this assume first that $\sigma(u,t) \in K_{R_0R_1}$ for all $t \in [0, t_0)$ and $\|\sigma(u,t_0)\| = R_0$ for some $t_0 \in \mathbf{R}_+$. If $\sigma(u,t_0) \in \widetilde{Q_0}$, then (2.26) guarantees that

$$\frac{d}{dt}\|\sigma(u,t)\|^2 = 2\left\langle \frac{d}{dt}\sigma(u,t), \sigma(u,t)\right\rangle = \big(W\big(\sigma(u,t)\big), J\sigma(u,t)\big) > 0$$

for t in a neighborhood of t_0. If $\sigma(u,t_0) \notin \widetilde{Q_0}$, then $\eta(\sigma(u,t)) = 0$ in a neighborhood of t_0. Hence $d\|\sigma(u,t)\|^2/dt \geqslant 0$ in a neighborhood of t_0, which means that $\|\sigma(u,t)\|$ is nondecreasing on some interval $[t_0, t_0 + \varepsilon)$. Similarly, if $|\sigma(u,t_0)| = R_1$, then $d|\sigma(u,t)|^2/dt \leqslant 0$ in a neighborhood of t_0, which means that $|\sigma(u,t)|$ is nonincreasing on some interval $[t_0, t_0 + \varepsilon)$. Therefore $\sigma(u,t)$ does not exit $K_{R_0R_1}$ for $t \in \mathbf{R}_+$.

Let us denote by E_λ the level set $(E \leqslant \lambda)$, i.e.,

$$E_\lambda = \big\{u \in K_{R_0R_1} :\ E(u) \leqslant \lambda\big\}.$$

We have

$$\begin{aligned} \frac{dE(\sigma(u,t))}{dt} &= \left\langle E'\big(\sigma(u,t)\big), \frac{d}{dt}\sigma(u,t)\right\rangle \\ &= \left(JE'\big(\sigma(u,t)\big), \frac{d}{dt}\sigma(u,t)\right) \\ &= \eta\big(\sigma(u,t)\big)\big(JE'\big(\sigma(u,t)\big), H\big(\sigma(u,t)\big)\big) \\ &\leqslant -\eta\big(\sigma(u,t)\big)\alpha a. \end{aligned} \tag{2.28}$$

Let $t_1 > 2\delta/(\alpha a)$ and let u be any element of $E_{c+\delta}$. If there is $t_0 \in [0, t_1]$ with $\sigma(u,t_0) \notin Q_1$, then

$$E\big(\sigma(u,t_1)\big) \leqslant E\big(\sigma(u,t_0)\big) < c - \delta.$$

Hence $\sigma(u,t_1) \in E_{c-\delta}$. Otherwise, $\sigma(u,t) \in Q_1$ for all $t \in [0,t_1]$, and so $\eta(\sigma(u,t)) \equiv 1$. Then (2.28) implies

$$E\big(\sigma(u,t_1)\big) \leqslant E(u) - \alpha a t_1 < c + \delta - 2\delta = c - \delta.$$

Thus

$$\sigma(E_{c+\delta}, t_1) \subset E_{c-\delta}. \tag{2.29}$$

Now by the definition of c, there is $\gamma \in \Gamma$ with

$$\gamma(t) \in E_{c+\delta} \quad \text{for all } t \in [0,1]. \tag{2.30}$$

We define a new path γ_1 joining u_0 and u_1 by $\gamma_1(t) = \sigma(\gamma(t), t_1)$, $t \in [0,1]$. Since η vanishes in the neighborhood of u_0 and u_1, we have $\sigma(u_0, t) \equiv u_0$ and $\sigma(u_1, t) \equiv u_1$. Hence $\gamma_1(0) = u_0$ and $\gamma_1(1) = u_1$ and so $\gamma_1 \in \Gamma$. On the other hand, from (2.29) and (2.30), we have

$$E(\gamma_1(t)) \leqslant c - \delta \quad \text{for all } t \in [0,1],$$

which contradicts the definition of c. □

Proof of Theorem 2.1. Assume that do not exist sequences satisfying (2.5) and (2.7) or (2.8). Then there are $a, \delta > 0$ such that

$$\left| JE'(u) - \frac{(JE'(u), Ju)}{|Ju|^2} Ju \right| \geqslant a \tag{2.31}$$

for all $u \in K_{R_0 R_1}$ satisfying $|E(u) - c| \leqslant \delta$, with $\|u\| = R_0$ and $(JE'(u), Ju) \geqslant 0$, and

$$\left| JE'(u) - \frac{(JE'(u), u)}{|u|^2} u \right| \geqslant a$$

for all $u \in K_{R_0 R_1}$ satisfying $|E(u) - c| \leqslant \delta$, with $|u| = R_1$ and $(JE'(u), u) \leqslant 0$.

Let $\theta > 0$ be such that

$$0 < \theta^{-2} - 1 \leqslant a^2 R^2 \nu_0^{-2},$$

where R comes from (1.2). Then from (2.31), also using (2.2) and (2.3), we obtain

$$\begin{aligned}
\big(JE'(u), Ju\big)^2 \big(\theta^{-2} - 1\big) &\leqslant \big(JE'(u), Ju\big)^2 a^2 R^2 \nu_0^{-2} \\
&\leqslant \big(JE'(u), Ju\big)^2 \left| JE'(u) - \frac{(JE'(u), Ju)}{|Ju|^2} Ju \right|^2 |Ju|^2 \nu_0^{-2} \\
&\leqslant \left| JE'(u) - \frac{(JE'(u), Ju)}{|Ju|^2} Ju \right|^2 |Ju|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\big(JE'(u), Ju\big)^2 \theta^{-2} &\leqslant \big(JE'(u), Ju\big)^2 + \big|JE'(u)\big|^2 |Ju|^2 - \big(JE'(u), Ju\big)^2 \\
&= \big|JE'(u)\big|^2 |Ju|^2.
\end{aligned}$$

Hence

$$\big(JE'(u), Ju\big) \leqslant \theta |Ju| \big|JE'(u)\big|$$

which is also true if $(JE'(u), Ju) < 0$. Thus, for $u \in K_{R_0 R_1}$ satisfying $\|u\| = R_0$ and $|E(u) - c| \leqslant \delta$, one has

$$\big(JE'(u), Ju\big) \leqslant \theta |Ju| \big|JE'(u)\big|.$$

Similarly, for $u \in K_{R_0 R_1}$ satisfying $|u| = R_1$ and $|E(u) - c| \leqslant \delta$, one has

$$-\big(JE'(u), u\big) \leqslant \theta |u| \big|JE'(u)\big|.$$

Now the conclusion of the first part follows from Lemma 2.3.

Finally we remark that (2.9), (2.10) make impossible the existence of a sequence as in (2.7) and (2.8), respectively. □

The next critical point result (together with the remark which follows) can be compared to the fixed point Theorem 20.2 in [4].

Theorem 2.2. *Assume that there exist $u_0, u_1 \in K_{R_0R_1}$ and $\nu_0, r > 0$, $|u_0| < r < |u_1|$, such that conditions* (2.1), (2.4), (2.9) *and* (2.10) *hold. In addition assume that $N := I - JE'$ and J are compact from X to X. Then there exists a point $u \in K_{R_0R_1}$ with $E'(u) = 0$ and $E(u) = c$.*

Proof. First note that (2.2) and (2.3) trivially hold since the maps N, J are bounded on $K_{R_0R_1}$. It remains to prove that any sequence (u_k) like in Theorem 2.1 has a convergent subsequence. Let $(u_k) \subset K_{R_0R_1}$ be such a sequence. Since both N, J are compact, passing if necessary to a subsequence, we may assume that $N(u_k) \to v$ and $Ju_k \to w$ for some $v, w \in X$. If (2.6) is satisfied, then from $JE'(u_k) = u_k - N(u_k) \to 0$ we deduce that $u_k \to v$ as we wished. Assume (2.7). Passing to another sequence we may suppose that $\frac{(JE'(u_k), Ju_k)}{|Ju_k|^2} \to a$ for some real number $a \geqslant 0$. Then (2.7) implies

$$u_k - N(u_k) - aJu_k \to 0 \tag{2.32}$$

and in consequence $u_k \to v + aw$. Next assume that (2.8) holds. As above we may assume that $\frac{(E'(u_k), u_k)}{R_1^2} \to a$ this time for some real number $a \leqslant 0$. Now from (2.8) we have

$$(1 - a)u_k - N(u_k) \to 0 \tag{2.33}$$

and so $u_k \to \frac{1}{1-a} v$. $\square$

Remark 2.2. In case that $X = H$, when $|.| = \|.\|$ and $J = I$, the conclusion of Theorem 2.2 is also true even though I is not compact, if we add the condition

$$\inf\left\{\left|N(u)\right|:\ u \in K,\ |u| = R_0\right\} > 0. \tag{2.34}$$

Indeed, in this case, (2.7) also implies (2.33) with $a \geqslant 0$. Notice $a \neq 1$, since otherwise (2.33) would imply that $N(u_k) \to 0$, where $|u_k| = R_0$, which contradicts (2.34). Then $u_k \to \frac{1}{1-a} v$.

Remark 2.3. If the imbedding $X \subset H$ is compact, then J is compact from X to X.

The following result is the compression type mountain pass theorem accompanying the corresponding fixed point theorem of Krasnoselskii [11] (see also [7, p. 325]).

Theorem 2.3. *Assume that there exist $u_0, u_1 \in K_{R_0R_1}$ and $\nu_0, r > 0$, $|u_0| < r < |u_1|$, such that conditions* (2.1) *and* (2.4) *hold. In addition assume that norm $\|.\|$ is increasing with respect to K, i.e.,*

$$\|u + v\| > \|u\| \quad \text{for all } u, v \in K,\ v \neq 0,$$

the maps J and $N := I - JE'$ are compact from X to X, and

(a) $\|N(u)\| \geqslant \|u\|$ *for* $\|u\| = R_0$,
(b) $|N(u)| \leqslant |u|$ *for* $|u| = R_1$.

Then there exists a point $u \in K_{R_0R_1}$ with $E'(u) = 0$ and $E(u) = c$.

Proof. First observe that (a) guarantees (2.11). Indeed, if $N(u) + \lambda Ju = u$ for some $u \in K_{R_0R_1}$, $\|u\| = R_0$ and $\lambda > 0$, then since $Ju \in K \setminus \{0\}$ and $\|.\|$ is increasing with respect to K, we deduce

$$\|u\| = \left\|N(u) + \lambda Ju\right\| > \left\|N(u)\right\|,$$

which contradicts (a).

Next observe that (b) guarantees (2.12). Indeed, if $N(u) = (1 + \lambda)u$ for some $u \in K_{R_0R_1}$, $|u| = R_1$ and $\lambda > 0$, then

$$\left|N(u)\right| = (1 + \lambda)|u| > |u|,$$

in contradiction with (b).

Thus the result follows from Theorem 2.2. $\square$

Remark 2.4. The result in Theorem 2.3 remains true if $X = H$, $J = I$, $|.| = \|.\|$ and K is a cone, without assuming that $|.|$ is increasing with respect to K. Indeed, in this case, if $N(u) + \lambda u = u$ for some $u \in K_{R_0R_1}$, $|u| = R_0$ and $\lambda > 0$, then since $N(u) \in K$, we have $\lambda \leqslant 1$. The case $\lambda = 1$ being excluded by (a), we obtain that

$$\left|N(u)\right| = (1-\lambda)|u| < |u|,$$

which contradicts (a). Also (a) guarantees (2.34). Now we use Remark 2.2.

Now instead of critical points of mountain pass type we seek critical points of minimum type.

Theorem 2.4. *Assume that conditions* (2.1)*,* (2.2)*,* (2.3) *are satisfied and that*

$$m := \inf_{K_{R_0R_1}} E > -\infty. \tag{2.35}$$

Then there exists a sequence (u_k) with $u_k \in K_{R_0R_1}$ such that

$$E(u_k) \to m \quad \text{as } k \to \infty \tag{2.36}$$

and one of conditions (2.6)–(2.8) *holds. If in addition, any sequence (u_k) as above has a convergent subsequence and* (2.9)*,* (2.10) *hold, then there exists $u \in K_{R_0R_1}$ with*

$$E'(u) = 0 \quad \text{and} \quad E(u) = m.$$

For the proof we need the following lemma.

Lemma 2.4. *Assume all the assumptions of Theorem* 2.4 *hold. In addition assume that there are constants $\delta > 0$ and $\theta \in [0, 1)$ such that for $u \in K_{R_0R_1}$ satisfying $E(u) - m \leqslant \delta$, one has*

$$\begin{aligned} &\left(JE'(u), Ju\right) \leqslant \theta|Ju|\left|JE'(u)\right| \quad \text{if } \|u\| = R_0 \quad \text{and} \\ &-\left(JE'(u), u\right) \leqslant \theta|u|\left|JE'(u)\right| \quad \text{if } |u| = R_1. \end{aligned} \tag{2.37}$$

Then there exists a sequence of elements $u_k \in K_{R_0R_1}$ with

$$E(u_k) \to m \quad \text{and} \quad E'(u_k) \to 0 \quad \text{as } k \to \infty.$$

Proof. We follow the proof of Lemma 2.3 with the only difference that one has m instead of c. Thus we obtain $\sigma(u,t)$ which does not exit $K_{R_0R_1}$ for $t \geqslant 0$. We fix any $u \in Q_1 = \{v \in K_{R_0R_1}\colon\ E(v) < m + \delta\}$ and take $t_1 > 2\delta/(\alpha a)$. Then (2.28) guarantees that $\sigma(u,t) \in Q_1$ for all $t \geqslant 0$. Then

$$E\left(\sigma(u, t_1)\right) \leqslant m + \delta - \alpha a t_1 < m - \delta,$$

contradicting the definition of m. □

Proof of Theorem 2.4. Assume there are no sequences satisfying (2.36) and (2.7) or (2.8). Then, as in the proof of Theorem 2.1, there are $\delta, \theta > 0$ such that for $u \in K_{R_0R_1}$ satisfying $E(u) \leqslant m + \delta$, one has (2.37). Now the conclusion of the first part follows from Lemma 2.4. □

Obviously, the comments on the compactness of the sequences in Theorem 2.1 making the object of Theorems 2.2 and 2.3, remain true for Theorem 2.4.

Remark 2.5. If both conditions (2.4) and (2.35) are satisfied, then Theorems 2.1 and 2.4 guarantee the existence of two distinct critical points of E in $K_{R_0R_1}$.

3. Application

Consider the two-point boundary value problem

$$\begin{cases} u''(t) + f\big(u(t)\big) = 0, & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.1}$$

Here f is a continuous function from $\mathbf{R}$ into $\mathbf{R}$, with $f(\mathbf{R}_+) \subset \mathbf{R}_+$. Let $X = H_0^1(0,1)$ with inner product and norm

$$(u,v) = \int_0^1 u'v'\,dt, \quad |u| = \left(\int_0^1 u'^2\,dt\right)^{1/2}$$

and let $H = L^2(0,1)$ with inner product and norm

$$\langle u,v\rangle = \int_0^1 uv\,dt, \quad \|u\| = \left(\int_0^1 u^2\,dt\right)^{1/2}.$$

We also denote by $|u|_\infty$ the max norm in $C[0,1]$ and by $|u|_{L^2(a,b)}$ the usual norm of $L^2(a,b)$.

Here $E : H_0^1(0,1) \to \mathbf{R}$ is given by

$$E(u) = \int_0^1 \left(\frac{1}{2}u'(t)^2 - F\big(u(t)\big)\right) dt, \quad u \in H_0^1(0,1),$$

where $F(u) = \int_0^u f(\tau)\,d\tau$. One has that $E'(u) = -u'' - f(u)$ in $H^{-1}(0,1)$,

$$(Jv, w) = \langle v, w\rangle \quad \text{for all } v \in H^{-1}(0,1),\ w \in H_0^1(0,1),$$

and $Jv = \int_0^1 G(t,s)v(s)\,ds$ for $v \in L^2(0,1)$, where $G(t,s)$ is the corresponding Green's function given by

$$G(t,s) = \begin{cases} s(1-t) & \text{for } 0 \leqslant s \leqslant t \leqslant 1, \\ t(1-s) & \text{for } 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Also $N(u) := u - JE'(u) = Jf(u)$ and

$$Jf(u) = \int_0^1 G(t,s) f\big(u(s)\big)\,ds.$$

Notice, since the imbedding of $H_0^1(0,1)$ into $C[0,1]$ is compact, N and J are compact from $H_0^1(0,1)$ to itself. Also note that

$$G(t,s) \leqslant G(s,s) \quad \text{for all } t,s \in [0,1], \tag{3.2}$$

and for every interval $[a,b]$ with $0 < a < b < 1$, there is a constant $M > 0$ with

$$MG(s,s) \leqslant G(t,s) \quad \text{for all } s \in [0,1],\ t \in [a,b]. \tag{3.3}$$

These properties of Green's function guarantee that for every nonnegative function $v \in L^2(0,1)$, one has

$$(Jv)(t) \geqslant M\|Jv\| \quad \text{for all } t \in [a,b]. \tag{3.4}$$

Indeed, if $v \geqslant 0$ on $[0,1]$, $t \in [a,b]$ and $t^* \in [0,1]$, then from (3.2), (3.3), we obtain

$$(Jv)(t) = \int_0^1 G(t,s)v(s)\,ds \geqslant M\int_0^1 G(s,s)v(s)\,ds \geqslant M\int_0^1 G\big(t^*,s\big)v(s)\,ds = M(Jv)\big(t^*\big).$$

This proves (3.4) if we choose t^* with $(Jv)(t^*) = |Jv|_\infty$ and we take into account that $|u|_\infty \geqslant \|u\|$ for all $u \in C[0,1]$.

Now we consider a cone K in $H_0^1(0,1)$, defined by

$$K = \{u \in H_0^1(0,1)\colon\ u \geqslant 0 \text{ on } [0,1] \text{ and } u(t) \geqslant M\|u\| \text{ for } t \in [a,b]\}.$$

If $u \geqslant 0$ on $[0,1]$, then $f(u) \geqslant 0$ on $[0,1]$ since $f(\mathbf{R}_+) \subset \mathbf{R}_+$ and so, according to (3.4), $Jf(u) \in K$. Consequently, $u - JE'(u) \in K$ for every $u \in K$.

Before we state our hypotheses, we recall that constant c_0 is such that $\|u\| \leqslant c_0|u|$ for all $u \in H_0^1(0,1)$ and we denote by c_∞ the imbedding constant of the inclusion $H_0^1(0,1) \subset C[0,1]$, i.e., $|u|_\infty \leqslant c_\infty|u|$ for all $u \in H_0^1(0,1)$. Also, for the subinterval $[a,b]$ of $[0,1]$, we let $\chi_{[a,b]}$ be the characteristic function of $[a,b]$, i.e., $\chi_{[a,b]}(t) = 1$ if $t \in [a,b]$, $\chi_{[a,b]}(t) = 0$ otherwise.

Our assumptions are as follows:

(H1) There exist R_0, R_1 with $0 < R_0 < c_0 R_1$ such that

$$\frac{\min_{\tau \in [MR_0, c_\infty R_1]} f(\tau)}{R_0} \geqslant \frac{1}{|J\chi_{[a,b]}|_{L^2(a,b)}}, \tag{3.5}$$

$$\frac{\max_{\tau \in [0, c_\infty R_1]} f(\tau)}{R_1} \leqslant \frac{1}{c_0}. \tag{3.6}$$

(H2) There are $u_0, u_1 \in K_{R_0 R_1}$ and r such that $|u_0| < r < |u_1|$ and

$$\max\{E(u_0), E(u_1)\} < \inf_{\substack{u \in K_{R_0 R_1} \\ |u| = r}} E(u).$$

Remark 3.1.

(1^0) If f is nondecreasing on $[0, c_\infty R_1]$, then (3.5) and (3.6) become

$$\frac{f(MR_0)}{MR_0} \geqslant \frac{1}{M|J\chi_{[a,b]}|_{L^2(a,b)}} \tag{3.7}$$

and, respectively,

$$\frac{f(c_\infty R_1)}{c_\infty R_1} \leqslant \frac{1}{c_0 c_\infty}. \tag{3.8}$$

Therefore, in this case, in order to guarantee (3.5) and (3.6), we only need to know how the nonlinearity f behaves at two points MR_0 and $c_\infty R_1$.

(2^0) We can even precise constants c_0, c_∞, M and $|J\chi_{[a,b]}|_{L^2(a,b)}$. For example, from Wirtinger's inequality, the best constant c_0 is $\frac{1}{\pi}$. Also we may take $c_\infty = 1$ and $M = \min\{a, 1-b\}$.

Theorem 3.1. *Assume that* (H1) *and* (H2) *hold. Then* (3.1) *has at least two positive solutions in* $K_{R_0 R_1}$.

Proof. We shall apply Theorems 2.3 and 2.4. First we show that (3.5) guarantees condition (a) in Theorem 2.3. Let $u \in K_{R_0 R_1}$ and $\|u\| = R_0$. Then for every $s \in [a,b]$ one has

$$MR_0 = M\|u\| \leqslant u(s) \leqslant |u|_\infty \leqslant c_\infty|u| \leqslant c_\infty R_1.$$

Furthermore, for every $t \in [a,b]$, we have

$$N(u)(t) = \int_0^1 G(t,s) f\big(u(s)\big)\,ds \geqslant \int_a^b G(t,s) f\big(u(s)\big)\,ds$$

$$\geqslant \min_{\tau \in [MR_0, c_\infty R_1]} f(\tau) \int_a^b G(t,s)\,ds = \min_{\tau \in [MR_0, c_\infty R_1]} f(\tau) J\chi_{a,b}(t).$$

Consequently, also using (3.5), we deduce that

$$\|N(u)\| \geqslant |N(u)|_{L^2(a,b)} \geqslant \min_{\tau\in[MR_0,c_\infty R_1]} f(\tau)|J\chi_{a,b}|_{L^2(a,b)} \geqslant R_0.$$

Hence

$$\|N(u)\| \geqslant \|u\|.$$

Next we show that (3.6) guarantees condition (b) in Theorem 2.3. Assume $u \in K_{R_0R_1}$ and $|u| = R_1$. Then $u(t) \leqslant |u|_\infty \leqslant c_\infty R_1$ and

$$\begin{aligned} |N(u)|^2 &= |Jf(u)|^2 = (Jf(u), Jf(u)) = \langle f(u), N(u)\rangle \leqslant \|f(u)\|\,\|N(u)\| \\ &\leqslant c_0|f(u)|_\infty|N(u)| \leqslant c_0 \max_{\tau\in[0,c_\infty R_1]} f(\tau)|N(u)| \leqslant R_1|N(u)|. \end{aligned}$$

Hence

$$|N(u)| \leqslant |u|.$$

Therefore Theorem 2.3 applies.

Finally we note that condition (2.35) in Theorem 2.4 is satisfied. Indeed, for $u \in K_{R_0R_1}$ one has that $\frac{1}{c_0}R_0 \leqslant |u| \leqslant R_1$ and $|u|_\infty \leqslant c_\infty R_1$. Consequently

$$E(u) = \int_0^1 \left(\frac{1}{2}u'^2 - F(u)\right) dt \geqslant \frac{1}{2c_0^2}R_0^2 - A,$$

where $A \geqslant F(\tau)$ for $0 \leqslant \tau \leqslant c_\infty R_1$. Hence $\inf_{K_{R_0R_1}} E(u) > -\infty$. $\square$

Example. Let

$$f(u) = \begin{cases} \frac{1}{2}\sqrt{u} & \text{if } 0 \leqslant u \leqslant 1, \\ \frac{1}{2}u^2 & \text{if } 1 < u \leqslant b, \\ \frac{1}{2}(\sqrt{u-b} + b^2) & \text{if } u > b. \end{cases} \tag{3.9}$$

Here $b > 2$ and will be specified later. Obviously f is increasing on $\mathbf{R}_+$ and

$$F(u) = \begin{cases} \frac{1}{3}u^{3/2} & \text{if } 0 \leqslant u \leqslant 1, \\ \frac{1}{6}(u^3 + 1) & \text{if } 1 < u \leqslant b. \end{cases}$$

First note that if we choose $r = 2$, then $\inf_{u\in K,\,|u|=r} E(u) \geqslant \frac{1}{2}$. Indeed, if $u \in K$ and $|u| = 2$, then since $|u|_\infty \leqslant |u|$, we have that $0 \leqslant u(t) \leqslant 2$ and so $F(u(t)) \leqslant \frac{3}{2}$ for all $t \in [0, 1]$. Hence

$$E(u) = \frac{1}{2}|u|^2 - \int_0^1 F(u(t))\, dt \geqslant 2 - \frac{3}{2} = \frac{1}{2}.$$

Let $u_0 = \phi$, where ϕ is the positive eigenfunction corresponding to the first eigenvalue λ_1, i.e.,

$$\begin{aligned} &\phi'' + \lambda_1\phi = 0, \quad t \in (0, 1), \\ &\phi(0) = \phi(1) = 0, \end{aligned}$$

$\phi \geqslant 0$ and $|\phi| = 1$. Then $|u_0| = 1 < r$ and

$$E(u_0) = \frac{1}{2}|\phi|^2 - \int_0^1 F(\phi(t))\, dt = \frac{1}{2} - \int_0^1 F(\phi(t))\, dt < \frac{1}{2}.$$

Next we take $u_1 := b\phi$ and we assume that $b > \frac{1}{|\phi|_\infty}$. We have $|u_1| = b > r$ and

$$E(u_1) < \frac{1}{2}b^2 - \frac{1}{6} \int\limits_{(b\phi(t)>1)} \big(b\phi(t)\big)^3\, dt. \tag{3.10}$$

Since the limit of the right side of (3.10) as $b \to \infty$ is equal to $-\infty$, we may choose b large enough that $E(u_1) < \frac{1}{2}$. Hence condition (H2) is satisfied. Finally, since

$$\lim_{\tau\to 0} \frac{f(\tau)}{\tau} = \frac{1}{2} \lim_{\tau\to 0} \frac{\sqrt{\tau}}{\tau} = \infty \quad \text{and} \quad \lim_{\tau\to\infty} \frac{f(\tau)}{\tau} = \frac{1}{2} \lim_{\tau\to\infty} \frac{\sqrt{\tau - b} + b^2}{\tau} = 0,$$

we may find R_0, R_1 such that $u_0, u_1 \in K_{R_0 R_1}$ and (3.7), (3.8) hold.

Therefore, according to Theorem 3.1, problem (3.1) with f given by (3.9) and b sufficiently large has two positive solutions.

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