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EXISTENCE OF SOLUTIONS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS AND SYSTEMS ON INFINITE INTERVALS

TOUFIK MOUSSAOUI, RADU PRECUP

Abstract. We study the existence of nontrivial solutions to the boundary-value problem

$$-u'' + cu' + \lambda u = f(x,u), \quad -\infty < x < +\infty,$$
$$u(-\infty) = u(+\infty) = 0$$

and to the system

$$-u'' + c_1u' + \lambda_1 u = f(x,u,v), \quad -\infty < x < +\infty,$$
$$-v'' + c_2v' + \lambda_2 v = g(x,u,v), \quad -\infty < x < +\infty,$$
$$u(-\infty) = u(+\infty) = 0, \quad v(-\infty) = v(+\infty) = 0,$$

where $c, c_1, c_2, \lambda, \lambda_1, \lambda_2$ are real positive constants and the nonlinearities f and g satisfy suitable conditions. The proofs are based on fixed point theorems.

1. Introduction

In this article, we consider the existence of nontrivial solutions to the boundary-value problem

$$\begin{gathered} -u'' + cu' + \lambda u = f(x,u), \quad -\infty < x < +\infty, \\ u(-\infty) = u(+\infty) = 0 \end{gathered} \tag{1.1}$$

and to the system

$$\begin{gathered} -u'' + c_1u' + \lambda_1 u = f(x,u,v), \quad -\infty < x < +\infty. \\ -v'' + c_2v' + \lambda_2 v = g(x,u,v), \quad -\infty < x < +\infty. \\ u(-\infty) = u(+\infty) = 0, \quad v(-\infty) = v(+\infty) = 0. \end{gathered} \tag{1.2}$$

Here $c, c_1, c_2, \lambda, \lambda_1, \lambda_2$ are positive numbers.

By a solution of Problem (1.1) we mean a function $u \in C^1(\mathbb{R},\mathbb{R})$ satisfying (1.1) and by a solution of Problem (1.2) we mean a vector-valued function $(u,v) \in C^1(\mathbb{R},\mathbb{R}^2) := C^1(\mathbb{R},\mathbb{R}) \times C^1(\mathbb{R},\mathbb{R})$ satisfying (1.2).

Some of the ideas used in this paper are motivated by Djebali and Moussaoui [1, 2] and Djebali and Mebarki [3].

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In [1], the authors study the boundary value problem (1.1). According to the behavior of the nonlinear source term, existence results of bounded solutions, positive solutions, classical as well as weak solutions are provided. They mainly use fixed point arguments.

Under some relations upon the real parameters and coefficients, the authors in [2] present some existence and nonexistence results for Problem (1.1). They use a variational method and fixed point arguments.

In [3], the authors study existence of positive nontrivial solutions for a boundary value problem on the positive half-line arising from epidemiology. They mainly use the fixed point theorem of cone expansion and compression of functional type.

In this paper, we study existence of nontrivial solutions for both Problems (1.1) and (1.2). Our arguments are based on fixed point theory. So, let us recall for the sake of completeness, Krasnosel'sk'ii-Zabreiko's and Schauder's fixed point theorems:

Theorem 1.1 ([4]). *Let $(E, \|.\|)$ be a Banach space, and $T : E \to E$ be completely continuous. Assume that $A : E \to E$ is a bounded linear operator such that 1 is not an eigenvalue of A and*

$$\lim_{\|u\| \to \infty} \frac{\|Tu - Au\|}{\|u\|} = 0.$$

Then T has a fixed point in E.

Theorem 1.2 ([5]). *Let E be a Banach space and $K \subset E$ a nonempty, bounded, closed and convex subset of E. Let $T : K \to K$ be a completely continuous operator. Then T has a fixed point in K.*

In what follows, $C_0(\mathbb{R}, \mathbb{R})$ stands for the Banach space of continuous functions defined on the real line and vanishing at infinity, endowed with the sup-norm

$$\|u\|_0 = \sup_{x \in \mathbb{R}} |u(x)|.$$

Recall that $L^1(\mathbb{R}, \mathbb{R})$ is the Banach space of integrable functions on $\mathbb{R}$ endowed with norm

$$|u|_1 = \int_{-\infty}^{+\infty} |u(s)|\, ds.$$

In the sequel, we put

$$k = \sqrt{c^2 + 4\lambda}, \quad k_1 = \sqrt{c_1^2 + 4\lambda_1}, \quad k_2 = \sqrt{c_2^2 + 4\lambda_2}.$$

This paper is organized as follows. In Section 2, we state the main result concerning the existence of solutions for the boundary value problem (1.1) (Theorem 2.1) by using the fixed point theorem of Krasnosel'sk'ii-Zabreiko. In Section 3, we generalize the result obtained in Section 2 to systems of differential equations by using the same technique (Theorem 3.1). In Section 4, we present two other results concerning Problem (1.2) by using Schauder's fixed point theorem (Theorem 4.1 and Theorem 4.2). Finally, in the last section, we give some examples to illustrate our results.

2. Existence result for a generalized Fisher-like equation

In this section, we study the boundary-value problem

$$\begin{gathered} -u'' + cu' + \lambda u = p(x)f(u), \quad -\infty < x < +\infty, \\ u(-\infty) = u(+\infty) = 0 \end{gathered} \tag{2.1}$$

where $p \in L^1(\mathbb{R}, \mathbb{R}_+)$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) \neq 0$. Our main result in this section is the following theorem.

Theorem 2.1. *Assume that*

$$\lim_{u\to\infty} \frac{f(u)}{u} = f_\infty \quad \text{with } |f_\infty| < \frac{k}{|p|_1}. \tag{2.2}$$

Then Problem (2.1) *has at least one nontrivial solution* $u \in C_0(\mathbb{R}, \mathbb{R})$.

Proof. It is clear that Problem (2.1) is equivalent to the integral equation

$$u(x) = \int_{-\infty}^{+\infty} G(x,s)p(s)f(u(s))\, ds$$

with the Green function

$$G(x,s) = \frac{1}{\rho_1 - \rho_2} \begin{cases} e^{\rho_1(x-s)} & \text{if } x \le s \\ e^{\rho_2(x-s)} & \text{if } x \ge s \end{cases}$$

and characteristic roots

$$\rho_1 = \frac{c + \sqrt{c^2 + 4\lambda}}{2}, \quad \rho_2 = \frac{c - \sqrt{c^2 + 4\lambda}}{2}.$$

Notice that

$$0 < G(x,s) \le \frac{1}{k} \quad \text{for all } x, s \in \mathbb{R}.$$

Define the mapping $T : C_0(\mathbb{R}, \mathbb{R}) \to C_0(\mathbb{R}, \mathbb{R})$ by

$$Tu(x) = \int_{-\infty}^{+\infty} G(x,s)p(s)f(u(s))\, ds.$$

In view of Krasnosel'sk'ii-Zabreiko's fixed point theorem, we look for fixed points of the operator T in the Banach space $C_0(\mathbb{R}, \mathbb{R})$. The proof is split into four steps.

Claim 1: The mapping T is well defined; indeed, for any $u \in C_0(\mathbb{R}, \mathbb{R})$, by Assumption (2.2), we obtain the following estimates:

$$|Tu(x)| \le \int_{-\infty}^{+\infty} G(x,s)|p(s)f(u(s))|\, ds \le \frac{1}{k}|p|_1 \max_{|y|\le\|u\|_0} |f(y)|.$$

The convergence of the integral defining $(Tu)(x)$ is then established. In addition for any $s \in \mathbb{R}$, $G(\pm\infty, s) = 0$, and then, taking the limit in $(Tu)(x)$, we obtain $Tu(\pm\infty) = 0$. Therefore, the mapping $T : C_0(\mathbb{R}, \mathbb{R}) \to C_0(\mathbb{R}, \mathbb{R})$ is well defined.

Claim 2: The operator T is continuous. Let $(u_n)_n \subset C_0(\mathbb{R}, \mathbb{R})$ be a sequence which converges uniformly to u_0 on each compact subinterval of $\mathbb{R}$. For some fixed $a > 0$, we will prove the uniform convergence of $(Tu_n)_n$ to Tu_0 on the interval $[-a, a]$. Let $\varepsilon > 0$ and choose some $b > a$ large enough. By the uniform convergence of the sequence $(u_n)_n$ on $[-b, b]$, there exists an integer $N = N(\varepsilon, b)$ satisfying

$$I_1 := \sup_{x\in\mathbb{R}} \int_{-b}^{+b} G(x,s)|p(s)||f(u_n(s)) - f(u_0(s))|\, ds < \frac{\varepsilon}{2} \quad \text{for all } n \ge N.$$

For $x \in [-a, a]$, we have that $|(Tu_n)(x) - (Tu_0)(x)| \leq I_1 + I_2 + I_3$, where

$$I_2 := \sup_{x \in \mathbb{R}} \int_{\mathbb{R}-[-b,+b]} G(x,s)|p(s)f(u_0(s))|\, ds \leq \frac{\varepsilon}{4}$$

(by the Cauchy Convergence Criterion and $\lim_{|s|\to+\infty} p(s)f(u_0(s)) = 0$),

$$I_3 := \sup_{x \in \mathbb{R}} \int_{\mathbb{R}-[-b,+b]} G(x,s)|p(s)f(u_n(s))|\, ds \leq \frac{\varepsilon}{4}$$

(by the Lebesgue Dominated Convergence Theorem). This proves the uniform convergence of the sequence $(Tu_n)_n$ to Tu_0 on $[-a, a]$.

Claim 3: For any $M > 0$, the set $\{Tu, \|u\|_0 \leq M\}$ is relatively compact in $C_0(\mathbb{R}, \mathbb{R})$. By the Ascoli-Arzela Theorem, it is sufficient to prove that all the functions of this set are equicontinuous on every subinterval $[-a, a]$ and that there exists a function $\gamma \in C_0(\mathbb{R}, \mathbb{R})$ such that for any $x \in \mathbb{R}$, $|Tu(x)| \leq \gamma(x)$. Let $x_1, x_2 \in [-a, a]$. We have successively the estimates

$$\begin{aligned} |Tu(x_2) - Tu(x_1)| &\leq \int_{-\infty}^{+\infty} |G_1(x_2,s) - G_1(x_1,s)||p(s)f(u(s))|\, ds \\ &\leq \max_{y\in[-M,M]} |f(y)| \int_{-\infty}^{+\infty} |G_1(x_2,s) - G_1(x_1,s)||p(s)|\, ds. \end{aligned}$$

By the continuity of the Green function G, the last term tends to 0, as x_2 tends to x_1, whence comes the equicontinuity of the functions from $\{Tu; \|u\|_0 \leq M\}$. Analogously we have

$$\begin{aligned} |Tu(x)| &\leq \int_{-\infty}^{+\infty} G(x,s)|p(s)f(u(s))|\, ds \\ &\leq \max_{y\in[-M,M]} |f(y)| \int_{-\infty}^{+\infty} G(x,s)|p(s)|\, ds := \gamma(x). \end{aligned}$$

Clearly, $\gamma \in C_0(\mathbb{R}, \mathbb{R})$.

Now, we consider the boundary-value problem

$$\begin{gathered} -u'' + cu' + \lambda u = f_\infty p(x)u(s), \quad -\infty < x < +\infty \\ u(-\infty) = u(+\infty) = 0, \end{gathered} \tag{2.3}$$

and define operator A by

$$Au(x) = f_\infty \int_{-\infty}^{+\infty} G(x,s)p(s)u(s)\, ds.$$

Obviously, A is a bounded linear operator. Furthermore, any fixed point of A is a solution of Problem (2.3), and conversely.

We now claim that 1 is not an eigenvalue of A. In fact, if $f_\infty = 0$, then Problem (2.3) has no nontrivial solutions. Let $f_\infty \neq 0$ and assume that Problem (2.3) has a nontrivial solution u. Then

$$\begin{aligned} |Au(x)| &\leq |f_\infty| \int_{-\infty}^{+\infty} G(x,s)|p(s)u(s)|\, ds \\ &\leq \frac{1}{k}|f_\infty||p|_1\|u\|_0 \\ &< \|u\|_0. \end{aligned}$$

Hence $\|Au\|_0 < \|u\|_0$. This contradiction means that Problem (2.3) has no nontrivial solution. Thus 1 is not an eigenvalue of A.

Finally, we prove that

$$\lim_{\|u\|_0\to\infty} \frac{\|Tu - Au\|_0}{\|u\|_0} = 0.$$

According to $\lim_{u\to\infty} f(u)/u = f_\infty$, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$|f(u) - f_\infty u| < \varepsilon|u|, \quad |u| > R.$$

Set $R^* = \max_{|u|\le R} |f(u)|$ and select $M > 0$ such that $R^* + |f_\infty|R < \varepsilon M$. Denote

$$I_1 = \{x \in \mathbb{R} : |u(x)| \le R\}, \quad I_2 = \{x \in \mathbb{R} : |u(x)| > R\}.$$

Thus for any $u \in C_0(\mathbb{R}, \mathbb{R})$ with $\|u\|_0 > M$, when $x \in I_1$, we have

$$|f(u(x)) - f_\infty u(x)| \le |f(u(x))| + |f_\infty||u(x)| \le R^* + |f_\infty|R < \varepsilon M < \varepsilon\|u\|_0.$$

Similarly, we conclude that for any $u \in C_0(\mathbb{R}, \mathbb{R})$ with $\|u\|_0 > M$, when $x \in I_2$, we also have that

$$|f(u(x)) - f_\infty u(x)| < \varepsilon\|u\|_0.$$

We conclude that for any $u \in C_0(\mathbb{R}, \mathbb{R})$ with $\|u\|_0 > M$, we have

$$|f(u(x)) - f_\infty u(x)| < \varepsilon\|u\|_0.$$

Then for any $u \in C_0(\mathbb{R}, \mathbb{R})$ with $\|u\|_0 > M$, one has

$$\begin{aligned}
|f(u(x))| &\le |f(u(x)) - f_\infty u(x)| + |f_\infty u(x)| \\
&\le \varepsilon\|u\|_0 + |f_\infty|\|u\|_0 \\
&\le (f_\infty| + \varepsilon)\|u\|_0.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
|Tu(x)| &= \Big| \int_{-\infty}^{+\infty} G(x,s)p(s)f(u(s))\,ds - f_\infty \int_{-\infty}^{+\infty} G(x,s)p(s)u(s)\,ds \Big| \\
&\le \frac{1}{k} \int_{-\infty}^{+\infty} |p(s)||f(u(s)) - f_\infty u(s)|\,ds \\
&< \frac{1}{k}|p|_1\varepsilon\|u\|_0.
\end{aligned}$$

Then we have

$$\lim_{\|u\|_0\to\infty} \frac{\|Tu - Au\|_0}{\|u\|_0} = 0.$$

Theorem 1.1 now guarantees that Problem (2.1) has a nontrivial solution. This completes the proof. □

3. Existence result for a generalized Fisher-like system

In this section, we study the system

$$\begin{gathered}
-u'' + c_1u' + \lambda_1 u = p(x)f(u,v), \quad -\infty < x < +\infty, \\
-v'' + c_2v' + \lambda_2 v = q(x)g(u,v), \quad -\infty < x < +\infty, \\
u(-\infty) = u(+\infty) = 0, \quad v(-\infty) = v(+\infty) = 0.
\end{gathered} \tag{3.1}$$

where $p, q \in L^1(\mathbb{R}, \mathbb{R}_+)$, $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions with $f(0,0) \ne 0$ or $g(0,0) \ne 0$.

Our main theorem in this section reads as follows.

Theorem 3.1. *Assume that*

$$\lim_{u+v\to\infty}\frac{f(u,v)}{u+v}=f_\infty \quad \text{with } |f_\infty|<\frac{k_1}{\alpha|p|_1}, \tag{3.2}$$

$$\lim_{u+v\to\infty}\frac{g(u,v)}{u+v}=g_\infty \quad \text{with } |g_\infty|<\frac{k_2}{\beta|q|_1}, \tag{3.3}$$

for some positive real numbers α and β satisfying $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Then Problem (3.1) has at least one nontrivial solution $(u,v)\in C_0(\mathbb{R},\mathbb{R}^2)$.

Proof. Let the Banach space $C_0(\mathbb{R},\mathbb{R}^2):=C_0(\mathbb{R},\mathbb{R})\times C_0(\mathbb{R},\mathbb{R})$ be endowed with the norm $\|(.,.)\|$ given by

$$\|(u,v)\|=\|u\|_0+\|v\|_0.$$

It is clear that Problem (3.1) is equivalent to the integral equation:

$$\begin{aligned}&\Big(u(x),v(x)\Big)\\ &=\Big(\int_{-\infty}^{+\infty}G_1(x,s)p(s)f(u(s),v(s))\,ds,\int_{-\infty}^{+\infty}G_2(x,s)q(s)g(u(s),v(s))\,ds\Big)\end{aligned}$$

with Green functions

$$G_1(x,s)=\frac{1}{r_1-r_1'}\begin{cases}e^{r_1(x-s)} & \text{if } x\le s\\ e^{r_1'(x-s)} & \text{if } x\ge s\end{cases}$$

and

$$G_2(x,s)=\frac{1}{r_2-r_2'}\begin{cases}e^{r_2(x-s)} & \text{if } x\le s\\ e^{r_2'(x-s)} & \text{if } x\ge s\end{cases}$$

and characteristic roots

$$\begin{gathered}r_1=\frac{c_1+\sqrt{c_1^2+4\lambda_1}}{2},\quad r_1'=\frac{c_1-\sqrt{c_1^2+4\lambda_1}}{2},\\ r_2=\frac{c_2+\sqrt{c_2^2+4\lambda_2}}{2},\quad r_2'=\frac{c_2-\sqrt{c_2^2+4\lambda_2}}{2}.\end{gathered}$$

Define the mapping $T:C_0(\mathbb{R},\mathbb{R}^2)\to C_0(\mathbb{R},\mathbb{R}^2)$ by $T=(T_1,T_1)$ where

$$\begin{gathered}T_1(u,v)(x)=\int_{-\infty}^{+\infty}G_1(x,s)p(s)f(u(s),v(s))\,ds\,,\\ T_2(u,v)(x)=\int_{-\infty}^{+\infty}G_2(x,s)q(s)g(u(s),v(s))\,ds.\end{gathered}$$

In view of Krasnosel'sk'ii-Zabreiko's fixed point theorem, we look for fixed points for the operator T in the Banach space $C_0(\mathbb{R},\mathbb{R}^2)$. The proof is split in four steps.

Claim 1: The mapping $T=(T_1,T_2)$ is well defined; indeed, for any $(u,v)\in C_0(\mathbb{R},\mathbb{R}^2)$, we get, by Assumptions (3.2), (3.3), the following estimate

$$\begin{aligned}|T_1(u,v)(x)|&\le\int_{-\infty}^{+\infty}G_1(x,s)|p(s)f(u(s),v(s))|\,ds\\ &\le\frac{1}{k_1}|p|_1\max_{|y|\le\|u\|_0,|z|\le\|v\|_0}|f(y,z)|.\end{aligned}$$

In the same way, we find that

$$|T_2(u,v)(x)| \le \frac{1}{k_2}|q|_1 \max_{|y|\le\|u\|_0,|z|\le\|v\|_0} |g(y,z)|.$$

The convergence of the integrals defining $T_1(u,v)(x)$ and $T_2(u,v)(x)$ is then established. In addition, for any $s \in \mathbb{R}$, $G_1(\pm\infty,s)=0, G_2(\pm\infty,s)=0$, and then, taking the limit in $T_1(u,v)(x)$ and $T_2(u,v)(x)$, we get

$$T(u,v)(\pm\infty) = \Big(T_1(u,v)(\pm\infty), T_2(u,v)(\pm\infty)\Big) = (0,0).$$

Therefore, the mapping $T : C_0(\mathbb{R},\mathbb{R}^2) \to C_0(\mathbb{R},\mathbb{R}^2)$ is well defined.

Claim 2: The operator T is continuous. It suffices to prove that both T_1 and T_1 are continuous. Let $\big((u_n,v_n)\big)_n \subset C_0(\mathbb{R},\mathbb{R}^2)$ be a sequence which converges uniformly to (u_0,v_0) on each compact subinterval of $\mathbb{R}$. For some fixed $a>0$, we will prove the uniform convergence of $(T(u_n,v_n))_n$ to $T(u_0,v_0)$ on $[-a,a]$. Let $\varepsilon > 0$ and choose some $b > a$ large enough. By the uniform convergence of the sequence $\big((u_n,v_n)\big)_n$ on $[-b,b]$, there exists an integer $N = N(\varepsilon,b)$ satisfying

$$\begin{aligned} I_1 :&= \sup_{x\in\mathbb{R}} \int_{-b}^{+b} G_1(x,s)|p(s)f(u_n(s),v_n(s)) - p(s)f(u_0(s),v_0(s))|\,ds \\ &< \frac{\varepsilon}{2} \quad \text{for } n \ge N. \end{aligned}$$

For $x \in [-a,a]$, we have that $|T_1(u_n,v_n)(x) - T_1(u_0,v_0)(x)| \le I_1 + I_2 + I_3$ with

$$I_2 := \sup_{x\in\mathbb{R}} \int_{\mathbb{R}-[-b,+b]} G_1(x,s)|p(s)f(u_0(s),v_0(s))|\,ds \le \frac{\varepsilon}{4}$$

(by the Cauchy Convergence Criterion and $\lim_{|s|\to+\infty} p(s)f(u_0(s),v_0(s)) = 0$) and

$$I_3 := \sup_{x\in\mathbb{R}} \int_{\mathbb{R}-[-b,+b]} G_1(x,s)|p(s)f(u_n(s),v_n(s))|\,ds \le \frac{\varepsilon}{4}$$

(by the Lebesgue Dominated Convergence Theorem). This proves the uniform convergence of $(T_1(u_n,v_n))_n$ to $T_1(u_0,v_0)$ on $[-a,a]$. Similarly, one can prove the uniform convergence of $(T_2(u_n,v_n))_n$ to $T_2(u_0,v_0)$ on $[-a,a]$.

Claim 3: For any $M > 0$, the set $\{T(u,v);\ \|(u,v)\| \le M\}$ is relatively compact in $C_0(\mathbb{R},\mathbb{R}^2)$. By the Ascoli-Arzela Theorem, it is sufficient to prove that the functions of this set are equicontinuous on every subinterval $[-a,a]$ and that there exist functions $\gamma_1,\gamma_2 \in C_0(\mathbb{R},\mathbb{R})$ such that for any $x \in \mathbb{R}$, $|T_1(u,v)(x)| \le \gamma_1(x)$ and $|T_2(u,v)(x)| \le \gamma_2(x)$. Let $x_1,x_2 \in [-a,a]$. Then

$$\begin{aligned} &|T_1(u,v)(x_2) - T_1(u,v)(x_1)| \\ &\le \int_{-\infty}^{+\infty} |G_1(x_2,s) - G_1(x_1,s)||p(s)f(u(s),v(s))|\,ds \\ &\le \max_{y+z\in[-M,M]} |f(y,z)| \int_{-\infty}^{+\infty} |G_1(x_2,s) - G_1(x_1,s)||p(s)|\,ds. \end{aligned}$$

By the continuity of the Green function G_1, the last term tends to 0, as x_2 tends to x_1. Similarly,

$$\begin{aligned}&|T_2(u,v)(x_2)-T_2(u,v)(x_1)|\\&\le \max_{y+z\in[-M,M]}|g(y,z)|\int_{-\infty}^{+\infty}|G_2(x_2,s)-G_2(x_1,s)||q(s)|\,ds.\end{aligned}$$

By the continuity of the Green function G_2, the last term tends to 0, as x_2 tends to x_1, whence comes the equicontinuity of $\{T(u,v);\|(u,v)\|\le M\}$. Now, we check analogously the second statement:

$$\begin{aligned}|T_1(u,v)(x)|&\le \int_{-\infty}^{+\infty}G_1(x,s)|p(s)f(u(s),v(s))|\,ds\\&\le \max_{y+z\in[-M,M]}|f(y,z)|\int_{-\infty}^{+\infty}G_1(x,s)|p(s)|\,ds:=\gamma_1(x)\end{aligned}$$

and $\gamma_1\in C_0(\mathbb{R},\mathbb{R})$. Also

$$|T_2(u,v)(x)|\le \max_{y+z\in[-M,M]}|g(y,z)|\int_{-\infty}^{+\infty}G_2(x,s)|q(s)|\,ds:=\gamma_2(x)$$

and $\gamma_2\in C_0(\mathbb{R},\mathbb{R})$.

Now we consider the boundary-value problem

$$\begin{gathered}-u''+c_1u'+\lambda_1u=p(x)(u(x)+v(x)),\quad -\infty<x<+\infty,\\-v''+c_2v'+\lambda_2v=q(x)(u(x)+v(x)),\quad -\infty<x<+\infty,\\u(-\infty)=u(+\infty)=0,\quad v(-\infty)=v(+\infty)=0.\end{gathered}\tag{3.4}$$

Define the operator $A=(A_1,A_2)$ by

$$\begin{aligned}A_1(u,v)(x)&=f_\infty\int_{-\infty}^{+\infty}G_1(x,s)p(s)(u(s)+v(x))\,ds,\\A_2(u,v)(x)&=g_\infty\int_{-\infty}^{+\infty}G_2(x,s)q(s)(u(s)+v(x))\,ds.\end{aligned}$$

Obviously, A is a bounded linear operator. Furthermore, any fixed point of A is a solution of the Problem (3.4), and conversely.

We now assert that 1 is not an eigenvalue of A. In fact, if $f_\infty=0$ and $g_\infty=0$, then the Problem (3.4) has no nontrivial solutions. If $f_\infty\ne 0$ or $g_\infty\ne 0$, suppose that the Problem (3.4) has a nontrivial solution (u,v). Then

$$\begin{aligned}|A_1(u,v)(x)|&\le |f_\infty|\int_{-\infty}^{+\infty}G_1(x,s)|p(s)(u(s)+v(s))|\,ds\\&\le\frac{1}{k_1}|f_\infty||p|_1(\|u\|_0+\|v\|_0)\\&=\frac{1}{k_1}|f_\infty||p|_1\|(u,v)\|\\&<\frac{1}{\alpha}\|(u,v)\|\end{aligned}$$

and

$$\begin{aligned}
|A_2(u,v)(x)| &\le |g_\infty| \int_{-\infty}^{+\infty} G_2(x,s)|q(s)(u(s)+v(s))|\,ds \\
&\le \frac{1}{k_2}|g_\infty||q|_1(\|u\|_0+\|v\|_0) \\
&= \frac{1}{k_2}|g_\infty||q|_1\|(u,v)\| \\
&< \frac{1}{\beta}\|(u,v)\|.
\end{aligned}$$

Hence

$$\|A(u,v)\| = \|A_1(u,v)\|_0 + \|A_2(u,v)\|_0 < (\frac{1}{\alpha}+\frac{1}{\beta})\|(u,v)\| = \|(u,v)\|.$$

This contradiction proves that Problem (3.4) has no nontrivial solution. Thus, 1 is not an eigenvalue of A.

Finally, we prove that

$$\lim_{\|(u,v)\|\to\infty} \frac{\|T(u,v)-A(u,v)\|}{\|(u,v)\|} = 0$$

which is equivalent to

$$\lim_{\|u\|_0+\|v\|_0\to\infty} \frac{\|T_1(u,v)-A_1(u,v)\|_0}{\|u\|_0+\|v\|_0} + \lim_{\|u\|_0+\|v\|_0\to\infty} \frac{\|T_2(u,v)-A_2(u,v)\|_0}{\|u\|_0+\|v\|_0} = 0.$$

According to $\lim_{u+v\to\infty} \frac{f(u,v)}{u+v} = f_\infty$ and $\lim_{u+v\to\infty} \frac{g(u,v)}{u+v} = g_\infty$, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\begin{gathered}
|f(u,v) - f_\infty(u+v)| < \varepsilon|u+v|, \quad \text{for } |u+v| > R, \\
|g(u,v) - g_\infty(u+v)| < \varepsilon|u+v|, \quad \text{for } |u+v| > R.
\end{gathered}$$

Set $R^* = \max\Big\{\max_{|u+v|\le R}|f(u,v)|, \max_{|u+v|\le R}|g(u,v)|\Big\}$ and select $M > 0$ such that $R^* + \max\{|f_\infty|, |g_\infty|\}R < \varepsilon M$. Denote

$$I_1 = \{x\in\mathbb{R} : |u(x)+v(x)| \le R\}, \quad I_2 = \{x\in\mathbb{R} : |u(x)+v(x)| > R\}.$$

Thus for any $(u,v)\in C_0(\mathbb{R},\mathbb{R}^2)$ with $\|(u,v)\| > M$, when $x\in I_1$, we have

$$\begin{aligned}
|f(u(x),v(x)) - f_\infty(u(x)+v(x))| &\le |f((u(x),v(x))| + |f_\infty||u(x)+v(x)| \\
&\le R^* + |f_\infty|R \\
&< \varepsilon M < \varepsilon\|(u,v)\|.
\end{aligned}$$

Similarly, we conclude that for any $(u,v)\in C_0(\mathbb{R},\mathbb{R}^2)$ with $\|(u,v)\| > M$, when $x\in I_2$, we also have that

$$|f(u(x),v(x)) - f_\infty(u(x)+v(x)| < \varepsilon\|(u,v)\|.$$

We conclude that for any $(u,v)\in C_0(\mathbb{R},\mathbb{R}^2)$ with $\|(u,v)\| > M$, one has

$$|f(u(x),v(x)) - f_\infty(u(x)+v(x))| < \varepsilon\|(u,v)\|.$$

Then for any $(u,v)\in C_0(\mathbb{R},\mathbb{R}^2)$ with $\|(u,v)\|>M$, we have

$$\begin{aligned}|f(u(x),v(x))| &\le |f(u(x),v(x))-f_\infty(u(x)+v(x))|+|f_\infty(u(x)+v(x))|\\ &\le \varepsilon\|(u,v)\|+|f_\infty|\|(u,v)\|\\ &\le (|f_\infty|+\varepsilon)\|(u,v)\|.\end{aligned}$$

In the same way, we find that for any $(u,v)\in C_0(\mathbb{R},\mathbb{R}^2)$ with $\|(u,v)\|>M$, we have

$$|g(u(x),v(x))|\le(|g_\infty|+\varepsilon)\|(u,v)\|.$$

Hence we obtain

$$\begin{aligned}&|T_1(u,v)(x)-A_1(u,v)(x)|\\ &=|\int_{-\infty}^{+\infty}G_1(x,s)p(s)f(u(s),v(s))\,ds-f_\infty\int_{-\infty}^{+\infty}G_1(x,s)p(s)(u(s)+v(s))\,ds|\\ &\le\frac{1}{k_1}\int_{-\infty}^{+\infty}|p(s)||f(u(s),v(s)))-f_\infty(u(s)+v(s))|\,ds\\ &<\frac{1}{k_1}|p|_1\varepsilon\|(u,v)\|\\ &=\frac{1}{k_1}|p|_1\varepsilon(\|u\|_0+\|v\|_0).\end{aligned}$$

Similarly,

$$|T_2(u,v)(x)-A_2(u,v)(x)|<\frac{1}{k_2}|q|_1\varepsilon(\|u\|_0+\|v\|_0).$$

Then we have

$$\begin{gathered}\lim_{\|u\|_0+\|v\|_0\to\infty}\frac{\|T_1(u,v)-A_1(u,v)\|_0}{\|u\|_0+\|v\|_0}=0,\\ \lim_{\|u\|_0+\|v\|_0\to\infty}\frac{\|T_2(u,v)-A_2(u,v)\|_0}{\|u\|_0+\|v\|_0}=0\end{gathered}$$

and hence

$$\lim_{\|u+v\|\to\infty}\frac{\|T(u,v)-A(u,v)\|}{\|u+v\|}=0.$$

Theorem 1.1 now guarantees that Problem (3.1) has at least one nontrivial solution. This completes the proof. □

4. Further results

In this section, we study the system

$$\begin{gathered}-u''+c_1u'+\lambda_1u=f(x,u,v),\quad -\infty<x<+\infty,\\ -v''+c_2v'+\lambda_2v=g(x,u,v),\quad -\infty<x<+\infty,\\ u(-\infty)=u(+\infty)=0,\quad v(-\infty)=v(+\infty)=0.\end{gathered}\tag{4.1}$$

where $f,g:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions with $f(x,0,0)\not\equiv0$ or $g(x,0,0)\not\equiv0$.

The main existence result of this section is as follows.

Theorem 4.1. *Assume that there exist two functions* $\varphi, \psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *continuous and nondecreasing with respect to their two variables and there exist two positive continuous functions* $p, q : \mathbb{R} \to \mathbb{R}_+$ *and* $M_0 > 0$ *such that*

$$\begin{aligned} &|f(x,u,v)| \le p(x)\varphi(|u|,|v|), \quad \text{for } (x,u,v) \in \mathbb{R}^3,\\ &|g(x,u,v)| \le q(x)\psi(|u|,|v|), \quad \text{for } (x,u,v) \in \mathbb{R}^3,\\ &\max\Big\{\frac{1}{k_1}|p|_1\varphi(M_0,M_0), \frac{1}{k_2}|q|_1\psi(M_0,M_0)\Big\} \le M_0. \end{aligned} \tag{4.2}$$

Then Problem (4.1) *admits at least one nontrivial solution* $(u,v) \in C_0(\mathbb{R},\mathbb{R}^2)$.

Proof. Define the mapping $T : C_0(\mathbb{R},\mathbb{R}^2) \to C_0(\mathbb{R},\mathbb{R}^2)$ by $T = (T_1, T_1)$ where

$$\begin{aligned} T_1(u,v)(x) &= \int_{-\infty}^{+\infty} G_1(x,s)f(s,u(s),v(s))\,ds,\\ T_2(u,v)(x) &= \int_{-\infty}^{+\infty} G_2(x,s)g(s,u(s),v(s))\,ds. \end{aligned}$$

In view of Schauder's fixed point theorem, we look for fixed points of T in the Banach space $C_0(\mathbb{R},\mathbb{R}^2)$. The proof is split into four steps.

Claim 1: The mapping T is well defined. Indeed, for any $(u,v) \in C_0(\mathbb{R},\mathbb{R}^2)$, we get, by Assumptions (4.2), the estimate

$$\begin{aligned} |T_1(u,v)(x)| &\le \int_{-\infty}^{+\infty} G_1(x,s)|f(s,u(s),v(s))|\,ds\\ &\le \int_{-\infty}^{+\infty} G_1(x,s)p(s)\varphi(|u(s)|,|v(s)|)\,ds\\ &\le \varphi(\|u\|,\|v\|)\int_{-\infty}^{+\infty} G_1(x,s)p(s)\,ds, \ \forall\, x \in \mathbb{R}\\ &\le \frac{1}{k_1}|p|_1\varphi(\|u\|,\|v\|). \end{aligned}$$

In the same way, one can prove that

$$|T_2(u,v)(x) \le \frac{1}{k_2}|q|_1\psi(\|u\|,\|v\|).$$

The convergence of the integrals defining $T(u,v)(x)$ is then established. In addition for any $s \in \mathbb{R}$, $G_1(\pm\infty,s) = 0$, $G_1(\pm\infty,s) = 0$, and then, taking the limit in $T(u,v)(x) = (T_1(u,v)(x), T_2(u,v)(x))$, we obtain $T(u,v)(\pm\infty) = 0$. Therefore, the mapping $T : C_0(\mathbb{R},\mathbb{R}^2) \to C_0(\mathbb{R},\mathbb{R}^2)$ is well defined.

Claim 2: As in Section 3, one can prove easily that the operator $T = (T_1, T_2)$ is continuous.

Claim 3: As in Section 3, one can prove easily that for any $M > 0$, the set $\{T(u,v) = (T_1(u,v), T_2(u,v));\ \|(u,v)\| \le M\}$ is relatively compact in $C_0(\mathbb{R},\mathbb{R}^2)$.

Claim 4: There exists a nonempty closed bounded convex K such that T maps K into itself. Let

$$K = \big\{(u,v) \in C_0(\mathbb{R},\mathbb{R}^2) : \|u\|_0 \le M_0, \|v\|_0 \le M_0\big\}.$$

From assumption (4.2), we know that

$$\frac{1}{k_1}|p|_1\varphi(M_0,M_0) \le 1, \quad \frac{1}{k_2}|q|_1\psi(M_0,M_0) \le 1.$$

If $\|u\| \le M_0$ and $\|v\| \le M_0$, then

$$\begin{aligned}\|T_1(u,v)\|_0 &\le \sup_{x\in\mathbb{R}} \int_{-\infty}^{+\infty} G_1(x,s)p(s)\varphi(|u(s)|,|v(s)|)\,ds \\ &\le \frac{1}{k_1}|p|_1\varphi(M_0,M_0) \\ &\le M_0.\end{aligned}$$

Similarly, $\|T_2(u,v)\|_0 \le M_0$. Therefore, the operator T maps K into itself. The proof of Theorem 3.1 then follows from Schauder's fixed point theorem. □

Using of Schauder's theorem, one can also prove the existence of a positive solution under some integral conditions on the nonlinear terms:

Theorem 4.2. *Suppose that the functions f and g are positive with $f(x,0,0) \not\equiv 0$ or $g(x,0,0) \not\equiv 0$, and satisfy the following two mean growth assumptions:*

$$\begin{aligned}|f(x,u,v)| &\le \varphi(x,|u|,|v|), \\ |g(x,u,v)| &\le \psi(x,|u|,|v|)\end{aligned}$$

where $\varphi,\psi : \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ are continuous, nondecreasing with respect to their two last arguments and verify

$$\int_{-\infty}^{+\infty} \varphi(x,M_0,M_0)\,dx \le k_1M_0, \quad \int_{-\infty}^{+\infty} \psi(x,M_0,M_0)\,dx \le k_2M_0 \tag{4.3}$$

for some constant $M_0 > 0$. Then Problem (4.1) has a positive solution $(u,v) \in C_0(\mathbb{R},\mathbb{R}_+\times\mathbb{R}_+)$.

Proof. To prove Theorem 4.2, we proceed as in Theorem 4.1 by taking the closed convex subset K of $C_0(\mathbb{R},\mathbb{R}^2)$ defined by:

$$K = \{(u,v) \in C_0(\mathbb{R},\mathbb{R}^2) : 0 \le u(x) \le M_0,\ 0 \le v(x) \le M_0 \text{ on } \mathbb{R}\}.$$

Using Assumption (4.3) and the fact that the mapping φ and ψ are nondecreasing with respect to their two last arguments, we find that T maps K into itself. Indeed, we derive the estimates:

$$\begin{aligned}0 \le T_1(u,v)(x) &\le \int_{-\infty}^{+\infty} G_1(x,s)\varphi(s,|u(y)|,|v(y)|)\,ds \\ &\le \frac{1}{k_1}\int_{-\infty}^{+\infty} \varphi(s,M_0,M_0)\,ds \le M_0\end{aligned}$$

and

$$0 \le T_2(u,v)(x) \le M_0.$$

In addition, the mapping T is continuous as can easily be seen and one can check that $T(K)$ is relatively compact. Then the claim of Theorem 4.2 follows. □

5. Examples

In this section, we give some examples to illustrate our results.

(1) Consider the boundary-value problem

$$\begin{gathered}-u'' + u' + 2u = \frac{1}{\pi(x^2+1)}[2u + 1 + \lg(1+|u|)], \\ u(-\infty) = u(+\infty) = 0.\end{gathered} \tag{5.1}$$

Here $p(x) = \frac{1}{\pi(x^2+1)}$ and $f(u) = 2u+1+\lg(1+|u|)$. Notice that $k = 3$, $|p|_1 = 1$ and $f_\infty = 2$. Thus, by Theorem 2.1, Problem (5.1) has at least one nontrivial solution $u \in C_0(R, R)$.

(2) Consider the boundary-value system

$$\begin{gathered} -u'' + u' + 2u = \frac{1}{2\pi(x^2+1)}\big[2(u+v)+1+\lg(1+|u+v|)\big], \\ -v'' + \sqrt{5}v' + v = \frac{1}{2\sqrt{\pi}}e^{-x^2}\big[2(u+v)+1+\sqrt{1+|u+v|}\big], \\ u(-\infty) = u(+\infty) = 0, \quad v(-\infty) = v(+\infty) = 0. \end{gathered} \tag{5.2}$$

Set $p(x) = \frac{1}{2\pi(x^2+1)}$, $q(x) = \frac{1}{2\sqrt{\pi}}e^{-x^2}$, $f(u,v) = 2(u+v)+1+\lg(1+|u+v|)$, and $g(u,v) = 2(u+v)+1+\sqrt{1+|u+v|}$. Notice that $k_1 = k_2 = 3$, $\alpha = \beta = 2$, $|p|_1 = |q|_1 = \frac{1}{2}$, and $f_\infty = g_\infty = 2$. Thus, by Theorem 3.1, Problem (5.3) has at least one nontrivial solution $(u,v) \in C_0(R, R^2)$.

(3) Consider the boundary-value system

$$\begin{gathered} -u'' + u' + 2u = \frac{1}{\pi(x^2+1)}\big(|v|^\mu + 1\big), \\ -v'' + \sqrt{5}v' + v = \frac{1}{\sqrt{\pi}}\exp^{-x^2}\big(|u|^\nu + 1\big), \\ u(-\infty) = u(+\infty) = 0, \quad v(-\infty) = v(+\infty) = 0. \end{gathered} \tag{5.3}$$

where μ and ν are real numbers such that $0 < \mu < 1$ and $0 < \nu < 1$. Set $p(x) = \frac{1}{\pi(x^2+1)}$, $q(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$, $\varphi(y,z) = z^\mu + 1$, and $\psi(y,z) = y^\nu + 1$. Notice that $k_1 = k_2 = 3$, $|p|_1 = |q|_1 = 1$ and if we choose any M_0 large enough, then condition (4.2) is satisfied. Thus, by Theorem 4.1, Problem (5.3) has at least one nontrivial solution $(u,v) \in C_0(R, R^2)$.

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Toufik Moussaoui

Department of Mathematics, E.N.S., P.O. Box 92, 16050 Kouba, Algiers, Algeria

E-mail address: `moussaoui@ens-kouba.dz`

Radu Precup

Department of Applied Mathematics, Babeş-Bolyai University, 400084 Cluj, Romania

E-mail address: `r.precup@math.ubbcluj.ro`