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DATA DEPENDENCE OF FIXED POINTS FOR NON-SELF GENERALIZED CONTRACTIONS

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Abstract. Data dependence of fixed points for several classes of non-self generalized contractions is studied. A fibre non-self contraction theorem is also established. An application to functional equations is included.
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1. Introduction

Let (X,d) be a metric space, $Y \subset X$ a nonempty subset of X and $f : Y \to X$ an operator. In what follow we shall use the following notations:
$F_f = \{x \in Y : f(x) = x\}$ – the fixed points set of f
$I(f) = \{Z \subset Y : f(Z) \subset Z, Z \neq \emptyset\}$ – the set of invariant subsets of f
$(MI)_f = \cup I(f)$ – the maximal invariant subset of f
$(AB)_f(x^*) = \{x \in Y : f^n(x)$ is defined for all $n \in \mathbb{N}$ and $f^n(x) \to x^* \in F_f\}$- the attraction basin of the fixed point x^* with respect to f
$(AB)_f = \bigcup\limits_{x^* \in F_f} (AB)_f(x^*)$ – the attraction basin of f
$(PH)_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ – the Pompeiu-Haudorff functional,

$$(PH)_d(A,B) = \max\left(\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right).$$

In [20] the author uses the weakly Picard operator technique for the study of data dependence of fixed points of self generalized contractions. The aim of this paper is to study the same problem in the case of non-self operators. In addition, we introduce the ψ-condition and we study the data dependence of the fixed points of operators satisfying the ψ-condition. A fibre non-self contraction theorem is also established. An application to functional equations is included.

2. Picard and weakly Picard non-self operators

We begin our considerations by some definitions. Let (X, d) be a metric space and $Y \subset X$ a nonempty subset of X.

Definition 2.1. *An operator $f : Y \to X$ is said to be a Picard operator (PO) if*

(i) $F_f = \{x_f^*\}$;
(ii) $(MI)_f = (BA)_f$.

Definition 2.2. *An operator $f : Y \to X$ is said to be a weakly Picard operator (WPO) if*

(i) $F_f \neq \emptyset$;
(ii) $(MI)_f = (BA)_f$.

Definition 2.3. *For each WPO $f : Y \to X$ we define the operator $f^\infty : (BA)_f \to (BA)_f$ by $f^\infty(x) = \lim_{n\to\infty} f^n(x)$.*

Remark 2.1. *It is clear that $f^\infty((BA)_f) = F_f$, so f^∞ is a set retraction of $(BA)_f$ to F_f.*

Remark 2.2. *In terms of weakly Picard self operators the above definitions take the following form:*

$$f : Y \to X \text{ is a WPO (PO) iff } f\,|_{(MI)_f} : (MI)_f \to (MI)_f \text{ is a WPO (PO).}$$

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$.

Definition 2.4. *An operator $f : Y \to X$ is said to be a ψ-WPO (ψ-PO) if f is a WPO (PO) and*

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x)) \quad \text{for every } x \in (MI)_f.$$

In case that $\psi(t) = ct$, $t \in \mathbb{R}_+$ *and* $c > 0$, *we say that* f *is a* c*-WPO (*c*-PO).*

Example 2.1. *Let* Y *be a nonempty subset of the metric space* (X, d) *and* $f : Y \to X$ *be an* α*-contraction* $(0 < \alpha < 1)$ *with* $F_f = \{x_f^*\}$. *Then* f *is a* $\frac{1}{1-\alpha}$*-PO.*

Indeed, for $x \in (MI)_f$ we have that $d(f^n(x), x_f^*) \to 0$ as $n \to \infty$, i.e., f is a PO. On the other hand, for $x \in (MI)_f$, we have

$$\begin{aligned} d(x, f^n(x)) &\leq d(x, f(x)) + d(f(x), f^2(x)) + ... + d(f^{n-1}(x), f^n(x)) \\ &\leq (1 + \alpha + \alpha^2 + ... + \alpha^{n-1})d(x, f(x)) \\ &\leq \frac{1}{1-\alpha}d(x, f(x)) \end{aligned}$$

whence

$$d(x, x_f^*) \leq \frac{1}{1-\alpha}d(x, f(x)).$$

Example 2.2. *Let* Y *be a nonempty subset of the metric space* (X, d) *and let* $f : Y \to X$ *be a generalized contraction of Ciric-Reich-Rus type, that is*

$$d(f(x), f(y)) \leq \alpha d(x, f(x)) + \beta d(y, f(y)) + \gamma d(x, y) \tag{2.1}$$

for all $x, y \in Y$, *where* α, β, γ *are non-negative numbers with* $\alpha + \beta + \gamma < 1$. *We suppose that* $F_f = \{x_f^*\}$. *Then* f *is a* c*-PO, where* $c = \dfrac{1-\beta}{1-\alpha-\beta-\gamma}$.

Indeed, if we let in (2.1) $y = f(x)$, $x \in (BA)_f$, we obtain

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)) + \beta d(f(x), f^2(x)) + \gamma d(x, f(x))$$

and so

$$d(f(x), f^2(x)) \leq \frac{\alpha+\gamma}{1-\beta}d(x, f(x)), \text{ for all } x \in (BA)_f.$$

Then, for every n, one has

$$\begin{aligned} d(x, f^n(x)) &\leq d(x, f(x)) + d\left(f(x), f^2(x)\right) + ... + d\left(f^{n-1}(x), f^n(x)\right) \\ &\leq \frac{1}{1-\frac{\alpha+\gamma}{1-\beta}}d(x, f(x)) \\ &= \frac{1-\beta}{1-\alpha-\beta-\gamma}d(x, f(x)). \end{aligned}$$

Consequently, f is a c-PO.

Example 2.3. *Let Y be a nonempty subset of the metric space (X, d) and let $f : Y \to X$ be a generalized contraction of Ciric type, that is*

$$d(f(x), f(y)) \leq q \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}$$

for all $x, y \in Y$ and some $q \in [0, \frac{1}{2})$. We suppose that $F_f = \{x_f^*\}$. Then A is a c-PO, where $c = \dfrac{1-q}{1-2q}$.

Example 2.4. *Let $f : Y \to X$ be a closed graphic contraction, i.e., f is closed (i.e. it has closed graph) and there exists $\alpha \in (0, 1)$ such that*

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x))$$

for all x for which $f^2(x)$ is defined. We suppose that $F_f \neq \emptyset$. Then f is a $\frac{1}{1-\alpha}$-WPO.

Indeed, the graphic contraction condition implies that for every $x \in (MI)_f$, the sequence $(f^n(x))$ is convergent. Since f is closed the limit of sequence $(f^n(x))$ is a fixed point of f. Thus f is a WPO. In addition, if $x \in (BA)_f$, then

$$\begin{aligned} d(x, f^n(x)) &\leq d(x, f(x)) + d(f(x), f^2(x)) + ... + d(f^{n-1}(x), f^n(x)) \\ &\leq (1 + \alpha + \alpha^2 + ... + \alpha^{n-1}) d(x, f(x)) \\ &\leq \frac{1}{1-\alpha} d\left(x, f\left(x\right)\right) \end{aligned}$$

and letting $n \to \infty$, we obtain

$$d\left(x, f^{\infty}\left(x\right)\right) \leq \frac{1}{1-\alpha} d\left(x, f\left(x\right)\right).$$

Example 2.5. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function (see* [19]*), i.e.,*

(a) *φ is increasing;*
(b) *$\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \in \mathbb{R}_+$;*
(c) *$t - \varphi(t) \to +\infty$ as $t \to \infty$.*

Let (X, d) be a metric space, $Y \subset X$ and $f : Y \to X$ a strict φ- contraction, i.e.,

$$d(f(x), f(y)) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in Y,$$

with $F_f \neq \emptyset$. Then f is a ψ_φ -PO, with respect to $\psi_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\psi_\varphi(\eta) = \sup\{t \in \mathbb{R}_+ : \ t - \varphi(t) \leq \eta\}.$$

Indeed, $f|_{(MI)_f}$ is a PO as follows from Matkowski's fixed point theorem (see [11], [10] and [19]). Let $F_f = \{x^*\}$. Then, for $x \in (BA)_f$, we have

$$d(x,x^*) \leq d(x,f(x)) + d(f(x),x^*)$$
$$\leq d(x,f(x)) + \varphi(d(x,x^*)).$$

Hence

$$d(x,x^*) - \varphi(d(x,x^*)) \leq d(x,f(x)).$$

So

$$d(x,x^*) \leq \psi_\varphi(d(x,f(x))).$$

Therefore f is a ψ_φ- PO.

Remark 2.3. *It is clear that if $f : X \to X$ is a WPO (PO), then $f|_Y : Y \to X$ is also a WPO (PO).*

3. Data dependence for ψ-WPOs and ψ-POs

Let (X, d) be a metric space, $Y \subset X$ a nonempty subset of X and $f, g : Y \to X$ two operators. Denote by $(PH)_d$ the Pompeiu-Hausdorff functional.

Theorem 3.1. *Assume that the following conditions are satisfied:*
(i) *f and g are ψ-WPOs;*
(ii) *$F_g \subset (BA)_f$ and $F_f \subset (BA)_g$;*
(iii) *there exists $\eta > 0$ such that*

$$d(f(x),g(x)) \leq \eta \quad \text{for all } x \in Y.$$

Then

$$(PH)_d(F_f,F_g) \leq \psi(\eta).$$

Proof. If $x \in F_g$, then

$$d(x,f^\infty(x)) \leq \psi(d(x,f(x))) = \psi(d(g(x),f(x))) \leq \psi(\eta).$$

If $y \in F_f$, then

$$d(y,g^\infty(y)) \leq \psi(d(y,g(y))) = \psi(d(f(y),g(y))) \leq \psi(\eta).$$

Now the conclusion follows from the next lemma from [19]. □

Lemma 3.1. *Let (X, d) be a metric space and A, $B \subset X$ two nonempty sets.If $\tau > 0$ is such that:*

(1) *for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq \tau$;*

(2) *for each $b \in B$ there exists $a \in A$ such that $d(a, b) \leq \tau$,*

then $(PH)_d(A, B) \leq \tau$.

A similar result holds for ψ-POs.

Theorem 3.2. *Assume that the following conditions are satisfied:*

(i) *f is a ψ-PO ($F_f = \{x_f^*\}$);*

(ii) $\emptyset \neq F_g \subset (BA)_f$;

(iii) *there exists $\eta > 0$ such that*

$$d(f(x), g(x)) \leq \eta \quad \text{for all } x \in Y.$$

Then

$$d(x_f^*, x_g^*) \leq \psi(\eta) \quad \text{for all } x_g^* \in F_g.$$

Proof. Let $x_g^* \in F_g$. Then

$$d(x_g^*, x_f^*) \leq \psi(d(x_g^*, f(x_g^*))) = \psi(d(g(x_g^*), f(x_g^*))) \leq \psi(\eta).$$ □

A better result holds in case of strict φ- contractions.

Theorem 3.3. *Assume that the following conditions are satisfied:*

(i) *f is a strict φ-contraction with $F_f = \{x_f^*\}$;*

(ii) $F_g \neq \emptyset$;

(iii) *there exists $\eta > 0$ such that*

$$d(f(x), g(x)) \leq \eta, \quad \text{for all } x \in Y.$$

Then

$$d(x_g^*, x_f^*) \leq \psi_\varphi(\eta), \quad \text{for all } x_g^* \in F_g.$$

(For the definition of ψ_φ see Example 2.5).

Proof. Let $x_g^* \in F_g$. We have

$$\begin{aligned} d(x_g^*, x_f^*) &\leq d(x_g^*, f(x_g^*)) + d(f(x_g^*), x_f^*) \\ &\leq \eta + \varphi(d(x_g^*, x_f^*)). \end{aligned}$$

Hence

$$d(x_g^*, x_f^*) - \varphi(d(x_g^*, x_f^*)) \leq \eta.$$

Then

$$d(x_g^*, x_f^*) \leq \psi_\varphi(\eta). \qquad \square$$

We also have the following result:

Theorem 3.4. *Assume that the following conditions are satisfied:*

(i) *there exist* $\alpha, \beta \in \mathbb{R}_+,\ \alpha + 2\beta < 1$ *such that*

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta[d(x, f(x)) + d(y, f(y))]$$

for all $x, y \in X$, *and let* $F_f = \{x_f^*\}$;

(ii) $F_g \neq \emptyset$;

(iii) *there exists* $\eta > 0$ *such that*

$$d(f(x), g(x)) \leq \eta, \quad \text{for all } x \in Y.$$

Then

$$d(x_g^*, x_f^*) \leq \frac{1+\beta}{1-\alpha}\eta, \quad \text{for all } x_g^* \in F_g. \tag{3.1}$$

Proof. Let $x_g^* \in F_g$. We have

$$\begin{aligned} d(x_g^*, x_f^*) &\leq d(x_g^*, f(x_g^*)) + d(f(x_g^*), x_f^*) \\ &\leq \eta + \alpha d(x_g^*, x_f^*) + \beta d(x_g^*, f(x_g^*)) \\ &\leq \eta + \alpha d(x_g^*, x_f^*) + \beta\eta. \end{aligned}$$

This immediately gives (3.1). $\square$

Remark 3.1. *In particular, condition (i) in Theorem* 3.1 *and Theorem* 3.2 *follows from a continuation principle* ([1], [2], [3], [4], [5], [6], [8], [12], [13], [14], [15], [16], [17], [18]). *For example, we have:*

Theorem 3.5. *Let* (X, d) *be a complete metric space,* $U \subset X$ *open and* $f : \overline{U} \to X$ *an operator. Assume that there exists* $H : \overline{U} \times [0, 1] \to X$ *continuous such that:*

(a) *there exists* $x_0 \in U$ *with* $H(\cdot, 0) = x_0$;

(b) $H(x, 1) = f(x)$ *for all* $x \in \overline{U}$;

(c) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(H(x, \lambda), H(y, \lambda)) \leq \alpha d(x, y)$$

for all $x, y \in \overline{U}$ *and* $\lambda \in [0, 1]$;

(d) $H(x, \lambda) \neq x$ *for all* $x \in \partial U$ *and* $\lambda \in [0, 1]$;

(e) $H(x,\lambda)$ *is continuous in* λ, *uniformly for* $x \in \overline{U}$.

Then:

(1) f *has a unique fixed point* x_f^* *and* f *is a* $\frac{1}{1-\alpha}$*-PO;*

(2) *if* g *is as in Theorem* 3.2, *then* $d(x_f^*, x_g^*) \leq \frac{\eta}{1-\alpha}$.

Proof. From conditions (c), (d) and (e) we have that the homotopy $H : \overline{U} \times [0,1] \to X$ has the properties from Granas' continuation principle for contractions on complete metric spaces ([8]). In addition we have that $H(.,0) = x_0$ has a unique fixed point. So H_λ has a unique fixed point in U for all $\lambda \in [0,1]$. Then, from condition (b), f has a unique fixed point x_f^*, that is $f(x_f^*) = x_f^*$, and using Example 2.1 f is a $\frac{1}{1-\alpha}$ -PO. Now (2) follows from Theorem 3.2. □

4. Data dependence for operators satisfying the ψ-condition

4.1. **The ψ- condition in the case $F_f = \{x_f^*\}$.** Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous in zero and $\psi(0) = 0$. Let (X,d) be a metric space, $Y \subset X$ and $f : Y \to X$ be any operator with $F_f = \{x_f^*\}$.

Definition 4.1. *The operator* f *satisfies the* ψ*-condition if*

$$d(x, x_f^*) \leq \psi(d(x, f(x))) \text{ for all } x \in Y.$$

Example 4.1. *If* $f : Y \to X$ *is an* α*-contraction* $(0 < \alpha < 1)$, *then* f *satisfies the* ψ*-condition with respect to* $\psi(t) = \frac{t}{1-\alpha}$.

Example 4.2. *If* $f : Y \to X$ *a strict* φ*-contraction with respect to some strict comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, *then* f *satisfies the* ψ*-condition with respect to* $\psi(r) = \sup\{t \in \mathbb{R}_+ : t - \varphi(t) \leq r\}$.

Example 4.3. *If* $Y = X$ *and* f *is a* ψ*-PO, then* f *satisfies the* ψ*-condition (see* [20] *and* [23]*).*

The above examples give rise to the following problems:

Problem 4.1. *Which metric conditions on* f *imply the* ψ*-condition with respect to some function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$*?*

Problem 4.2. *Let* $Y = X$. *For which generalized contractions we have that:*

(i) f *satisfies the* ψ*- condition with respect to some function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$;

(ii) f *is not a* ψ*-PO.*

Theorem 4.1. *If f satisfies the ψ-condition, then the fixed point problem is well posed for f.*

Proof. Let $x_n \in Y$ be such that (see [22])

$$d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

Then, from the ψ-condition, we have

$$d(x_n, x_f^*) \leq \psi(d(x_n, f(x_n))) \to 0 \text{ as } n \to \infty. \qquad \square$$

Theorem 4.2. *Assume that the following conditions are satisfied:*
(i) $Y = X$;
(ii) f satisfies the ψ- condition;
(iii) f is asymptotically regular.
Then f is a ψ-PO.

Proof. Let $x \in X$. We have

$$d(f^n(x), x_f^*) \leq \psi(d(f^n(x), f^{n+1}(x))) \to 0 \text{ as } n \to \infty.$$

So, f is a PO. Now (ii) implies that f is a ψ-PO. $\square$

Now we state a data dependence result for operators satisfying the ψ-condition.

Theorem 4.3. *Let (X, d) be a metric space, $Y \subset X$ and $f, g : Y \to X$ two operators. We suppose that:*
(i) f satisfies the ψ- condition;
(ii) there exists $\eta > 0$ such that

$$d(f(x), g(x)) \leq \eta, \qquad \text{for every } x \in Y.$$

Then

$$d(x_f^*, x_g^*) \leq \psi(\eta)$$

for every $x_g^* \in F_g$.

Proof. Let $x_g^* \in F_g$. Then (i) and (ii) guarantee that

$$d(x_g^*, x_f^*) \leq \psi(d(x_g^*, f(x_g^*))) = \psi(d(g(x_g^*), f(x_g^*))) \leq \psi(\eta). \qquad \square$$

In particular, for φ-contractions (see Example 2.5) we have the following result:

Theorem 4.4. *Let (X,d) be a metric space, $Y \subset X$ and $f,g : Y \to X$ be two operators. We suppose that:*

(i) f is a φ-contraction;

(ii) there exists $\eta > 0$ such that

$$d(f(x), g(x)) \leq \eta \qquad \text{for every } x \in Y.$$

Then

$$d(x_f^*, x_g^*) \leq \psi(\eta)$$

for every $x_g^* \in F_g$.

4.2. **The ψ-condition in the case $F_f \neq \emptyset$.** Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous in zero and $\psi(0) = 0$. Let (X,d) be a metric space, $Y \subset X$ and $f : Y \to X$ be any operator with $F_f \neq \emptyset$.

Definition 4.2. *The operator f satisfies the ψ- condition if there exists a set retraction $\chi_f : Y \to F_f$ such that*

$$d(x, \chi_f(x)) \leq \psi(d(x, f(x))) \text{ for every } x \in Y.$$

Example 4.4. *Let $Y = X$ and let $f : X \to X$ be a ψ-WPO. In this case we take $\chi_f = f^\infty$ and f satisfies the ψ-condition.*

Example 4.5. *Let (X,d) be a metric space, $Y \subset X$ and $f : Y \to X$. We suppose that*

(i) $Y = \bigcup_{i \in I} Y_i$ *is a partition of Y such that $F_f \cap Y_i = \{x_i^*\}$, $i \in I$;*

(ii) $f|_{Y_i} : Y_i \to X$ *is an α-contraction, $i \in I$.*

Then f satisfies the ψ- condition with respect to $\psi(t) = \frac{t}{1-\alpha}$.

Problem 4.3. *Which generalized contractions f satisfy the ψ-condition with respect to some function ψ?*

Problem 4.4. *In the case $Y = X$, for which generalized contractions we have that:*

(i) *f satisfies the ψ-condition;*

(ii) *f is not a $\psi-$WPO?*

We have the following data dependence result.

Theorem 4.5. *Let (X, d) be a metric space, $Y \subset X$ and $f, g : Y \to X$ be two operators. We suppose that:*

(i) *f, g satisfy the ψ-condition and $F_f \neq \emptyset$;*

(ii) *there exists $\eta > 0$ such that*

$$d(f(x), g(x)) \leq \eta, \quad \textit{for every } x \in Y;$$

(iii) *$F_g \neq \emptyset$.*

Then

$$(PH)_d(F_f, F_g) \leq \psi(\eta).$$

Proof. Let $x \in F_g$. Then

$$d(x, \chi_f(x)) \leq \psi(d(x, f(x))) = \psi(d(g(x), f(x))) \leq \psi(\eta).$$

Similarly, if $y \in F_f$, then

$$d(y, \chi_g(y)) \leq \psi(d(y, g(y))) = \psi(d(f(y), g(y))) \leq \psi(\eta).$$

Now from Lemma 3.1 we have

$$(PH)_d(F_f, F_g) \leq \psi(\eta).$$

□

5. Fibre non-self contraction theorems

In what follows we need the notion of L-space structure. Let X be a nonempty set. Let

$$s(X) = \{(x_n)_{n\in\mathbb{N}} : x_n \in X,\ n \in \mathbb{N}\}.$$

Let $c(X) \subset s(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ an operator.

Definition 5.1. *Following M. Fréchet* [7] (1905) *the triple $(X, c(X), Lim)$ is called an $L-$ space if the following conditions are satisfied:*

(i) *If $x_n = x$ for every $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$;*

(ii) *If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.*

In what follows an L-space $(X, c(X), Lim)$ will be simply denoted by $(X, \to)$.

Theorem 5.1. *Let $(X, \to)$ be an $L-$space, $X_1 \subset X$ a nonempty set and (Y, d) a complete metric space. Let $g : X_1 \to X$, $h : X_1 \times Y \to Y$ and $f : X_1 \times Y \to X \times Y$, $f(x, y) = (g(x), h(x, y))$. We suppose that:*

(i) *g is a WPO (PO);*

(ii) *there exists $\alpha \in (0, 1)$ such that*

$$d(h(x, y_1), h(x, y_2)) \le \alpha d(y_1, y_2)$$

for all $x \in (AB)_g$ and $y_1, y_2 \in Y$;

(iii) *f is continuous.*

Then f is a WPO (PO).

Proof. First of all we remark that $(MI)_f = (MI)_g \times Y$ and $(MI)_g = (AB)_g$. Let $x_0 \in (AB)_g$ and $y_0 \in Y$. Define $x_{n+1} = g(x_n)$, $y_{n+1} = h(x_n, y_n)$ for $n \in \mathbb{N}$. It is clear that $x_n \to x^* \in F_g$ as $n \to \infty$. Let $F_{h(x^*,\cdot)} = \{y^*\}$. Let us prove that $y_n \to y^*$. We have

$$\begin{aligned} d(y_{n+1}, y^*) &= d(h(x_n, y_n), y^*) \\ &\le d(h(x_n, y_n), h(x_n, y^*)) + d(h(x_n, y^*), y^*) \\ &\le \alpha d(y_n, y^*) + d(h(x_n, y^*), y^*) \\ &\ldots \\ &\le \alpha^{n+1} d(y_0, y^*) + \alpha^n d(h(x_0, y^*), y^*) + \\ &\ldots + \alpha d(h(x_{n-1}, y^*), y^*) + d(h(x_n, y^*), y^*). \end{aligned}$$

Then $d(y_{n+1}, y^*) \to 0$ by the Cauchy lemma, see [24] Indeed, in Cauchy lemma we take $a_k = \alpha^k$, $b_k = d(h(x_k, y^*), y^*)$ and we have $\sum_{k=0}^{\infty} a_k < \infty$, $b_n \to 0$ as $n \to 0$. So $\sum_{k=0}^{n+1} a_{n+1-k} b_k \to 0$ as $n \to \infty$. □

The above result is very useful to study the differentiability of solutions of operator equations with respect to a parameter. For example, let us to consider the following equation

$$x(t, \lambda) = F(t, x(t, \lambda), \lambda), \;\; t \in [a, b], \;\; \lambda \in J \subset \mathbb{R}, \tag{5.1}$$

where $F : [a, b] \times I \times J \to \mathbb{R}$. We suppose that:

(i) $I, J \subset \mathbb{R}$ are compact intervals;

(ii) $F \in C([a, b] \times I \times J)$;

(iii) $F(t,\cdot,\cdot) \in C^1(I \times J)$ for every $t \in [a,b]$;

(iv) $\left|\frac{\partial F}{\partial x}(t,u,\lambda)\right| \le \alpha < 1$ for every $t \in [a,b]$, $u \in I$, $\lambda \in J$.

(v) equation (5.1) has at least one solution.

Then we have:

Theorem 5.2. *Under the above conditions equation* (5.1) *has in* $C([a,b]\times J, I)$ *a unique solution* x^* *and* $x^*(t,\cdot) \in C^1(J)$ *for every* $t \in [a,b]$.

Proof. Let $X = C([a,b] \times J, I)$ with the supremum norm $\|.\|_C$ and let $B : C([a,b] \times J, I) \to C([a,b] \times J)$ be defined by $B(x)(t,\lambda) = F(t, x(t,\lambda), \lambda)$.

From the conditions (iv) and (v) it follows that $F_B = \{x^*\}$. Let $Y = \{x \in C([a,b] \times J, I) : B(x)(t,\lambda) \in I$ for all $t \in [a,b]$, $\lambda \in J\}$. It is clear that $x^* \in Y$, $B(Y) \subset Y$ and $B : Y \to Y$ is a PO. Let $x_0 \in Y$ be such that there exists $\frac{\partial x_0}{\partial \lambda}$ and $\frac{\partial x_0}{\partial \lambda} \in C([a,b] \times J)$. Let us suppose that there exists $\frac{\partial x^*}{\partial \lambda}$. Then we have that

$$\frac{\partial x^*(t,\lambda)}{\partial \lambda} = \frac{\partial F(t, x^*(t,\lambda),\lambda)}{\partial x} \cdot \frac{\partial x^*(t,\lambda)}{\partial \lambda} + \frac{\partial F(t, x^*(t,\lambda),\lambda)}{\partial \lambda}.$$

This relation suggests us to consider the following operators:

$$C : Y \times C([a,b] \times J) \to C([a,b] \times J)$$

defined by

$$C(x,y)(t,\lambda) = \frac{\partial F(t, x(t,\lambda),\lambda)}{\partial x} \cdot y(t,\lambda) + \frac{\partial F(t, x(t,\lambda),\lambda)}{\partial \lambda}$$

and

$$A : Y \times C([a,b] \times J) \to Y \times C([a,b] \times J)$$

with

$$A(x,y) = (B(x), C(x,y)).$$

From Theorem 3.2 we have that A is a PO. This implies that the sequences $x_{n+1} = B(x_n)$, $y_{n+1} = C(x_n, y_n)$ converge, $x_n \to x^*$, $y_n \to y^*$ and $x^* = B(x^*)$, $y^* = C(x^*, y^*)$.

Let us take $y_0 = \frac{\partial x_0}{\partial \lambda}$. Then $y_n = \frac{\partial x_n}{\partial \lambda}$. So

$$x_n \to x^* \quad \text{as } n \to \infty, \text{ with respect to the norm } \|.\|_C$$

and

$$\frac{\partial x_n}{\partial \lambda} \to y^* \quad \text{as } n \to \infty.$$

These imply that $y^* \in C^1([a,b] \times J)$ and $y^* = \dfrac{\partial x^*}{\partial \lambda}$. □

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