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Existence, localization and multiplicity results for positive radial solutions of semilinear elliptic systems

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Abstract

Existence, localization and multiplicity results of positive solutions to a system of singular second-order differential equations are established by means of the vector version of Krasnoselskii's cone fixed point theorem. The results are then applied for positive radial solutions to semilinear elliptic systems.

Keywords: Semilinear elliptic system; Dirichlet problem; Singular boundary value problem; Positive solution; Radial solution; Fixed point

1. Introduction

The paper is concerned with the existence, localization and multiplicity of positive radial solutions to the following semilinear elliptic system:

$$\begin{cases} \Delta u_1 + f_1(|x|)g_1(u_1, u_2) = 0, \\ \Delta u_2 + f_2(|x|)g_2(u_1, u_2) = 0 \end{cases} \tag{1.1}$$

in $\Omega := \{x \in \mathbf{R}^n\colon |x| > r_0\}$ $(n \geqslant 3)$, under the conditions

$$u_1 = u_2 = 0 \quad \text{for } |x| = r_0 \quad \text{and} \quad u_1, u_2 \to 0 \quad \text{as } |x| \to \infty. \tag{1.2}$$

Our interest in studying these systems comes from their applications in many areas from physics, biology, chemistry, etc. There exists an extensive literature devoted to scalar semilinear elliptic equations. We list here, for example, papers [1–3,8], and especially paper [7] which has mainly motivated us. As regards systems of type (1.1), we just name the works [4–6,9,11,12].

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Looking for radial solutions, we can write (1.1) in the radial variable $r = |x|$, as

$$\begin{cases} u_1''(r) + \dfrac{n-1}{r} u_1'(r) + f_1(r) g_1\big(u_1(r), u_2(r)\big) = 0, \\ u_2''(r) + \dfrac{n-1}{r} u_2'(r) + f_2(r) g_2\big(u_1(r), u_2(r)\big) = 0 \end{cases} \tag{1.3}$$

for $r > r_0$, and the boundary conditions (1.2) as

$$u_1(r_0) = u_2(r_0) = 0 \quad \text{and} \quad u_1(r), u_2(r) \to 0 \quad \text{as } r \to \infty. \tag{1.4}$$

Furthermore, following [7], we set

$$s := r^{2-n}, \qquad v_i(s) = u_i\big(r(s)\big) \quad (i = 1, 2),$$

and then

$$t := \big(r_0^{2-n} - s\big)/r_0^{2-n}, \qquad z_i(t) = v_i\big(s(t)\big) \quad (i = 1, 2),$$

in order to rewrite (1.3)–(1.4) as

$$\begin{cases} z_1''(t) + q_1(t) g_1\big(z_1(t), z_2(t)\big) = 0, \\ z_2''(t) + q_2(t) g_2\big(z_1(t), z_2(t)\big) = 0 \end{cases} \tag{1.5}$$

for $0 < t < 1$, and respectively

$$z_1(0) = z_1(1) = z_2(0) = z_2(1) = 0. \tag{1.6}$$

Here

$$q_i(t) = \frac{r_0^2}{(n-2)^2} (1-t)^{-\frac{2(n-1)}{n-2}} f_i\big(r_0(1-t)^{-\frac{1}{n-2}}\big). \tag{1.7}$$

Thus, the problem of radial solutions (1.1)–(1.2) reduces to the singular boundary-value problem (1.5)–(1.6).

Our approach to problem (1.5)–(1.6) is based on a new method to treat systems of operator equations which was established in [10], namely the vector version of Krasnoselskii's cone fixed point theorem:

Theorem 1.1. *(See [10].) Let $(X, |.|)$ be a normed linear space; $K_1, K_2 \subset X$ two cones; $K := K_1 \times K_2$; $r, R \in \mathbf{R}_+^2$ with $0 < r_i < R_i$ for $i = 1, 2$, $K_{r,R} := \{u = (u_1, u_2) \in K\colon\ r_i \leqslant |u_i| \leqslant R_i,\ i = 1, 2\}$, and let $N : K_{r,R} \to K$, $N = (N_1, N_2)$ be a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) *$u_i - N_i(u) \notin K_i$ if $|u_i| = r_i$, and $N_i(u) - u_i \notin K_i$ if $|u_i| = R_i$;*
(b) *$N_i(u) - u_i \notin K_i$ if $|u_i| = r_i$, and $u_i - N_i(u) \notin K_i$ if $|u_i| = R_i$.*

Then N has a fixed point $u = (u_1, u_2)$, i.e., $u_i = N_i(u_1, u_2)$ and $r_i < |u_i| < R_i$ for $i = 1, 2$.

Remark 1.1. Under the assumptions of Theorem 1.1 four cases are possible for $u \in K_{r,R}$:

(c1) $u_1 - N_1(u) \notin K_1$ if $|u_1| = r_1$, $N_1(u) - u_1 \notin K_1$ if $|u_1| = R_1$,
$u_2 - N_2(u) \notin K_2$ if $|u_2| = r_2$, $N_2(u) - u_2 \notin K_2$ if $|u_2| = R_2$;
(c2) $u_1 - N_1(u) \notin K_1$ if $|u_1| = r_1$, $N_1(u) - u_1 \notin K_1$ if $|u_1| = R_1$,
$N_2(u) - u_2 \notin K_2$ if $|u_2| = r_2$, $u_2 - N_2(u) \notin K_2$ if $|u_2| = R_2$;
(c3) $N_1(u) - u_1 \notin K_1$ if $|u_1| = r_1$, $u_1 - N_1(u) \notin K_1$ if $|u_1| = R_1$,
$u_2 - N_2(u) \notin K_2$ if $|u_2| = r_2$, $N_2(u) - u_2 \notin K_2$ if $|u_2| = R_2$;
(c4) $N_1(u) - u_1 \notin K_1$ if $|u_1| = r_1$, $u_1 - N_1(u) \notin K_1$ if $|u_1| = R_1$,
$N_2(u) - u_2 \notin K_2$ if $|u_2| = r_2$, $u_2 - N_2(u) \notin K_2$ if $|u_2| = R_2$.

2. Positive solutions for singular differential systems

In this section we discuss the boundary value (1.5)–(1.6). We shall assume that $g_i \in C(\mathbf{R}_+^2; \mathbf{R}_+)$ and $q_i \in C((0,1); (0,\infty)) \cap L^1(0,1)$ and that q_i are singular at 0 and/or 1.

By a positive solution of (1.5)–(1.6), we understand a function $z = (z_1, z_2) \in C^2((0,1); \mathbf{R}^2) \cap C^1([0,1]; \mathbf{R}^2)$ with $z_i(t) > 0$ for all $t \in (0,1)$ and $i = 1,2$, and which satisfies (1.5) on $(0,1)$ and the boundary condition (1.6).

Let $X = C[0,1]$ be endowed with norm $|v|_\infty = \max_{t\in[0,1]}|v(t)|$, and let P be the cone of all nonnegative functions from X. Let

$$G(t,s) = \begin{cases} t(1-s) & \text{if } 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t) & \text{if } 0 \leqslant s \leqslant t \leqslant 1 \end{cases}$$

be the Green function associated to the differential operator $-u''$ and the Dirichlet boundary condition.

Notice that condition $q_i \in L^1(0,1)$ guarantees that for every $v \in C[0,1]$ and $i \in \{1,2\}$, the function

$$u(t) := \int_0^1 G(t,s) q_i(s) v(s)\, ds$$

is well-defined and belongs to $C^1[0,1]$.

Now the problem of finding nonnegative solutions for (1.5)–(1.6) is equivalent to the integral system in P^2,

$$\begin{cases} z_1(t) = \displaystyle\int_0^1 G(t,s) q_1(s) g_1\big(z_1(s), z_2(s)\big)\, ds, \\ z_2(t) = \displaystyle\int_0^1 G(t,s) q_2(s) g_2\big(z_1(s), z_2(s)\big)\, ds. \end{cases} \tag{2.1}$$

Let $N : P^2 \to P^2$ be the completely continuous map $N = (N_1, N_2)$ given by

$$N_i(z)(t) = \int_0^1 G(t,s) q_i(s) g_i\big(z_1(s), z_2(s)\big)\, ds, \quad i = 1,2.$$

Then (2.1) is equivalent to the fixed point problem

$$z = N(z), \quad z \in P^2.$$

Now we fix any subinterval $[a,b]$ of $[0,1]$, with $0 < a < b < 1$, and we easily check that

$$\begin{aligned} &G(t,s) \leqslant G(s,s) \quad \text{for all } t,s \in [0,1], \quad \text{and} \\ &MG(s,s) \leqslant G(t,s) \quad \text{for } t \in [a,b],\ s \in [0,1], \end{aligned} \tag{2.2}$$

where $M = \min\{a, 1-b\}$.

If $v \in P$,

$$u(t) := \int_0^1 G(t,s) q_i(s) v(s)\, ds$$

and $u(t_0) = |u|_\infty$, then according to (2.2), for every $t \in [a,b]$, we have

$$u(t) \geqslant M \int_0^1 G(s,s) q_i(s) v(s)\, ds \geqslant M \int_0^1 G(t_0,s) q_i(s) v(s)\, ds = M u(t_0) = M|u|_\infty.$$

Thus, if in $X := C[0,1]$ we consider the cone $K_1 = K_2$ defined as

$$K_1 := \{v \in P\colon\ v(t) \geqslant M|v|_\infty \text{ for all } t \in [a,b]\}$$

and in X^2 the corresponding cone $K := (K_1)^2$, then we have that $N(K) \subset K$.

Before we state our main result, we introduce the following notations. For $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, we let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ $(i = 1, 2)$, and

$$\begin{aligned}
\gamma_1 &= \min\{g_1(u_1,u_2)\colon\ M\beta_1 \leqslant u_1 \leqslant \beta_1,\ Mr_2 \leqslant u_2 \leqslant R_2\},\\
\gamma_2 &= \min\{g_2(u_1,u_2)\colon\ Mr_1 \leqslant u_1 \leqslant R_1,\ M\beta_2 \leqslant u_2 \leqslant \beta_2\},\\
\Gamma_1 &= \max\{g_1(u_1,u_2)\colon\ 0 \leqslant u_1 \leqslant \alpha_1,\ 0 \leqslant u_2 \leqslant R_2\},\\
\Gamma_2 &= \max\{g_2(u_1,u_2)\colon\ 0 \leqslant u_1 \leqslant R_1,\ 0 \leqslant u_2 \leqslant \alpha_2\}.
\end{aligned} \tag{2.3}$$

Also, let

$$A_i = \max_{t\in[0,1]} \int_a^b G(t,s)q_i(s)\,ds, \qquad B_i = \max_{t\in[0,1]} \int_0^1 G(t,s)q_i(s)\,ds.$$

Clearly, $B_i > A_i > 0$ for $i = 1, 2$.

Theorem 2.1. *Assume that there exist $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, such that*

$$\begin{aligned}
&B_1\Gamma_1 < \alpha_1, \qquad A_1\gamma_1 > \beta_1,\\
&B_2\Gamma_2 < \alpha_2, \qquad A_2\gamma_2 > \beta_2.
\end{aligned} \tag{2.4}$$

Then (1.5)–(1.6) *has at least one positive solution $z = (z_1, z_2)$ with $r_i < |z_i|_\infty < R_i$, $i = 1, 2$, where $r_i = \min\{\alpha_i, \beta_i\}$ and $R_i = \max\{\alpha_i, \beta_i\}$. Moreover, the orbit of z for $t \in [a, b]$ is included in the rectangle $(Mr_1, R_1) \times (Mr_2, R_2)$.*

Proof. First note that if $z \in K_{r,R}$, $r_1 < |z_1|_\infty < R_1$ and $r_2 < |z_2|_\infty < R_2$, then by the definition of K,

$$Mr_1 < z_1(t) < R_1 \quad \text{and} \quad Mr_2 < z_2(t) < R_2$$

for all $t \in [a, b]$, showing that the orbit of z for $t \in [a, b]$ is included in the rectangle $(Mr_1, R_1) \times (Mr_2, R_2)$.

Also, if we know for example that $|z_i|_\infty = \alpha_i$, then $z_i(t) \leqslant \alpha_i$ for all $t \in [0, 1]$ and

$$M\alpha_i \leqslant z_i(t) \leqslant \alpha_i \quad \text{for all } t \in [a,b].$$

We claim that for every $z \in K_{r,R}$ and $i \in \{1, 2\}$, the following properties hold:

$$\begin{aligned}
&|z_i|_\infty = \alpha_i \quad \text{implies} \quad N_i(z) - z_i \notin K_i,\\
&|z_i|_\infty = \beta_i \quad \text{implies} \quad z_i - N_i(z) \notin K_i
\end{aligned} \tag{2.5}$$

guaranteeing the applicability of Theorem 1.1.

Indeed, if $|z_1|_\infty = \alpha_1$ and we would have that $N_1(z) - z_1 \in K_1$, then

$$z_1(t) \leqslant N_1(z)(t) \leqslant \Gamma_1 \int_0^1 G(t,s)q_1(s)\,ds \leqslant B_1\Gamma_1$$

for all $t \in [0, 1]$. This yields the contradiction $\alpha_1 < \alpha_1$. Now if $|z_1|_\infty = \beta_1$ and $z_1 - N_1(z) \in K_1$, then we obtain

$$z_1(t_1^*) \geqslant N_1(z)(t_1^*) \geqslant \int_a^b G(t_1^*,s)q_1(s)g_1(z_1(s), z_2(s))\,ds \geqslant A_1\gamma_1,$$

where $t_1^* \in [0, 1]$ is such that $A_1 = \int_a^b G(t_1^*, s)q_1(s)\,ds$. This implies $\beta_1 > \beta_1$, a contradiction. Hence (2.5) holds for $i = 1$. Similarly, (2.5) is true for $i = 2$. □

In particular, if g_1, g_2 have some monotonicity properties in z_1 and z_2, for $z_1 \in [0, R_1]$ and $z_2 \in [0, R_2]$, then we can precise the numbers $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$. For example,

(1) if g_1, g_2 are increasing in z_1 and z_2, then

$$\Gamma_1 = g_1(\alpha_1, R_2), \qquad \gamma_1 = g_1(M\beta_1, Mr_2),$$
$$\Gamma_2 = g_2(R_1, \alpha_2), \qquad \gamma_2 = g_2(Mr_1, M\beta_2);$$

(2) if g_1 is increasing in z_1 and z_2, g_2 is increasing in z_1 and decreasing in z_2, then

$$\Gamma_1 = g_1(\alpha_1, R_2), \qquad \gamma_1 = g_1(M\beta_1, Mr_2),$$
$$\Gamma_2 = g_2(R_1, 0), \qquad \gamma_2 = g_2(Mr_1, \beta_2);$$

(3) if g_1 is increasing in z_1 and decreasing in z_2, g_2 is decreasing in z_1 and increasing in z_2, then

$$\Gamma_1 = g_1(\alpha_1, 0), \qquad \gamma_1 = g_1(M\beta_1, R_2)$$
$$\Gamma_2 = g_2(0, \alpha_2), \qquad \gamma_2 = g_2(R_1, M\beta_2).$$

Notice that conditions (2.4) indicate the behavior of g_1, g_2 in some regions of $\mathbf{R}_+^2$, in order to establish the existence and the localization of at least one solution. Combined with monotonicity properties like those in (1)–(3), the hypotheses (2.4) show us how the nonlinearities g_1, g_2 behave at four points in $\mathbf{R}_+^2$. Under more restrictive monotonicity conditions on g_1, g_2 we can also prove the uniqueness of solution as shows the next theorem.

For the next result we say that g_i is increasing in both variables on $(0, R_1) \times (0, R_2)$ if

$$\begin{cases} 0 < u_1 \leqslant \bar{u}_1 < R_1, \\ 0 < u_2 \leqslant \bar{u}_2 < R_2 \end{cases} \Rightarrow \quad g_i(u_1, u_2) \leqslant g_i(\bar{u}_1, \bar{u}_2).$$

Also we say that the $\frac{g_1(u_1,u_2)}{u_1}$ is strictly increasing (decreasing) on $(0, R_1) \times (0, R_2)$ if

$$\begin{cases} 0 < u_1 < \bar{u}_1 < R_1, \\ 0 < u_2 \leqslant \bar{u}_2 < R_2 \end{cases} \Rightarrow \quad \frac{g_1(u_1, u_2)}{u_1} < (>) \frac{g_1(\bar{u}_1, \bar{u}_2)}{\bar{u}_1}.$$

Similarly, $\frac{g_2(u_1,u_2)}{u_2}$ is said to be strictly increasing (decreasing) on $(0, R_1) \times (0, R_2)$ if

$$\begin{cases} 0 < u_1 \leqslant \bar{u}_1 < R_1, \\ 0 < u_2 < \bar{u}_2 < R_2 \end{cases} \Rightarrow \quad \frac{g_2(u_1, u_2)}{u_2} < (>) \frac{g_1(\bar{u}_1, \bar{u}_2)}{\bar{u}_2}.$$

Theorem 2.2. *Assume that there exist* $0 < R_1, R_2 \leqslant \infty$ *such that* g_1, g_2 *are increasing in both variables and* $\frac{g_1(u_1,u_2)}{u_1}$, $\frac{g_2(u_1,u_2)}{u_2}$ *are strictly monotone on* $(0, R_1) \times (0, R_2)$. *Then problem* (1.5)–(1.6) *has at most one positive solution* $z = (z_1, z_2)$ *satisfying* $|z_i|_\infty < R_i$, $i = 1, 2$.

Proof. Assume that $z = (z_1, z_2)$ and $\bar{z} = (\bar{z}_1, \bar{z}_2)$ are two distinct positive solutions of (1.5)–(1.6) with $|z_i|_\infty < R_i$ and $|\bar{z}_i|_\infty < R_i$ for $i = 1, 2$. We may assume that $z_1 \leqslant \bar{z}_1$ and $z_2 \leqslant \bar{z}_2$. Indeed, otherwise, if we let

$$u_i(t) = \min\{z_i(t), \bar{z}_i(t)\} \quad (i = 1, 2), \qquad u = (u_1, u_2),$$

and we take into account that g_i is increasing in both variables, we obtain

$$N_i(u)(t) = \int_0^1 G(t, s) q_i(s) g_i\big(u_1(s), u_2(s)\big)\, ds \leqslant \int_0^1 G(t, s) q_i(s) g_i\big(z_1(s), z_2(s)\big)\, ds = z_i(t)$$

and similarly $N_i(u)(t) \leqslant \bar{z}_i(t)$. Then $N_i(u) \leqslant u_i$, $i = 1, 2$. Consequently, for each $i \in \{1, 2\}$, the sequence $(N_i^k(u))_k$ decreases to a positive function z_i^*, $z_i^* \leqslant u_i$, and $z^* = N(z^*)$, where $z^* = (z_1^*, z_2^*)$. Thus we may replace the couple of distinct solutions $[z, \bar{z}]$ by an ordered couple of distinct solutions, namely by $[z^*, z]$ or $[z^*, \bar{z}]$. This proves our claim.

Since $z, \bar{z}$ are distinct, there exists $i \in \{1, 2\}$ and a subinterval $[\alpha, \beta]$ of $[0, 1]$ with $z_i(t) < \bar{z}_i(t)$ on (α, β). Let

$$u_i(t) = z_i(t)\bar{z}_i'(t) - z_i'(t)\bar{z}_i(t).$$

Clearly, $u_i \in C^1(0,1) \cap C[0,1]$ and $u_i(0) = u_i(1) = 0$. Also, for $t \in (0,1)$, one has

$$u_i'(t) = z_i(t)\bar{z}_i''(t) - z_i''(t)\bar{z}_i(t) = q_i(t)z_i(t)\bar{z}_i(t)\left(\frac{g_i(z(t))}{z_i(t)} - \frac{g_i(\bar{z}(t))}{\bar{z}_i(t)}\right).$$

Since $\frac{g_i(z)}{z_i}$ is strictly monotone, we deduce that

$$u_i'(t) \geqslant 0 \quad \text{on } (0,1) \quad \text{and} \quad u_i'(t) > 0 \quad \text{on } (\alpha, \beta),$$

or

$$u_i'(t) \leqslant 0 \quad \text{on } (0,1) \quad \text{and} \quad u_i'(t) < 0 \quad \text{on } (\alpha, \beta),$$

which contradicts $u_i(0) = u_i(1) = 0$. □

Theorem 2.1 immediately yields multiplicity results provided that nonlinearities g_1, g_2 are oscillating functions.

Theorem 2.3. *Assume that there exist a natural number $N \geqslant 1$ and $\alpha_i^k, \beta_i^k > 0$ with $\alpha_i^k \neq \beta_i^k$ for $i = 1,2$ and $k = 1,2,\ldots,N$, such that*

$$R_1^k \leqslant r_1^{k+1} \quad \text{or} \quad R_2^k \leqslant r_2^{k+1} \tag{2.6}$$

for $k = 1,2,\ldots,N-1$, and

$$B_1\Gamma_1^k < \alpha_1^k, \qquad A_1\gamma_1^k > \beta_1^k,$$
$$B_2\Gamma_2^k < \alpha_2^k, \qquad A_2\gamma_2^k > \beta_2^k$$

for $k = 1,2,\ldots,N$. Here $r_i^k = \min\{\alpha_i^k, \beta_i^k\}$, $R_i^k = \max\{\alpha_i^k, \beta_i^k\}$ and γ_i^k, Γ_i^k are defined by (2.3)*, correspondingly. Then* (1.5)–(1.6) *has at least N distinct positive solutions $z^k = (z_1^k, z_2^k)$ with $r_i^k < |z_i^k|_\infty < R_i^k$ for $i = 1,2$ and $k = 1,2,\ldots,N$.*

Proof. Apply Theorem 2.1 for each $k \in \{1,2,\ldots,N\}$ to obtain a positive solution z^k satisfying

$$r_i^k < \left|z_i^k\right|_\infty < R_i^k, \quad i = 1,2. \tag{2.7}$$

From (2.6), we have that for each $k \in \{1,2,\ldots,N-1\}$,

$$\left(r_i^k, R_i^k\right) \cap \left(r_i^{k+1}, R_i^{k+1}\right) = \emptyset \quad \text{for } i = 1 \text{ or } i = 2. \tag{2.8}$$

Now (2.7) and (2.8) guarantee that z^k, $k = 1,2,\ldots,N$, are distinct solutions. □

Remark 2.1. In particular, the previous theorems established for a system reduce to results for a scalar equation. Indeed, the boundary value problem for a scalar equation

$$\begin{cases} z''(t) + q(t)g\big(z(t)\big) = 0, \\ z(0) = z(1) = 0 \end{cases} \tag{2.9}$$

can be viewed as a problem of type (1.5)–(1.6) if we take $q_i(t) = q(t)$ for $i = 1,2$, $g_1(z_1,z_2) = g(z_1)$ and $g_2(z_1,z_2) = g(z_2)$.

3. Positive radial solutions

Theorem 2.1 yields the following existence and localization result of positive radial solutions to problem (1.1)–(1.2).

Theorem 3.1. *Assume that* $g_i \in C(\mathbf{R}_+^2; \mathbf{R}_+)$, $f_i \in C([r_0, \infty); (0, \infty))$ *and*

$$\int_{r_0}^{\infty} \tau^{n-1} f_i(\tau)\, d\tau < \infty \tag{3.1}$$

for $i = 1, 2$. *In addition assume that there exist* $\alpha_i, \beta_i > 0$ *with* $\alpha_i \neq \beta_i$, $i = 1, 2$, *such that* (2.4) *holds. Then* (1.1)–(1.2) *has at least one positive radial solution* $u = (u_1, u_2)$ *with* $r_i < \sup_{|x| \geqslant r_0} |u_i(x)| < R_i$ *for* $i = 1, 2$.

Proof. Notice that (3.1) guarantees that functions q_i given by (1.7) belong to $L^1(0, 1)$. Now the result follows from Theorem 2.1. □

From Theorem 2.2 we immediately obtain a uniqueness result for (1.1)–(1.2).

Theorem 3.2. *If in addition to the assumptions of Theorem* 3.1, g_1, g_2 *are increasing in both variables and* $\frac{g_1(u_1,u_2)}{u_1}$, $\frac{g_2(u_1,u_2)}{u_2}$ *are strictly monotone on* $(0, R_1) \times (0, R_2)$, *then* (1.1)–(1.2) *has a unique positive radial solution* $u = (u_1, u_2)$ *satisfying* $r_i < \sup_{|x| \geqslant r_0} |u_i(x)| < R_i$ *for* $i = 1, 2$.

Proof. Apply Theorems 3.1 and 2.2. □

Finally, Theorem 2.3 implies the following multiplicity result for (1.1)–(1.2).

Theorem 3.3. *Assume that* g_i *and* f_i *are as in Theorem* 3.1 *and that there exist numbers* N, α_i^k *and* β_i^k, $k = 1, 2, \ldots, N$, *satisfying all the conditions from Theorem* 2.3. *Then* (1.1)–(1.2) *has at least* N *distinct positive radial solutions* $u^k = (u_1^k, u_2^k)$ *with* $r_i^k < \sup_{|x| \geqslant r_0} |u_i^k(x)| < R_i^k$ *for* $i = 1, 2$ *and* $k = 1, 2, \ldots, N$.

The next theorems can be viewed as examples of applicability of the previous results. For all these theorems we assume that the functions f_i, $i = 1, 2$, are like in Theorem 3.1.

Theorem 3.4. *Let* $g_1(u_1, u_2)$, $g_2(u_1, u_2)$ *be nondecreasing in* u_1 *and* u_2 *for* $u_1, u_2 \in \mathbf{R}_+$. *If*

$$\lim_{x \to \infty} \frac{g_i(x, x)}{x} = 0 \quad \textit{and} \quad \lim_{x \to 0} \frac{g_i(x, x)}{x} = \infty \tag{3.2}$$

for $i = 1, 2$, *then* (1.1)–(1.2) *has at least one positive radial solution.*

Proof. From (3.2) there are α_1, β_1 with $0 < \beta_1 < \alpha_1$ such that

$$\frac{g_i(\alpha_1, \alpha_1)}{\alpha_1} < \frac{1}{B_i}, \qquad \frac{g_i(M\beta_1, M\beta_1)}{M\beta_1} > \frac{1}{MA_i} \tag{3.3}$$

for $i = 1, 2$. Let $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. Then $r_i = \beta_1, R_i = \alpha_1$, and according to (1), $\Gamma_i = g_i(\alpha_1, \alpha_1)$, $\gamma_i = g_i(M\beta_1, M\beta_1)$ for $i = 1, 2$. Now (3.3) guarantees (2.4). □

Example 3.1. The functions $g_i(u_1, u_2) = (u_1 u_2)^{\frac{1}{3}}$, $i = 1, 2$, satisfy the conditions of Theorem 3.4.

Theorem 3.5. *Let* $g_1(u_1, u_2)$, $g_2(u_1, u_2)$ *be nondecreasing in* u_1 *and* u_2 *for* $u_1, u_2 \in \mathbf{R}_+$. *Assume that*

$$\lim_{x \to \infty} \frac{g_2(x, x)}{x} = 0, \qquad \lim_{x \to 0} \frac{g_2(x, x)}{x} = \infty, \tag{3.4}$$

$$\lim_{x \to \infty} \frac{g_1(x, 0)}{x} = \infty \quad \textit{and} \quad \lim_{x \to 0} \frac{g_1(x, y)}{x} = 0 \quad \textit{for every } y > 0. \tag{3.5}$$

Then (1.1)–(1.2) *has at least one positive radial solution.*

Proof. From (3.4) there are $\alpha_0, \beta_0 > 0$ such that

$$\frac{g_2(M\alpha_1, M\alpha_1)}{M\alpha_1} > \frac{1}{MA_2} \quad \text{and} \quad \frac{g_2(\beta_1, \beta_1)}{\beta_1} < \frac{1}{B_2} \tag{3.6}$$

for every $\alpha_1 \leqslant \alpha_0$ and $\beta_1 \geqslant \beta_0$. Let $\alpha_1 < \beta_1$, $\alpha_2 = \beta_1$ and $\beta_2 = \alpha_1$. Then $r_i = \alpha_1$, $R_i = \beta_1$ for $i = 0, 1$, and according to (1), $\Gamma_1 = g_1(\alpha_1, \beta_1)$, $\Gamma_2 = g_2(\beta_1, \beta_1)$, $\gamma_1 = g_1(M\beta_1, M\alpha_1)$ and $\gamma_2 = g_2(M\alpha_1, M\alpha_1)$. Clearly (3.6) guarantees that the inequalities in (2.4) corresponding to $i = 2$ hold for every $\alpha_1 \leqslant \alpha_0$ and $\beta_1 \geqslant \beta_0$. Now due to (3.5), since

$$\frac{g_1(M\beta_1, M\alpha_1)}{M\beta_1} \geqslant \frac{g_1(M\beta_1, 0)}{M\beta_1},$$

we may first choose $\beta_1 \geqslant \beta_0$ with $\frac{g_1(M\beta_1,0)}{M\beta_1} > \frac{1}{MA_1}$, and then $\alpha_1 \leqslant \alpha_0$, $0 < \alpha_1 < \beta_1$ with $\frac{g_1(\alpha_1,\beta_1)}{\alpha_1} < \frac{1}{B_1}$. Thus condition (2.4) is satisfied. □

Example 3.2. The functions $g_1(u_1, u_2) = u_1^2(1 + u_2^2)$, $g_2(u_1, u_2) = (u_1 u_2)^{\frac{1}{3}}$ satisfy the conditions of Theorem 3.5.

Theorem 3.6. *Let $g_1(u_1, u_2)$ be nondecreasing in u_1 and nonincreasing in u_2, and $g_2(u_1, u_2)$ be nonincreasing in u_1 and nondecreasing in u_2, for $u_1, u_2 \in \mathbf{R}_+$. Assume that*

$$\begin{aligned} &\lim_{x\to 0} \frac{g_1(x, Mx)}{x} = 0, \qquad \lim_{x\to\infty} \frac{g_1(Mx, x)}{x} = \infty, \\ &\lim_{x\to 0} \frac{g_2(Mx, x)}{x} = 0, \qquad \lim_{x\to\infty} \frac{g_2(x, Mx)}{x} = \infty. \end{aligned} \tag{3.7}$$

Then (1.1)–(1.2) *has at least one positive radial solution.*

Proof. From (3.7) it follows that there exist α_1 and β_1 with $0 < \alpha_1 < \beta_1$ such that

$$\begin{aligned} &\frac{g_1(\alpha_1, M\alpha_1)}{\alpha_1} < \frac{1}{B_1}, \qquad \frac{g_1(M\beta_1, \beta_1)}{\beta_1} > \frac{1}{A_1}, \\ &\frac{g_2(M\alpha_1, \alpha_1)}{\alpha_1} < \frac{1}{B_2}, \qquad \frac{g_2(\beta_1, M\beta_1)}{\beta_1} > \frac{1}{A_2}. \end{aligned} \tag{3.8}$$

Let $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. Then $r_i = \alpha_1$, $R_i = \beta_1$ for $i = 1, 2$. Also, by (3), $\Gamma_1 = g_1(\alpha_1, M\alpha_1)$, $\Gamma_2 = g_2(M\alpha_1, \alpha_1)$, $\gamma_1 = g_1(M\beta_1, \beta_1)$ and $\gamma_2 = g_2(\beta_1, M\beta_1)$. Now (3.8) guarantees (2.4). □

Example 3.3. The functions $g_1(u_1, u_2) = \frac{u_1^3}{u_2+1}$, $g_2(u_1, u_2) = \frac{u_2^3}{u_1+1}$ satisfy the conditions of Theorem 3.6.

Theorem 3.7. *Let $s, t, p, q \in \mathbf{R}_+$ satisfy $t > s + 1$ and $q > p + 1$. Then the system*

$$\begin{cases} \Delta u_1 + f_1(|x|) u_1^t u_2^s = 0, \\ \Delta u_2 + f_2(|x|) u_1^p u_2^q = 0 \end{cases}$$

has a unique positive radial solution satisfying (1.2).

Proof. In this case $g_1(u_1, u_2) = u_1^t u_2^s$, $g_2(u_1, u_2) = u_1^p u_2^q$ are increasing in both variables and since $t, q > 1$, the functions $\frac{g_1(u_1,u_2)}{u_1}$, $\frac{g_2(u_1,u_2)}{u_2}$ are strictly monotone on $(0, \infty) \times (0, \infty)$. Thus the problem has at most one positive radial solution. For the existence, we only have to find numbers $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, such that (2.4) holds. We shall look for these numbers such that $\alpha_i < \beta_i$, $i = 1, 2$. Take $\beta_i = \theta\alpha_i$, with any $\theta > 1$ large enough that

$$\theta^{t-s-1} > \frac{B_1}{A_1 M^{t+s}} \quad \text{and} \quad \theta^{q-p-1} > \frac{B_2}{A_2 M^{p+q}}.$$

Then we may choose numbers c_1, c_2 such that

$$\frac{1}{A_1 M^{t+s} \theta^{t-1}} < c_1 < \frac{1}{B_1 \theta^s}, \qquad \frac{1}{A_2 M^{p+q} \theta^{q-1}} < c_2 < \frac{1}{B_2 \theta^p}. \tag{3.9}$$

Now we let α_1, α_2 be the solution of the system

$$\alpha_1^{t-1}\alpha_2^{s} = c_1, \qquad \alpha_1^{p}\alpha_2^{q-1} = c_2. \tag{3.10}$$

Notice this system is solvable. To see this, we rewrite it as

$$\alpha_1^{(t-1)p}\alpha_2^{sp} = c_1^{p}, \qquad \alpha_1^{(t-1)p}\alpha_2^{(t-1)(q-1)} = c_2^{t-1}.$$

Then

$$\alpha_2^{(t-1)(q-1)-sp} = c_2^{t-1}c_1^{-p}. \tag{3.11}$$

Here the exponent $(t-1)(q-1) - sp \neq 0$ since $t-1 > s$ and $q-1 > p$. Hence α_2 is uniquely defined by (3.11). Then α_1 follows uniquely from the first equation in (3.10), since $t-1 > 0$. Now (3.9) and (3.10) give

$$\begin{aligned} &B_1\alpha_1^{t-1}\alpha_2^{s}\theta^{s} < 1, \qquad A_1M^{t+s}\theta^{t-1}\alpha_1^{t-1}\alpha_2^{s} > 1,\\ &B_2\alpha_1^{p}\theta^{p}\alpha_2^{q-1} < 1, \qquad A_2M^{p+q}\theta^{q-1}\alpha_1^{p}\alpha_2^{q-1} > 1. \end{aligned} \tag{3.12}$$

Since in our case $r_i = \alpha_i$, $R_i = \theta\alpha_i$ for $i = 1, 2$, and

$$\begin{aligned} &\Gamma_1 = g_1(\alpha_1, \theta\alpha_2), \qquad \gamma_1 = g_1(M\theta\alpha_1, M\alpha_2),\\ &\Gamma_2 = g_2(\theta\alpha_1, \alpha_2), \qquad \gamma_2 = g_2(M\alpha_1, M\theta\alpha_2), \end{aligned}$$

we easily see that (3.12) is equivalent to condition (2.4). □

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