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The role of matrices that are convergent to zero in the study of semilinear operator systems

Radu Precup

Department of Applied Mathematics, Babeş–Bolyai University, 400084 Cluj, Romania

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ABSTRACT

In this paper we explain the advantage of vector-valued norms and the role of matrices that are convergent to zero in the study of semilinear operator systems by means of some basic methods of nonlinear analysis: the contraction principle, Schauder's fixed point theorem, the Leray–Schauder continuation principle and Krasnoselskii's cone fixed point theorem. A vector version of Krasnoselskii's theorem is also established.

1. Introduction

In this paper we are concerning with the solvability of the semilinear operator system

$$\begin{cases} N_1(u_1, u_2) = u_1 \\ N_2(u_1, u_2) = u_2 \end{cases} \tag{1}$$

in a Banach space X with norm $|.|$. Here $N_i : X^2 \to X$ $(i = 1, 2)$ are given nonlinear operators. Systems of this type arise from mathematical modelling of many processes from a variety of disciplines, including physics, biology, chemistry, engineering and other sciences. Thus, initial value problems and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [4] and mathematical economics [3] can be put in the operator form (1).

It is obvious that system (1) can be viewed as a fixed point problem:

$$N(u) = u \tag{2}$$

in the space X^2, where $u = (u_1, u_2)$ and $N = (N_1, N_2)$. Therefore, we may think of applying to (2), in X^2 endowed with a norm induced by the norm of X, different abstract existence results from nonlinear functional analysis, such as the Banach contraction principle, the Schauder fixed point theorem, the Leray–Schauder continuation principle, Krasnoselskii's cone fixed point theorem, and so on. The aim of this paper is to point out that better results can be obtained for system (1) if in X^2 we consider the vector-valued norm

$$||u|| = \begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} \tag{3}$$

E-mail address: r.precup@math.ubbcluj.ro.

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for $u = (u_1, u_2) \in X^2$, instead of a usual scalar norm such as

$$|u|_l = |u_1| + |u_2|,$$
$$|u|_m = \max\{|u_1|, |u_2|\}, \text{ or}$$
$$|u|_e = \sqrt{|u_1|^2 + |u_2|^2}.$$

Historically, this was shown by Perov and Kibenko [7] (see also [1,8,9]) in connection with the contraction principle.

Theorem 1 (*Perov*). *Let (E, d) be a complete generalized metric space with $d : E \times E \to \mathbf{R}^n$ and let $N : E \to E$ be such that*

$$d(N(u), N(v)) \leq M\, d(u, v)$$

for all $u, v \in E$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \to 0$ as $k \to \infty$, then N has a unique fixed point u and

$$d\left(N^k(v), u\right) \leq M^k (I - M)^{-1} d(N(v), v)$$

for every $v \in E$ and $k \geq 1$.

Let us note the following properties of matrices that are convergent to zero:

Lemma 2. *Let M be a square matrix of nonnegative numbers. The following statements are equivalent:*

(i) *M is a matrix convergent to zero.*
(ii) *$I - M$ is non-singular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots.$$

(iii) $|\lambda| < 1$ *for every* $\lambda \in \mathbf{C}$ *with* $\det(M - \lambda I) = 0$.
(iv) *$I - M$ is non-singular and $(I - M)^{-1}$ has nonnegative elements.*

Proof. The equivalence of (i), (ii) and (iii) is well known; see for example [9,8]. Also the implication from (ii) to (iv) is obvious. Thus the proof will be complete if we show that (iv) implies (i). Assume (iv). Then from the identity

$$S_k(I - M) = I - M^{k+1}$$

which is true for $S_k = I + M + \cdots + M^k$, since M and $(I - M)^{-1}$ have nonnegative elements, we deduce that

$$\begin{aligned} S_k &= \left(I - M^{k+1}\right)(I - M)^{-1} \\ &= (I - M)^{-1} - M^{k+1}(I - M)^{-1} \\ &\leq (I - M)^{-1}. \end{aligned}$$

Thus $(S_k)_{k \geq 1}$ is a bounded sequence. Since it is nondecreasing (on elements) by its definition, we deduce that it is convergent. As a consequence $M^k \to 0$ as $k \to \infty$. ■

We conclude this introduction by three other well-known abstract results of nonlinear functional analysis. For proofs and more information we refer the reader to [2,6].

Theorem 3 (*Schauder*). *Let E be a Banach space, D a nonempty closed bounded and convex subset of E, and $N : D \to D$ a completely continuous operator. Then N has at least one fixed point.*

Theorem 4 (*Leray–Schauder*). *Let E be a Banach space, U a bounded open subset of E with $0 \in U$, and $N : \overline{U} \to E$ a completely continuous operator. If $u \neq \lambda N(u)$ for all $u \in \overline{U} \setminus U$ and $\lambda \in (0, 1)$, then N has at least one fixed point.*

Theorem 5 (*Krasnoselskii*). *Let $(E, |.|_0)$ be a Banach space, W a proper wedge of E (i.e., a closed convex set satisfying $\lambda W \subset W$ for all $\lambda \geq 0$ and which is not a linear subspace of E) and let $\alpha, \beta > 0$ with $\alpha \neq \beta$. Assume that $N : W \to W$ is a completely continuous operator such that*

$$x \neq \lambda N(x) \quad \text{for } |x|_0 = a \quad \text{and} \quad \lambda \in (0, 1), \tag{4}$$
$$x \neq \lambda N(x) \quad \text{for } |x|_0 = \beta \quad \text{and} \quad \lambda \in (1, \infty), \tag{5}$$

and

$$\inf\{|N(x)|_0 : |x|_0 = \beta\} > 0. \tag{6}$$

Then N has at least one fixed point x with

$$\min\{\alpha, \beta\} \leq |x|_0 \leq \max\{\alpha, \beta\}.$$

Notice that a sufficient condition for (4) is that

$$|N(x)|_0 \le |x|_0 \quad \text{for } |x|_0 = \alpha. \tag{7}$$

Also, a sufficient condition for (5) and (6) is that

$$|N(x)|_0 \ge |x|_0 \quad \text{for } |x|_0 = \beta. \tag{8}$$

2. Application of Perov's fixed point theorem

First we shall present an application of Perov's theorem to system (1).

Theorem 6. *Assume that for each $i \in \{1, 2\}$, there exist nonnegative numbers a_i and b_i such that*

$$|N_i(u_1, u_2) - N_i(v_1, v_2)| \le a_i |u_1 - v_1| + b_i |u_2 - v_2| \tag{9}$$

for all $u_1, u_2, v_1, v_2 \in X$. In addition assume that

$$M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \text{ is a convergent to zero matrix.} \tag{10}$$

Then (1) has a unique solution $u = (u_1, u_2)$ in X^2 and $N_i^k(v) \to u_i$ as $k \to \infty$ for every $v \in X^2$ and $i = 1, 2$.

Proof. Condition (9) can be rewritten as

$$\|N(u) - N(v)\| \le M \|u - v\|.$$

Thus Perov's fixed point theorem applies. Here $E := X^2$ and $d(u, v) := \|u - v\|$. ■

3. Application of Schauder's fixed point theorem

Now we shall relax assumption (9) to an at most linear growth condition, but we shall require instead a compactness property for N_i.

Theorem 7. *Assume that for each $i \in \{1, 2\}$, the operator N_i is completely continuous and that there exist nonnegative numbers a_i, b_i and c_i such that*

$$|N_i(u_1, u_2)| \le a_i |u_1| + b_i |u_2| + c_i \tag{11}$$

for all $u_1, u_2 \in X$. In addition assume that condition (10) is satisfied. Then (1) has at least one solution $u = (u_1, u_2)$ with

$$\begin{bmatrix} |u_1| \\ |u_2| \end{bmatrix} \le (I - M)^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{12}$$

Proof. Condition (11) can be written in a matrix form as

$$\begin{bmatrix} |N_1(u)| \\ |N_2(u)| \end{bmatrix} \le M \begin{bmatrix} |u_1| \\ |u_2| \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{13}$$

We shall apply Schauder's fixed point theorem to the restriction of N to a subset of X^2, of the form

$$D := \left\{ u = (u_1, u_2) \in X^2 : |u_1| \le R_1 \text{ and } |u_2| \le R_2 \right\}.$$

Thus the existence problem reduces to the invariance condition $N(D) \subset D$. Therefore, we have to find two nonnegative numbers R_1, R_2 such that

$$|u_1| \le R_1,\ |u_2| \le R_2 \quad \text{imply } |N_1(u)| \le R_1,\ |N_2(u)| \le R_2.$$

According to (13), if $|u_i| \le R_i$ for $i = 1, 2$, then

$$\begin{bmatrix} |N_1(u)| \\ |N_2(u)| \end{bmatrix} \le M \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Hence it would be enough that

$$M \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

that is

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = (I - M)^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{14}$$

Notice that (14) gives us nonnegative constants R_1, R_2, since $c_1, c_2 \geq 0$ and $(I - M)^{-1}$ has nonnegative elements as follows from Lemma 2(iv). ■

4. Application of the Leray–Schauder principle

In this section, we assume that X is a Hilbert space with inner product $\langle .,. \rangle$ and norm $|.|$ and we shall refine the result of Theorem 6 assuming that the operators N_i, $i = 1, 2$, split into $N_i' + N_i''$ with N_i' satisfying an at most linear growth condition of type (11) and N_i'' satisfying a sign-type condition. No growth conditions will be required for N_i''.

Theorem 8. *Assume that for each $i \in \{1, 2\}$, N_i is completely continuous and splits into $N_i' + N_i''$, where*

$$\left|N_i'(u)\right| \leq a_i |u_1| + b_i |u_2| + c_i \tag{15}$$

$$\left\langle N_i''(u), u_i \right\rangle \leq 0 \tag{16}$$

for all $u = (u_1, u_2) \in X^2$ and some nonnegative constants a_i, b_i and c_i. In addition assume that condition (10) is satisfied. Then (1) has at least one solution $u = (u_1, u_2)$ in X^2, and any solution of (1) satisfies (12).

Proof. According to the Leray–Schauder continuation principle, it suffices to show that the set of all solutions in X^2 to the equations

$$\begin{cases} \lambda N_1(u_1, u_2) = u_1 \\ \lambda N_2(u_1, u_2) = u_2 \end{cases} \tag{17}$$

when $\lambda \in [0, 1]$ is bounded. Let $u = (u_1, u_2)$ be any solution of (17). Then, for each $i \in \{1, 2\}$, from (15) and (16), we obtain

$$\begin{aligned} |u_i|^2 &= \lambda \langle N_i(u), u_i \rangle \leq \lambda \left\langle N_i'(u), u_i \right\rangle \leq \left|N_i'(u)\right| |u_i| \\ &\leq (a_i |u_1| + b_i |u_2| + c_i) |u_i|. \end{aligned}$$

Hence

$$|u_i| \leq a_i |u_1| + b_i |u_2| + c_i.$$

These inequalities can be put in the form

$$\begin{bmatrix} |u_1| \\ |u_2| \end{bmatrix} \leq M \begin{bmatrix} |u_1| \\ |u_2| \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which by (10) immediately yields (12). ■

5. Application of Krasnoselskii's theorem

In this section the notation $|.|_0$ stands for any norm on X^2. Also, for a square matrix M of nonnegative numbers, the notation $M \leq 1$ means that μM is convergent to zero for every $\mu \in (0, 1)$, or equivalently, that $|\lambda| \leq 1$ for every $\lambda \in \mathbf{C}$ with $\det(M - \lambda I) = 0$.

Theorem 9. *Let K be a proper wedge of the Banach space $(X, |.|)$ and let $N_i : K^2 \to K$ be completely continuous maps, $i = 1, 2$. Assume that there are $\alpha, \beta \in (0, \infty)$ with $\alpha \neq \beta$, and nonnegative numbers $a_i, b_i, a_i', b_i' (i = 1, 2)$ such that*

$$|N_i(u)| \leq a_i |u_1| + b_i |u_2| \quad \text{for } |u|_0 = \alpha, \ i = 1, 2 \tag{18}$$

$$|u_i| \leq a_i' |N_1(u)| + b_i' |N_2(u)| \quad \text{for } |u|_0 = \beta, \ i = 1, 2. \tag{19}$$

If in addition $M \leq 1$ and $M' \leq 1$ where

$$M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \quad \text{and} \quad M' = \begin{bmatrix} a_1' & b_1' \\ a_2' & b_2' \end{bmatrix}$$

then (1) has at least one solution $u = (u_1, u_2) \in K \times K$ satisfying

$$\min\{\alpha, \beta\} \leq |u|_0 \leq \max\{\alpha, \beta\}.$$

Proof. Notice that conditions (18) and (19) can be written in the form

$$\|N(u)\| \leq M\|u\| \quad \text{for } |u|_0 = \alpha \tag{20}$$

and respectively

$$M'\|N(u)\| \geq \|u\| \quad \text{for } |u|_0 = \beta. \tag{21}$$

First we show that $u \neq \lambda N(u)$ for all $u \in K^2$ with $|u|_0 = \alpha$ and $\lambda \in (0, 1)$. Indeed, otherwise, if $u = \lambda N(u)$ for some u with $|u|_0 = \alpha$ and $\lambda \in (0, 1)$, then $\|u\| = \lambda\|N(u)\| \leq \lambda M\|u\|$. Thus $(I - \lambda M)\|u\| \leq 0$. Since λM is convergent to zero, we deduce that $\|u\| \leq (I - \lambda M)^{-1} 0 = 0$, whence $u = 0$. This contradicts $|u|_0 = \alpha > 0$.

Next we prove that $u \neq \lambda N(u)$ for all $u \in K^2$ with $|u|_0 = \beta$ and $\lambda \in (1, \infty)$. Indeed, if $u = \lambda N(u)$ for some u with $|u|_0 = \beta$ and $\lambda \in (1, \infty)$, then $\|u\| = \lambda\|N(u)\|$ and so $M'\|u\| = \lambda M'\|N(u)\| \geq \lambda\|u\|$. Consequently, $\left(I - \frac{1}{\lambda}M'\right)\|u\| \leq 0$, whence $\|u\| \leq \left(I - \frac{1}{\lambda}M'\right)^{-1} 0$. Hence $u = 0$, which contradicts $|u|_0 = \beta > 0$.

Finally, we note that (21) guarantees that $\inf\{|N(u)|_0 : |u|_0 = \beta\} > 0$. Therefore we may apply Krasnoselskii's theorem to $E = X^2$ and $W = K^2$. ■

6. The vector-valued norm versus scalar norms

The aim of this section is to show that the results in Sections 2–4, obtained by using the vector-valued norm (3), are better than those established by means of any scalar norm in X^2.

1^0. Let us first consider the scalar norm $|u|_l = |u_1| + |u_2|$. Then, if N_1, N_2 satisfy the Lipschitz conditions (9), we obtain

$$|N(u) - N(v)|_l \leq \max\{a_1 + a_2, b_1 + b_2\}\,|u - v|_l \tag{22}$$

for all $u, v \in X^2$. Similarly, if N_1, N_2 satisfy (11), then

$$|N(u)|_l \leq \max\{a_1 + a_2, b_1 + b_2\}\,|u|_l + c_1 + c_2. \tag{23}$$

Finally, if N_1, N_2 are as in Theorem 7, and u is any solution of (17), then

$$|u|_l \leq \max\{a_1 + a_2, b_1 + b_2\}\,|u|_l + c_1 + c_2. \tag{24}$$

Thus, the Banach contraction principle, Schauder's fixed point theorem and the Leray–Schauder continuation principle can be applied, provided that

$$\alpha := \max\{a_1 + a_2, b_1 + b_2\} < 1. \tag{25}$$

2^0. If in X^2 we consider the scalar norm $|u|_m = \max\{|u_1|, |u_2|\}$, then the corresponding formulas for (22)–(24) are, respectively,

$$|N(u) - N(v)|_m \leq \max\{a_1 + b_1, a_2 + b_2\}\,|u - v|_m$$
$$|N(u)|_m \leq \max\{a_1 + b_1, a_2 + b_2\}\,|u|_m + \max\{c_1, c_2\}$$
$$|u|_m \leq \max\{a_1 + b_1, a_2 + b_2\}\,|u|_m + \max\{c_1, c_2\}.$$

Thus, with this choice of a scalar norm in X^2, the above three results of nonlinear functional analysis apply provided that

$$\beta := \max\{a_1 + b_1, a_2 + b_2\} < 1. \tag{26}$$

3^0. Similarly, if in X^2 we consider the euclidean norm $|u|_e = \sqrt{|u_1|^2 + |u_2|^2}$, then the corresponding formulas for (22)–(24) are, respectively,

$$|N(u) - N(v)|_e \leq \sqrt{a_1^2 + a_2^2 + b_1^2 + b_2^2}\,|u - v|_e$$
$$|N(u)|_e \leq \sqrt{a_1^2 + a_2^2 + b_1^2 + b_2^2}\,|u|_e + \sqrt{c_1^2 + c_2^2}$$
$$|u|_e \leq \sqrt{a_1^2 + a_2^2 + b_1^2 + b_2^2}\,|u|_e + \sqrt{c_1^2 + c_2^2}.$$

Thus, the applicability condition for the above abstract results is the following inequality:

$$\gamma := a_1^2 + a_2^2 + b_1^2 + b_2^2 < 1. \tag{27}$$

The following examples show that, in general, the condition that M is a matrix convergent to zero is weaker than conditions (25)–(27).

Example 1. Let

$$M = \begin{bmatrix} a & a \\ b & b \end{bmatrix}.$$

Then the characteristic roots of M are $\lambda_1 = 0$ and $\lambda_2 = a + b$. Hence M is convergent to zero if and only if $a + b < 1$. On the other hand,

$$\alpha = a + b, \qquad \beta = \max\{2a, 2b\}, \qquad \gamma = 2\left(a^2 + b^2\right).$$

Hence each one of the conditions $\beta < 1$ and $\gamma < 1$ is more restrictive than the condition that M is convergent to zero.

Example 2. Let

$$M = \begin{bmatrix} a & b \\ a & b \end{bmatrix}.$$

Now the characteristic roots of M are $\lambda_1 = 0$, $\lambda_2 = a + b$ and so again M is convergent to zero if and only if $a + b < 1$. Also, in this case

$$\alpha = \max\{2a, 2b\}, \qquad \beta = a + b, \qquad \gamma = 2\left(a^2 + b^2\right).$$

Hence each one of the conditions $\alpha < 1$ and $\gamma < 1$ is more restrictive than the condition that M is convergent to zero.

Example 3. Let

$$M = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

Then, the characteristic roots of M are $\lambda_1 = a - b$, $\lambda_2 = a + b$ and so again M is convergent to zero if and only if $a + b < 1$. Now

$$\alpha = a + b, \qquad \beta = a + b, \qquad \gamma = 2\left(a^2 + b^2\right).$$

Hence the condition $\gamma < 1$ is more restrictive than the condition that M is convergent to zero.

Example 4. Assume

$$M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

Then, the characteristic roots of M are $\lambda_1 = a$, $\lambda_2 = c$ and M is convergent to zero if and only if $\max\{a, c\} < 1$. Here

$$\alpha = \max\{a, b + c\}, \qquad \beta = \max\{a + b, c\}, \qquad \gamma = a^2 + b^2 + c^2$$

which show that in general, each one of the conditions $\alpha < 1$, $\beta < 1$ and $\gamma < 1$ is more restrictive than the condition that M is convergent to zero.

Similar considerations can be given in connection with Theorem 9. More exactly, one can test that for different choices of the norm $|.|_0$ on X^2, and different types of matrices $M, M' \leq 1$, conditions (20) and (21) are less restrictive than (7) and (8). Therefore, we may conclude that for different types of estimations, the use of the vector-valued norm and, correspondingly, of the matrices convergent to zero, is more appropriate when treating systems of equations.

Notice that our approach can be adapted in order to treat systems of n equations ($n \geq 3$). For related topics we refer the reader to the forthcoming paper [5].

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