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AN ASYMPTOTIC FORMULA FOR A CLASS OF APPROXIMATION PROCESSES OF KING'S TYPE

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Abstract

In this paper we present a general class of linear positive operators of discrete type reproducing the third test function of Korovkin theorem. In a certain weighted space it forms an approximation process. A Voronovskaja-type result is established and particular cases are analyzed.

1. Introduction

Regarding a sequence $(L_n)_{n\geqq 1}$ of positive linear operators defined on the Banach space $C\big([a,b]\big)$, Bohman–Korovkin criterion says: if $(L_n e_k)_{n\geqq 1}$ converges to e_k uniformly on $[a,b]$, $k\in\{0,1,2\}$, for the test functions $e_0(x)=1$, $e_1(x)=x$, $e_2(x)=x^2$, then $(L_n f)_{n\geqq 1}$ converges to f uniformly on $[a,b]$, for each f belonging to $C\big([a,b]\big)$.

Many classical linear positive operators preserve e_0 and e_1, in other words they have the degree of exactness one. In [4] J. P. King defined linear and positive operators which generalize the classical Bernstein operators and reproduce the test functions e_0 and e_2. From Approximation Theory point of view, the construction is useful. In spite of the fact that they have the degree of exactness null, the order of approximation is at least as good as the order of the initial operators.

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In the present paper we indicate a general class of operators $(L_n^*)_{n\geqq 1}$ depending on two sequences of real numbers with the properties $L_n^* e_0 = e_0$ and $L_n^* e_2 = e_2$. After studying its approximation properties in a weighted space, we establish an asymptotic formula. As particular cases, we obtain Voronovskaja-type results for the modified variants in the King sense of some classical approximation processes: Szász–Mirakjan, Baskakov, Bernstein operators.

2. The operators L_n^*

We set $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Following [1], we consider a sequence $(L_n)_{n\geqq 1}$ of linear positive operators of discrete type acting on a subspace of $C(\mathbb{R}_+)$ and defined by

$$(L_n f)(x) = \sum_{k=0}^{\infty} u_{n,k}(x) f(x_{n,k}), \quad x \geqq 0,\ f \in \mathcal{F} \cap E_\alpha, \tag{2.1}$$

where $u_{n,k} \in C(\mathbb{R}_+)$ is a positive real valued function for each $(n,k) \in \mathbb{N} \times \mathbb{N}_0$, $(x_{n,k})_{k\geqq 0}$ is a mesh of nodes on $\mathbb{R}_+$ and

$$\mathcal{F} := \big\{ f : \mathbb{R}_+ \to \mathbb{R}_+ : \ \text{the series in (2.1) is convergent} \big\},$$

$$E_\alpha := \big\{ f \in C(\mathbb{R}_+) : (1+x^\alpha)^{-1} f(x) \text{ is convergent as } x \to \infty \big\},$$

$\alpha \geqq 2$ being fixed. For each $n \in \mathbb{N}$, we assume that the following identities

$$(L_n e_0)(x) = 1, \quad (L_n e_1)(x) = x, \quad (L_n e_2)(x) = a_n x^2 + b_n x, \quad x \geqq 0, \tag{2.2}$$

are fulfilled, where $a_n > 0$, $b_n \geqq 0$. As regards the sequences of real numbers $(a_n)_{n\geqq 1}$ and $(b_n)_{n\geqq 1}$ we assume

$$\lim_{n\to\infty} a_n = 1, \quad \lim_{n\to\infty} b_n = 0. \tag{2.3}$$

Based on Bohman–Korovkin theorem these relations guarantee that $(L_n)_{n\geqq 1}$ is a strong positive approximation process on any compact, $\mathcal{K} \subset \mathbb{R}_+$, this meaning $\lim_{n\to\infty} (L_n f)(x) = f(x)$ uniformly for every $f \in \mathcal{F} \cap E_\alpha$ and every $x \in \mathcal{K}$.

For each $n \in \mathbb{N}$, we define the continuous function $v_n : \mathbb{R}_+ \to \mathbb{R}_+$,

$$v_n(x) = \frac{1}{2a_n}\Big(\sqrt{b_n^2 + 4a_n x^2} - b_n\Big), \quad x \geqq 0. \tag{2.4}$$

Starting from (2.1), we introduce the operators

$$(L_n^* f)(x) = \sum_{k=0}^{\infty} u_{n,k}\big(v_n(x)\big) f(x_{n,k}), \quad x \geqq 0,\ f \in \mathcal{F} \cap E_\alpha,\ n \in \mathbb{N}, \tag{2.5}$$

where v_n is given by (2.4).

Based on (2.2), the following identities

$$L_n^* e_0 = e_0, \quad L_n^* e_1 = v_n, \quad L_n^* e_2 = e_2 \tag{2.6}$$

hold. Relations (2.3) and Bohman–Korovkin criterion imply $\lim_{n\to\infty} L_n^* f = f$ uniformly on compact intervals of $\mathbb{R}_+$ for every $f \in \mathcal{F} \cap E_\alpha$.

Since $(Le_1)^2 \leqq (Le_0)(Le_2)$ is a common property of any linear positive operator L of summation type, relations (2.6) lead us to the following inequality

$$v_n(x) \leqq x, \quad x \geqq 0. \tag{2.7}$$

Endowing the weighted space E_α with the norm $\|\cdot\|_\alpha$,

$$\|f\|_\alpha := \sup_{x \geqq 0} \frac{|f(x)|}{1+x^\alpha}, \quad f \in E_\alpha,$$

an approximation result is indicated below.

THEOREM 2.1. *Let $(L_n^*)_{n\geqq 1}$ be given by (2.5). For every $f \in \mathcal{F} \cap E_\alpha$ ($\alpha \geqq 2$) we have*

$$\lim_{n\to\infty} \|L_n^* f - f\|_\alpha = 0.$$

PROOF. It is known that $\{e_0, e_1, e_2\}$ is a Korovkin set in E_2. On account of (2.6) it is enough to prove

$$\lim_{n\to\infty} \|L_n^* e_1 - e_1\|_\alpha = 0. \tag{2.8}$$

Clearly, for each $x \geqq 0$, one has $x^\alpha - x + 1 > 0$. Let us fix $n \in \mathbb{N}$.

By using (2.6), (2.7) and (2.4), for any $x \geqq 0$ we get

$$\frac{|(L_n^* e_1)(x) - x|}{1+x^\alpha} = \frac{x - v_n(x)}{1+x^\alpha} \leqq \frac{b_n}{a_n} + \frac{|\sqrt{a_n} - 1|}{\sqrt{a_n}}.$$

Based on (2.3), it is evident that the right-hand side in the above estimate tends to 0 as n tends to infinity. This fact implies (2.8) and the proof is finished. □

Theorem 2.1 indicates that the sequence $(L_n^*)_{n\geqq 1}$ furnishes a new strong approximation process on the weighted space E_α, $\alpha \geqq 2$.

We notice the operator L_n^* is non-expansive on the space E_α. Indeed,

$$\frac{1}{1+x^\alpha}\big|(L_n^* f)(x)\big| \leqq \sum_{k=0}^{\infty} u_{n,k}\big(v_n(x)\big)\,\|f\|_\alpha = \|f\|_\alpha,$$

and consequently, $\|L_n^* f\|_\alpha \leqq \|f\|_\alpha$.

Further on, we present the relationship between the local smoothness of f and the local approximation.

Theorem 2.2. *Let $(L_n^*)_{n\geq 1}$ be given by (2.5). Let f be locally* Lip α *on E, where $0<\alpha\leqq 1$ and $E\subset\mathbb{R}_+$. One has*

$$\big|(L_n^* f)(x) - f(x)\big| \leqq c_f\big(\overline{v}_n^{\alpha/2}(x) + 2d^\alpha(x,E)\big), \quad x \geqq 0,$$

where $\overline{v}_n(x) = 2x\big(x - v_n(x)\big)$, $d(x,E)$ represents the distance between x and E and c_f is a constant depending only on f.

Proof. Since f is locally Lip α on E, it satisfies the condition

$$\big|f(x) - f(y)\big| \leqq c_f|x-y|^\alpha, \quad (x,y)\in\mathbb{R}_+\times E.$$

Clearly, this holds for any $x\in\mathbb{R}_+$ and $y\in\overline{E}$, the closure of the set E in $\mathbb{R}$. Let $(x,x_0)\in\mathbb{R}_+\times\overline{E}$ such that $|x-x_0| = d(x,E) = \inf\big\{|x-y| : y\in E\big\}$. Since $\big|f - f(x)\big| \leqq \big|f - f(x_0)\big| + \big|f(x_0) - f(x)\big|$ and L_n^* is a linear positive operator reproducing the constants, we get

$$\begin{aligned}(2.9)\qquad \big|(L_n^* f)(x) - f(x)\big| &\leqq L_n^*\big(\big|f - f(x_0)\big|, x\big) + \big|f(x) - f(x_0)\big| \\ &\leqq L_n^*\big(c_f|e_1 - x_0|^\alpha, x\big) + c_f|x - x_0|^\alpha.\end{aligned}$$

Based on Hölder's inequality, one has $L_n^* h^\alpha \leqq (L_n^* h^2)^{\alpha/2}$ for every function $h\in\mathbb{R}_+^{\mathbb{R}_+}$. Consequently, for every $x\geqq 0$ we deduce

$$(2.10)\qquad L_n^*\big(|e_1 - x|^\alpha, x\big) \leqq \Big(L_n^*\big((e_1 - x)^2, x\big)\Big)^{\alpha/2} = \overline{v}_n^{\alpha/2}(x),$$

see (2.6). Since $|t-x_0| \leqq |t-x| + |x-x_0|$ and L_n^* is monotone, the inequality $(a+b)^\alpha \leqq a^\alpha + b^\alpha$, $a\geqq 0$, $b\geqq 0$, $0<\alpha\leqq 1$, and relation (2.10) imply

$$\begin{aligned}L_n^*\big(c_f|e_1 - x_0|^\alpha, x\big) &\leqq c_f\Big(L_n^*\big(|e_1 - x|^\alpha, x\big) + |x - x_0|^\alpha\Big) \\ &\leqq c_f\big(\overline{v}^{\alpha/2}(x) + |x - x_0|^\alpha\big).\end{aligned}$$

Returning to (2.9), the conclusion follows. □

REMARK 2.3. In [3] the authors had examined the statistical convergence in Approximation Theory establishing some Korovkin type theorems. Our operator L_n^* maps $C_b(\mathbb{R}_+)$ in $C_b(\mathbb{R}_+)$ because of the first identity of relation (2.6). Here $C_b(\mathbb{R}_+)$ denotes the space of all real-valued continuous and bounded functions defined on $\mathbb{R}_+$. On the basis of [3] Theorem 1, relations (2.6) lead us to the following result.

$$\text{If } \operatorname{st-}\lim_n \|v_n - e_1\|_K = 0, \quad \text{then } \operatorname{st-}\lim_n \|L_n^* f - f\|_K = 0,$$

for any function f belonging to $C_b(\mathbb{R}_+)$, where $K \subset \mathbb{R}_+$ is a compact and $\|\cdot\|_K$ is the norm of the uniform convergence on K.

3. The main result

In what follows, let $(\lambda_n)_{n\geqq 1}$ be a sequence of strictly positive real numbers such that

$$\lim_{n\to\infty} \lambda_n = \infty, \quad \lim_{n\to\infty} \lambda_n \frac{a_n - 1}{a_n} = a, \quad \lim_{n\to\infty} \lambda_n \frac{b_n}{a_n} = b, \tag{3.1}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}$, and the sequences $(a_n)_{n\geqq 1}$, $(b_n)_{n\geqq 1}$ have been introduced by (2.2)–(2.3). The sequence $(\lambda_n)_{n\geqq 1}$ plays an important role in an investigation of saturation of these operators.

For each $x \geqq 0$, φ_x is the function defined by $\varphi_x(t) = t - x$, $t \geqq 0$, and $L\varphi_x^s$ ($s \in \mathbb{N}_0$) represents the s-th order central moment of the linear positive operator L.

By $E_\alpha^{(2)}$ we denote the space of all twice continuously differentiable functions $f \in E_\alpha$ such that its derivatives f', f'' belong to E_α.

THEOREM 3.1. *Let $(L_n^*)_{n\geqq 1}$ be defined by* (2.5). *If*

$$\lim_{n\to\infty} \lambda_n^2 \big(L_n^* \varphi_x^4\big)(x) \quad \textit{exists and is finite,} \tag{3.2}$$

then, for any function f belonging to $E_\alpha^{(2)}$, $(\alpha \geqq 4)$, such that $\{f, f', f''\} \subset \mathcal{F}$, one has

$$\lim_{n\to\infty} \lambda_n \big((L_n^* f)(x) - f(x)\big) = \frac{ax+b}{2}\big(xf''(x) - f'(x)\big), \tag{3.3}$$

where $x > 0$ and a, b are given at (3.1).

PROOF. We take a fixed point $x > 0$. For any $f \in E_\alpha^{(2)}$ and $t \geqq 0$ define

$$\phi_f(x;t) = \begin{cases} \dfrac{f(t) - f(x) - (t-x)f'(x) - 2^{-1}(t-x)^2 f''(x)}{(t-x)^2}, & \text{if } \ t \neq x, \\ 0, & \text{if } \ t = x. \end{cases}$$

Since $\lim_{t\to x} \phi_f(x;t) = 0 = \phi_f(x;x)$ and $\{f, f', f''\} \subset E_\alpha \cap \mathcal{F}$, one has $\phi_f(x;\cdot) \in E_\alpha \cap \mathcal{F}$. We can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 \phi_f(x;t).$$

Applying L_n^* and knowing that $L_n^* e_0 = e_0$, we get

$$(L_n^* f)(x) - f(x) = (L_n^* \varphi_x)(x) f'(x) + \frac{1}{2}(L_n^* \varphi_x^2)(x) f''(x) + L_n^*\big(\varphi_x^2 \phi_f(x;\cdot)\big)(x).$$

By using (2.6) we have

$$(L_n^* \varphi_x)(x) = v_n(x) - x, \quad \left(L_n^* \varphi_x^2\right)(x) = 2x\big(x - v_n(x)\big),$$

and hence

$$\lambda_n\big((L_n^* f)(x) - f(x)\big) \tag{3.4}$$
$$= \lambda_n\big(x - v_n(x)\big)\big(xf''(x) - f'(x)\big) + \lambda_n L_n^*\big(\varphi_x^2 \phi_f(x;\cdot)\big)(x).$$

The Cauchy inequality allows us to write

$$\lambda_n L_n^*\big(\varphi_x^2 \phi_f(x;\cdot)\big)(x) \leqq \big\{ L_n^* \phi_f^2(x;\cdot)(x)\big\}^{1/2} \big\{\lambda_n^2 \left(L_n^* \varphi_x^4\right)(x)\big\}^{1/2}. \tag{3.5}$$

Since L_n^* is an approximation process on the space E_α, here $\alpha \geqq 4$, the definition of $\phi_f(x;\cdot)$ implies $\lim_{n\to\infty} L_n^*\big(\phi_f^2(x;\cdot)\big)(x) = 0$. Starting from (3.5), both assumption (3.2) and the previous relation imply

$$\lim_{n\to\infty} \lambda_n L_n^*\big(\varphi_x^2 \phi_f(x;\cdot)\big)(x) = 0.$$

On the other hand, in view of (2.4), (3.1) and (2.3), for any $x > 0$ we get

$$\lim_{n\to\infty} \lambda_n\big(x - v_n(x)\big) = \lim_{n\to\infty} \frac{\lambda_n \frac{a_n - 1}{a_n} x^2 + \lambda_n \frac{b_n}{a_n} x}{x + \frac{b_n}{2a_n} + \sqrt{\left(\frac{b_n}{2a_n}\right)^2 + \frac{x^2}{a_n}}} = \frac{ax + b}{2}.$$

Returning at (3.4), we obtain the desired pointwise convergence. □

4. Examples

In the following two examples, $\mathcal{F}$ may coincide with E_2.

4.1. As L_n, $n \geqq 1$, in (2.1) we consider the Szász–Mirakjan operators defined by

$$(S_n f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \quad x \geqq 0,$$

where $f \in E_2$. For this class, in (2.3) the identities $a_n = 1$, $b_n = 1/n$, $n \in \mathbb{N}$, hold. Consequently, we get

$$v_n(x) = \frac{\sqrt{1+4n^2x^2}-1}{2n}, \quad x \geqq 0,\ n \in \mathbb{N}. \tag{4.1}$$

By simple computation, see e.g. [2], we obtain

$$(S_n\varphi_x)(x) = 0, \quad \left(S_n\varphi_x^2\right)(x) = \frac{x}{n}, \quad \left(S_n\varphi_x^3\right)(x) = \frac{x}{n^2}, \tag{4.2}$$

$$\left(S_n\varphi_x^4\right)(x) = \frac{3x^2}{n^2} + \frac{x}{n^3}, \quad x \geqq 0.$$

Choosing in (3.1) $\lambda_n = n$, clearly $a = 0$, $b = 1$. Examining (4.1) and (4.2), condition (3.2) is fulfilled, more precisely $\lim_{n\to\infty} \lambda_n^2 \left(S_n^*\varphi_x^4\right)(x) = 3x^2$. Based on (3.3) the asymptotic formula for Szász–Mirakjan modified operators runs as follows

$$\lim_{n\to\infty} n\big((S_n^* f)(x) - f(x)\big) = \frac{x}{2}f''(x) - \frac{1}{2}f'(x), \quad x > 0,$$

for any $f \in E_\alpha^{(2)}$, $\alpha \geqq 4$.

4.2. This time we start from classical Baskakov operators given by

$$(B_n f)(x) = \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right),\ b_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k},\ x \geqq 0,$$

where $f \in E_2$. It is known that $a_n = 1 + 1/n$ and $b_n = 1/n$, $n \in \mathbb{N}$, consequently

$$(4.3) \qquad v_n(x) = \frac{\sqrt{1 + 4n(n+1)x^2} - 1}{2(n+1)}, \quad x \geqq 0,\ n \in \mathbb{N}.$$

In order to present the first four central moments of B_n, we can use the following recurrence relation

$$n\big(B_n\varphi_x^{m+1}\big)(x) = x(1+x)\left[\frac{d}{dx}\big(B_n\varphi_x^m\big)(x) + m\big(B_n\varphi_x^{m-1}\big)(x)\right] \quad \text{for} \quad m \geqq 2,$$

with $(B_n\varphi_x^0)(x) = 1$, $(B_n\varphi_x)(x) = 0$, see, for example, [5]. Hence,

$$(4.4) \qquad \big(B_n\varphi_x^2\big)(x) = \frac{x(1+x)}{n}, \quad \big(B_n\varphi_x^3\big)(x) = \frac{x(1+x)(1+2x)}{n^2},$$

$$\big(B_n\varphi_x^4\big)(x) = \frac{x(1+x)}{n^3}\big(3(n+2)x(1+x) + 1\big), \quad x \geqq 0.$$

Choosing again in (3.1) $\lambda_n = n$, one has $a = b = 1$. Since

$$\lim_{n\to\infty} \lambda_n^2\big(B_n^*\varphi_x^4\big)(x) = 3x^2(1+x)^2,$$

relation (3.2) takes place and, in this case, the asymptotic formula will be read as follows

$$\lim_{n\to\infty} n\big((B_n^*f)(x) - f(x)\big) = \frac{1+x}{2}\big(xf''(x) - f'(x)\big), \quad x > 0,$$

for any $f \in E_\alpha^{(2)}$, $\alpha \geqq 4$.

Remark 4.1. The same construction as in (2.1)–(2.5) can be made and the same results can be obtained under the hypothesis that L_n, $n \in \mathbb{N}$, are defined on the Banach space $C\big([0,\gamma]\big)$, $\gamma > 0$ fixed. Roughly speaking, we can replace $\mathbb{R}_+$ by the compact $[0,\gamma]$. Actually, the reasoning is simple: for every function $f \in \mathcal{F} \cap C\big([0,\gamma]\big)$, we introduce the function $f_\gamma \in \mathcal{F} \cap C(\mathbb{R}_+)$ given by

$$f_\gamma(x) = \begin{cases} f(x), & x \in [0,\gamma], \\ f(\gamma), & x > \gamma. \end{cases}$$

It is evident $f_\gamma \in E_\alpha$ and $(L_nf_\gamma)(x) = f(\gamma)(L_ne_0)(x) = f(\gamma)$, for each $x > \gamma$.

On the other hand, the series in (2.1) can be replaced by a finite sum, such as $\sum_{k=0}^{n} u_{n,k}(x)f(x_{n,k})$. Practically, this can be obtained by choosing $u_{n,k}$, $k > n$, the null functions.

4.3. With these remarks in mind, in (2.1) we consider $L_n = P_n$, $n \geqq 1$, Bernstein operators, where

$$(P_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

For this sequence, the approximation process described by (2.5) has been established by King [4], Eq. (2.2). One has

$$v_n(x) = \begin{cases} x^2, & n = 1, \\ -\dfrac{1}{2(n-1)} + \sqrt{\left(\dfrac{n}{n-1}\right) x^2 + \dfrac{1}{4(n-1)^2}}, & n = 2, 3, \dots \end{cases}$$

For the original Bernstein polynomials we identify $a_n = 1 - 1/n$, $b_n = 1/n$ for any $n \in \mathbb{N}$. Also, the first four explicit moments are the following

(4.5)
$$(P_n \varphi_x)(x) = 0, \quad \left(P_n \varphi_x^2\right)(x) = \frac{x(1-x)}{n}, \quad \left(P_n \varphi_x^3\right)(x) = \frac{x(1-x)(1-2x)}{n^2},$$
$$\left(P_n \varphi_x^4\right)(x) = \frac{x(1-x)}{n^3}\big(3(n-2)x(1-x) + 1\big), \quad x \in [0,1].$$

Taking in (3.1) $\lambda_n = n$, we get $a = -1$ and $b = 1$. Condition (3.2) holds uniformly with respect to $x \in [0,1]$ and the Bernstein–King non-polynomial operators P_n^*, $n \geqq 1$, enjoy the following property

$$\lim_{n\to\infty} n\big((P_n^* f)(x) - f(x)\big) = \frac{1-x}{2}\big(x f''(x) - f'(x)\big), \quad x \in (0,1],$$

for every $f \in E_\alpha^{(2)}$, $\alpha \geqq 4$. We may remark in passing that $P_n^* f$ interpolates f at $x = 0$ and $x = 1$.

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