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Asymptotic formulae for Baskakov-Mastroianni operators based on q-integers

Octavian Agratini and Cristina Radu

Abstract. We establish an asymptotic formula for a general sequence of positive linear operators of discrete type. This class represents a generalization in q-Calculus of the operators introduced by G. Mastroianni, the construction taking its origin in a paper of Baskakov. We also mark out Voronovskaja-type formulas for two particular cases which are q-extensions of the Szász-Mirakjan operator and the ordinary Baskakov operator.

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1 Introduction

The roots of the paper are in connection with a general class of operators introduced by Baskakov [6] and developed by Mastroianni [8]. For comparing, [3, p. 344, p. 351] can also be consulted. A q-analogue of these operators has been introduced in [9]. The author investigated their weighted statistical approximation properties. For the sake of completeness, we recall: a *q-analogue* or a *q-extension* of a mathematical object X is a family of objects $X(q)$ (usually, $0 < q < 1$) such that $\lim\limits_{q \to 1^-} X(q) = X$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Deeping the researches on the mentioned q-operators, in [1] all their moments were explicitly expressed with the help of a new q-analogue of the Stirling numbers of the second kind. At the same time, the rate of convergence was established in two cases: for bounded functions and for functions having a polynomial growth.

The aim of this note is to present an asymptotic formula of Voronovskaja-type for q-Baskakov-Mastroianni operators. As applications, we deduce asymptotic formulae for two special cases which are q-extensions of the Szász-Mirakjan operators and the ordinary Baskakov operators, respectively.

The paper is organized as follows. Section 2 includes elements of q-Calculus, the form of the announced class of operators and some results obtained previously. Section 3 comprises the statement of the main result and two particular asymptotic formulae. The last Section is devoted to the proofs of our results.

2 The $T_{n,q}$ operators

First of all we collect some useful formulas in q-Calculus, see, e.g., [4], [7].

Let $q > 0$. For any $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the q-integer $[n]_q$ and the q-factorial $[n]_q!$ are respectively defined by

$$[n]_q = \sum_{j=0}^{n-1} q^j, \qquad [n]_q! = \prod_{j=1}^{n} [j]_q, \quad n \in \mathbb{N},$$

and $[0]_q = 0$, $[0]_q! = 1$. The q-binomial coefficients are denoted by $\left[\begin{array}{c} n \\ k \end{array} \right]_q$ and are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k = 0, 1, \ldots, n.$$

For $q = 1$ one has $[n]_1 = n$, $[n]_1! = n!$ and $\left[\begin{array}{c} n \\ k \end{array} \right]_1$ turns into the ordinary binomial coefficient $\left(\begin{array}{c} n \\ k \end{array} \right)$.

We also recall the significance of the notation $(x-a)_q^r$.

$$(x-a)_q^r = \prod_{s=0}^{r-1} (x - q^s a) = \sum_{k=0}^{r} \left[\begin{array}{c} r \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} x^{r-k} (-a)^k.$$

In the sequel we always will assume that $q \in (0,1)$.

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \quad D_q f(0) := \lim_{x \to 0} D_q f(x),$$

and the high q-derivatives $D_q^0 f := f$, $D_q^n f := D_q \left(D_q^{n-1} f \right)$, $n = 1, 2, \ldots$.

A function f is q-differentiable on a real interval I if for any $q \in (0,1)$ the q-derivative of f exists and is finite in every $x \in I$. Also, we recall the product rule

$$D_q \left(f(x) g(x) \right) = g(x) D_q f(x) + f(qx) D_q g(x).$$

For each $(m,r) \in \mathbb{N}_0 \times \mathbb{N}_0$, a q-analogue of Stirling numbers of the second kind can be considered the following

$$\sigma_q(m,r) = \frac{1}{[r]_q!} \sum_{j=0}^{r} (-1)^j q^{(r-j)(r-j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}}. \tag{2.1}$$

Lemma 2.1 ([1, Lemma 2]) *The numbers $\sigma_q(m,r)$, $(m,r) \in \mathbb{N}_0 \times \mathbb{N}_0$, given by (2.1) enjoy the following properties.*

$$\sigma_q(m,0) = 0, \quad m \in \mathbb{N}, \quad \text{and} \quad \sigma_q(0,0) = 1, \tag{2.2}$$

$$q^r \sigma_q(m+1,r) = [r]_q \sigma_q(m,r) + \sigma_q(m,r-1), m \in \mathbb{N}_0, \quad r \in \mathbb{N}, \tag{2.3}$$

$$\sigma_q(m,r) = 0, \quad r > m. \tag{2.4}$$

It is known that a q-analogue is not unique. In this direction it is worth to be mentioned that others q-Stirling numbers denoted by $S_q(m,r)$ have been used by Ali Aral [5] in studying a q-generalization of Szász-Mirakjan operators.

Now we are in position to present $T_{n,q}$, $n \in \mathbb{N}$, $q \in (0,1)$, operators [1, Section 3]. Let $(\phi_n)_{n \geq 1}$ be a sequence of real valued functions defined on $\mathbb{R}_+$, continuously infinitely q-differentiable on $\mathbb{R}_+$ and satisfying the following conditions.

$(P1)$ $\quad \phi_n(0) = 1, \quad n \in \mathbb{N}, \quad$ (2.5)

$(P2)$ $\quad (-1)^k D_q^k \phi_n(x) \geq 0, \quad n \in \mathbb{N}, k \in \mathbb{N}_0, x \geq 0, \quad$ (2.6)

$(P3)$ $\quad$ For every $(n,k) \in \mathbb{N} \times \mathbb{N}_0$ there exists a positive integer i_k, $0 \leq i_k \leq k$, and a real function $\beta_{n,k,i_k,q} : \mathbb{R}_+ \to \mathbb{R}$ such that

$$D_q^{k+1} \phi_n(x) = (-1)^{i_k+1} D_q^{k-i_k} \phi_n(q^{i_k+1} x) \beta_{n,k,i_k,q}(x), \tag{2.7}$$

where

$$\lim_n \frac{\beta_{n,k,i_k,q}(0)}{[n]_q^{i_k+1}\, q^{k-i_k}} = 1. \tag{2.8}$$

We define the operators

$$(T_{n,q}f)(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{\frac{k(k-1)}{2}} D_q^k \phi_n(x) f\left(\frac{[k]_q}{[n]_q\, q^{k-1}}\right), \quad x \geq 0, \tag{2.9}$$

where $f \in \mathcal{F}(\mathbb{R}_+) := \{f : \mathbb{R}_+ \to \mathbb{R},\ \text{ the series in (2.9) is absolutely convergent for all } n \in \mathbb{N}\}$.

Clearly, for each $n \in \mathbb{N}$, $T_{n,q}$ is linear and positive operator. With the help of q-Stirling numbers defined by (2.1), we are able to indicate all moments of our operators.

Lemma 2.2 ([1, Lemma 4]) *Let $T_{n,q}$, $n \in \mathbb{N}$, be defined by (2.9). One has*

$$(T_{n,q}e_m)(x) = \sum_{r=0}^{m} \frac{(-x)^r}{[n]_q^m} q^m D_q^r \phi_n(0) \sigma_q(m,r), \quad x \geq 0, \tag{2.10}$$

where e_m stands for the monomial of m degree, $m \in \mathbb{N}_0$.

We easily deduce

$$(T_{n,q}e_0)(x) = 1, \tag{2.11}$$

$$(T_{n,q}e_1)(x) = -x\frac{D_q\phi_n(0)}{[n]_q}, \tag{2.12}$$

$$(T_{n,q}e_2)(x) = x^2\frac{D_q^2\phi_n(0)}{q[n]_q^2} - x\frac{D_q\phi_n(0)}{[n]_q^2}. \tag{2.13}$$

These three particular moments have been obtained by a straightforward calculation in [9, Lemma 1], without involving q-Stirling numbers $\sigma_q(m,r)$.

Since $\lim_n [n]_q = (1-q)^{-1}$, relation (2.13) implies

$$\lim_n (T_{n,q}e_2)(x) = q^{-i_1}x^2 + (1-q)x \neq e_2(x),$$

where the index i_1 can be 0 or 1, see relation (2.7). Clearly, $(T_{n,q})_n$ does not form an approximation process.

Remark 2.3 *To become $(T_{n,q})_n$ an approximation process, for each $n \in \mathbb{N}$, the constant q will be replaced by a number $q_n \in (0,1)$ such that $\lim_n q_n = 1$. Under this assumption, for any compact $\mathcal{K} \subset \mathbb{R}_+$ and for each $f \in \mathcal{F}(\mathbb{R}_+)$ which is continuous and bounded, one has*

$$\lim_n (T_{n,q_n} f)(x) = f(x), \text{ uniformly in } x \in \mathcal{K}. \tag{2.14}$$

The fastest motivation of the above statement can be made by using a more general result obtained by Altomare [2] in the setting of locally compact metric space. It reads as follows.

Theorem 2.4 ([2, Theorem 3.5]) *Let J be a real interval and consider a lattice subspace E of the space $F(J)$ of all real-valued functions on J, containing the functions $e_0(x) = 1$, $e_1(x) = x$ and $e_2(x) = x^2$ $(x \in J)$. Consider a sequence $(L_n)\, n \geq 1$ of positive linear operators from E into $F(J)$ and assume that for every $k = 0, 1, 2$*

$$\lim_{n\to\infty} L_n(e_k) = e_k \text{ uniformly on compact subsets of } J.$$

Then

$$\lim_{n\to\infty} L_n(f) = f \text{ uniformly on compact subsets of } J$$

for every $f \in E \cap C_b(J)$ ($C_b(J)$ denoting the space of all continuous and bounded functions on J).

Choosing $J = \mathbb{R}_+$, $E = \mathcal{F}(\mathbb{R}_+)$, and taking into account relations (2.11), (2.12), (2.13), Theorem 2.4 leads us to the identity (2.14).

3 Results

Examining (2.6) and (2.7), for the sake of brevity we denote

$$d_{n,k} = D_q^k \phi_n(0), \quad n \in \mathbb{N}, \quad k \in \mathbb{N}_0. \tag{3.1}$$

For each $x \geq 0$, let φ_x be the function defined by $\varphi_x(t) = t - x$, $t \geq 0$. $T_{n,q}\varphi_x^s$ represents the s-th order central moment of the operator $T_{n,q}$, where $s \in \mathbb{N}_0$.

On the basis of relations (2.5), (2.7) and (2.8) one has $d_{n,k} \neq 0$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Also $\sigma_q(s,0) = 0$, $s \in \mathbb{N}$, see (2.2). By inspecting relation (2.10), we conclude

Remark 3.1 *For a given $s \in \mathbb{N}$, both $T_{n,q}e_s$ and $T_{n,q}\varphi_x^s$ are polynomials in x of s degree having the constant term null.*

Consequently, we can write

$$(T_{n,q}\varphi_x^s)(x) = \sum_{k=1}^{s} m_{n,s,k}(q)x^k, \quad x \geq 0. \tag{3.2}$$

Lemma 3.2 *Let $T_{n,q}$, $n \in \mathbb{N}$, be defined by (2.9). For $s = 4$, the coefficients of the polynomial introduced by (3.2) have the following values.*

$$m_{n,4,1}(q) = -\frac{d_{n,1}}{[n]_q^4}, \qquad m_{n,4,2}(q) = \frac{1+3q+3q^2}{q^3}\frac{d_{n,2}}{[n]_q^4} + 4\frac{d_{n,1}}{[n]_q^3}, \tag{3.3}$$

$$m_{n,4,3}(q) = -\frac{1+2q+3q^2}{q^5}\frac{d_{n,3}}{[n]_q^4} - \frac{4(1+2q)}{q^2}\frac{d_{n,2}}{[n]_q^3} - 6\frac{d_{n,1}}{[n]_q^2}, \tag{3.4}$$

$$m_{n,4,4}(q) = \frac{d_{n,4}}{q^6[n]_q^4} + 4\frac{d_{n,3}}{q^3[n]_q^3} + 6\frac{d_{n,2}}{q[n]_q^2} + 4\frac{d_{n,1}}{[n]_q} + 1. \tag{3.5}$$

Lemma 3.3 *Let $(q_n)_{n\geq 1}$, $0 < q_n < 1$, be a sequence such that $\lim_n q_n = 1$. Supposing that the properties $(P1)$, $(P2)$, $(P3)$ take place for $q = q_n$, one has*

$$\lim_n \frac{d_{n,k}}{[n]_{q_n}^k} = (-1)^k, \quad k \in \mathbb{N}_0, \tag{3.6}$$

where $d_{n,k}$ is defined as in (3.1).

Lemma 3.4 *Let $(q_n)_{n\geq 1}$, $0 < q_n < 1$, and $(\lambda_n)_{n\geq 1}$, $\lambda_n > 0$, be sequences such that $\lim_n q_n = 1$, $\lim_n \lambda_n = \infty$ and $\lim_n \frac{\lambda_n}{[n]_{q_n}}$ is finite. Let T_{n,q_n}, $n \in \mathbb{N}$, be defined as in (2.9). If the sequences*

$$\left(\lambda_n^2 m_{n,4,3}(q_n)\right)_n \quad \text{and} \quad \left(\lambda_n^2 m_{n,4,4}(q_n)\right)_n \quad \text{are bounded,} \tag{3.7}$$

then $\left(\lambda_n^2 \left(T_{n,q_n}\varphi_x^4\right)(x)\right)_n$ is bounded with respect to n. In the above, $m_{n,4,3}$ and $m_{n,4,4}$ are given by (3.4) and (3.5), respectively.

As usual, $C^2(\mathbb{R}_+)$ denotes the space of all real-valued continuous functions on $\mathbb{R}_+$ which are twice continuously differentiable in $\mathbb{R}_+$.

Returning at the sequence $(\lambda_n)_{n\geq 1}$ of strictly positive real numbers with the property $\lim_n \lambda_n = \infty$, we suppose that the real numbers τ_1, τ_2, τ_3 exist verifying the following relations

$$\begin{cases} \lim_n \lambda_n \left(\frac{d_{n,1}}{[n]_{q_n}} + 1 \right) = \tau_1, \\ \lim_n \frac{\lambda_n}{[n]_{q_n}} = \tau_2, \\ \lim_n \lambda_n \left(1 + 2\frac{d_{n,1}}{[n]_{q_n}} + \frac{d_{n,2}}{q_n [n]_{q_n}^2} \right) = \tau_3, \end{cases} \tag{3.8}$$

where $d_{n,1}, d_{n,2}$ are defined as in (3.1) with $q = q_n$. The sequence $(\lambda_n)_n$ will play a crucial role in establishing the Voronovskaja-type theorem.

Theorem 3.5 *Let $(q_n)_{n\geq 1}$, $0 < q_n < 1$, and $(\lambda_n)_{n\geq 1}$, $\lambda_n > 0$, be sequences such that $\lim_n q_n = 1$, $\lim_n \lambda_n = \infty$ and the conditions (3.8) are fulfilled. Let T_{n,q_n}, $n \in \mathbb{N}$, be defined as in (2.9). If (3.7) takes place, then for any function $f \in \mathcal{F}(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$ with f'' bounded, one has*

$$\lim_{n\to\infty} \lambda_n \left((T_{n,q_n} f)(x) - f(x) \right) = -\tau_1 f'(x) + \frac{1}{2} \left(\tau_2 x + \tau_3 x^2 \right) f''(x), \tag{3.9}$$

for any $x > 0$.

The theorem shows that $(T_{n,q_n} f)(x) - f(x)$ is of order not better then $1/\lambda_n$, if $f'(x)$ and $f''(x)$ are not simultaneous null.

Two special cases of T_{n,q_n}, $n \in \mathbb{N}$, operators have been exhibited in [9, Section 5]. In order to obtain an asymptotic formula of Voronovskaja-type for these classes of q-operators, we will check the possibility to apply Theorem 3.5.

Application 1. We choose $\phi_n(x) := E_{q_n}\left(-[n]_{q_n} x \right)$, $x \geq 0$, $n \in \mathbb{N}$. Here E_{q_n} is the known expansion in q-Calculus of the exponential function being defined as follows

$$E_{q_n}(x) = \sum_{k=0}^{\infty} q_n^{k(k-1)/2} \frac{x^k}{[k]_{q_n}!}, \quad x \in \mathbb{R},$$

see, e.g., [7, p. 31]. T_{n,q_n} operators turn into S^*_{n,q_n}, a q-analogue of Szász-Mirakjan operators. For all $(n,k) \in \mathbb{N} \times \mathbb{N}_0$, we have $\phi_n(0) = 1$,

$$D^k_{q_n} \phi_n(x) = (-1)^k [n]^k_{q_n} q_n^{\frac{k(k-1)}{2}} E_{q_n}\left(-[n]_{q_n} q_n^k x \right), \quad x \geq 0,$$

and

$$D^{k+1}_{q_n} \phi_n(x) = -D^k_{q_n} \phi_n(q_n x) \beta_{n,k,0,q_n}(x).$$

Consequently, relations (2.5)-(2.8) hold, where, for every $(n,k)\in\mathbb{N}\times\mathbb{N}_0$, $i_k=0$ and $\beta_{n,k,0,q_n}(x)=[n]_{q_n}q_n^k$ is a constant function. Taking $\lambda_n=[n]_{q_n}$, all we have to do is to verify conditions (3.8) and (3.7).

Since $d_{n,k}=(-1)^k[n]_{q_n}^k q_n^{\frac{k(k-1)}{2}}$, we get $\tau_1=0$, $\tau_2=1$ and $\tau_3=0$. Further on, by using (3.4) and (3.5), we easily deduce

$$m_{n,4,3}(q_n)=\frac{(1-q_n)^2}{q_n^2[n]_{q_n}},\quad m_{n,4,4}(q_n)=0,$$

and (3.7) takes place. The asymptotic formula (3.9) for S^*_{n,q_n} will be read as follows

$$\lim_{n\to\infty}[n]_{q_n}\left(\left(S^*_{n,q_n}f\right)(x)-f(x)\right)=\frac{x}{2}f''(x),\quad x>0, \tag{3.10}$$

where $f\in\mathcal{F}(\mathbb{R}_+)\cap C^2(\mathbb{R}_+)$ with f'' bounded.

Application 2. Choosing $\phi_n(x):=(1+q_n^nx)_{q_n}^{-n}$, $x\ge 0$, $n\in\mathbb{N}$, T_{n,q_n} operators become a q-analogue of the ordinary Baskakov operators, say V^*_{n,q_n}. With the help of the known formulas $[-n]_q=-[n]_q q^{-n}$, $n\in\mathbb{N}$, and $D_q(1+ax)_q^{\alpha}=[\alpha]_q a(1+aqx)_q^{\alpha-1}$ for any real numbers a,α, we deduce

$$D_{q_n}^k\phi_n(x)=(-1)^k\left(\prod_{j=0}^{k-1}[n+j]_{q_n}\right)\left(1+q_n^{n+k}x\right)_{q_n}^{-n-k},\quad x\ge 0.$$

Conditions (2.5) and (2.6) are satisfied. Choosing $i_k=k$ and $\beta_{n,k,k,q_n}(x)=\left(\prod_{j=0}^{k}[n+j]_{q_n}\right)\left(1+q_n^{k+1}x\right)_{q_n}^{-k-1}$, $n\in\mathbb{N}$, $k\in\mathbb{N}_0$, (2.7) and (2.8) are also fulfilled. In the above we used the formula $(1+y)_q^{\alpha}(1+q^{\alpha}y)_q^{\beta}=(1+y)_q^{\alpha+\beta}$ for $q=q_n$, $\alpha=-n$, $\beta=-k-1$ and $y=q_n^{n+k+1}x$.

Again, we choose $\lambda_n=[n]_{q_n}$. This time, $d_{n,k}=(-1)^k[n]_{q_n}\cdot\ldots\cdot[n+k-1]_{q_n}$ and, in accordance with (3.8), one gets $\tau_1=0$, $\tau_2=1$, $\tau_3=1$. With a little more effort we find

$$\begin{aligned}\lambda_n^2 m_{n,4,3}(q_n) &= \frac{(q_n-1)^2}{q_n^2}[n]_{q_n}-\frac{(1+2q_n)(q_n^2-2q_n-1)}{q_n^4}\\ &+ \frac{(1+q_n)(1+2q_n+3q_n^2)}{q_n^5[n]_{q_n}}\end{aligned}$$

and

$$\lambda_n^2 m_{n,4,4}(q_n)=\frac{(q_n-1)^2}{q_n^3}[n]_{q_n}-\frac{q_n^3-3q_n-1}{q_n^5}+\frac{(1+q_n)(1+q_n+q_n^2)}{q_n^6[n]_{q_n}}.$$

We used the identities $[n+k]_{q_n} = 1+q_n+\ldots+q_n^k[n]_{q_n}$ for $k=1,2,3$. Since $q_n \in (0,1)$ and $q_n \to 1$, the requirement (3.7) is fulfilled. Actually, both sequences are convergent.

Applying Theorem 3.5, we obtain the following asymptotic formula.

$$\lim_n [n]_{q_n} \left(\left(V_{n,q_n}^* f \right)(x) - f(x) \right) = \frac{x(x+1)}{2} f''(x), \quad x > 0, \tag{3.11}$$

where $f \in \mathcal{F}(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$ with f'' bounded.

Remark 3.6 *For $q_n = 1$, $S_{n,1}^*$ and $V_{n,1}^*$ turn into the classical Szász-Mirakjan and Baskakov operators, respectively. Our formulae (3.10), (3.11) become the known Voronovskaja-type identities verified by these discrete operators. In these two special cases the order of approximation is $1/n$.*

4 Proofs

Proof of Lemma 3.2

First of all, examining relations (2.2)-(2.4), we deduce $\sigma_q(m,m) = q^{-\frac{m(m+1)}{2}}$, $m \in \mathbb{N}_0$.

To calculate all required coefficients, we need to know some particular values of q-Stirling numbers $\sigma_q(m,r)$ described by (2.1). The below table may be useful.

$m \setminus r$	0	1	2	3	4
0	1	0	0	0	0
1	0	q^{-1}	0	0	0
2	0	q^{-2}	q^{-3}	0	0
3	0	q^{-3}	$(1+2q)q^{-5}$	q^{-6}	0
4	0	q^{-4}	$(1+3q+3q^2)q^{-7}$	$(1+2q+3q^2)q^{-9}$	q^{-10}

The identities (3.3)-(3.5) result after a boring calculation based on (2.10) and taking in view that

$$\left(T_{n,q} \varphi_x^4 \right)(x) = \sum_{m=0}^{4} (-1)^m \binom{4}{m} x^m \left(T_{n,q} e_m \right)(x).$$

□

Proof of Lemma 3.3

For $k = 0$ the conclusion is evident, see (2.5). Further on, choosing $x = 0$ in (2.7), we can write

$$\frac{d_{n,k+1}}{[n]_{q_n}^{k+1}} = (-1)^{i_k+1} q_n^{k-i_k} \frac{\beta_{n,k,i_k,q_n}(0)}{[n]_{q_n}^{i_k+1} q_n^{k-i_k}} \frac{d_{n,k-i_k}}{[n]_{q_n}^{k-i_k}}, \tag{4.1}$$

for a certain index $i_k \in \{0, 1, \ldots, k\}$.

The proof runs by mathematical induction with respect to k. Assuming $\lim\limits_n \frac{d_{n,j}}{[n]_{q_n}^j} = (-1)^j$ for $j = \overline{0,k}$, relations (4.1) and (2.8) imply $\lim\limits_n \frac{d_{n,k+1}}{[n]_{q_n}^{k+1}} = (-1)^{k+1}$. Consequently, relation (3.6) holds.

□

Proof of Lemma 3.4

Since $\lim\limits_n \frac{1}{[n]_{q_n}} = 0$, setting $\lim\limits_n \frac{\lambda_n}{[n]_{q_n}} = \tau_2 \in \mathbb{R}$, we get $\lim\limits_n \lambda_n^2 m_{n,4,1}(q_n) = 0$ and $\lim\limits_n \lambda_n^2 m_{n,4,2}(q_n) = 7\tau_2^2 - 4\tau_2$, see (3.3). Taking into account (3.2), the assumptions' lemma guarantee the achievement of the statement.

□

Proof of Theorem 3.5

Let $x > 0$ be fixed. For any $f \in \mathcal{F}(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$ with f'' bounded, we define

$$\phi_f(x;t) = \begin{cases} \frac{f(t)-f(x)-\varphi_x(t)f'(x)-2^{-1}\varphi_x^2(t)f''(x)}{(t-x)^2}, & \text{if } t \neq x, \\ 0, & \text{if } t = x, \end{cases}$$

where $t \in \mathbb{R}_+$. We get $\lim\limits_{t\to x} \phi_f(x;t) = 0 = \phi_f(x;x)$, consequently $\phi_f(x;\cdot) \in C(\mathbb{R}_+)$. Moreover, φ_x^2 and $\varphi_x^2\phi_f(x;\cdot)$ belong to $\mathcal{F}(\mathbb{R}_+)$. For the function f, we can write the Lagrange form of the Taylor formula

$$f(t) = f(x) + \varphi_x(t)f'(x) + \frac{1}{2}\varphi_x^2(t)f''(x) + \varphi_x^2(t)\phi_f(x;t).$$

Applying T_{n,q_n} and using (2.11), we obtain

$$(T_{n,q_n}f)(x) - f(x)$$

$$= (T_{n,q_n}\varphi_x)(x)f'(x) + \frac{1}{2}\left(T_{n,q_n}\varphi_x^2\right)(x)f''(x) + T_{n,q_n}\left(\varphi_x^2\phi_f(x;\cdot)\right)(x).$$

Taking into account relations (2.12)-(2.13), we get

$$\begin{aligned} \left(T_{n,q_n}\varphi_x\right)(x) &= -\left(\frac{d_{n,1}}{[n]_{q_n}}+1\right)x, \\ \left(T_{n,q_n}\varphi_x^2\right)(x) &= \left(\frac{d_{n,2}}{q_n[n]_{q_n}^2}+2\frac{d_{n,1}}{[n]_{q_n}}+1\right)x^2-\frac{d_{n,1}}{[n]_{q_n}^2}x, \end{aligned}$$

and hence

$$\begin{aligned} &\lambda_n\left(\left(T_{n,q_n}f\right)(x)-f(x)\right)=-\lambda_n\left(\frac{d_{n,1}}{[n]_{q_n}}+1\right)xf'(x) \\ &+\frac{1}{2}\left(\lambda_n\left(\frac{d_{n,2}}{q_n[n]_{q_n}^2}+2\frac{d_{n,1}}{[n]_{q_n}}+1\right)x^2-\frac{\lambda_n d_{n,1}}{[n]_{q_n}^2}x\right)f''(x) \\ &+\lambda_n T_{n,q_n}\left(\varphi_x^2\phi_f(x;\cdot)\right)(x). \qquad (4.2) \end{aligned}$$

Since $\varphi_x^2\phi_f(x,\cdot)\in\mathcal{F}(\mathbb{R}_+)$ and the series in (2.9) is absolutely convergent, we deduce $\varphi_x^2\left|\phi_f(x,\cdot)\right|\in\mathcal{F}(\mathbb{R}_+)$. Further on, by applying Cauchy inequality for the last term of (4.2), one has

$$0\le\lambda_n\left|T_{n,q_n}\left(\varphi_x^2\phi_f(x;\cdot)\right)(x)\right|\le\lambda_n T_{n,q_n}\left(\varphi_x^2\left|\phi_f(x;\cdot)\right|,x\right)$$

$$\le\left\{\lambda_n^2\left(T_{n,q_n}\varphi_x^4\right)(x)\right\}^{1/2}\left\{\left(T_{n,q_n}\phi_f^2(x;\cdot)\right)(x)\right\}^{1/2}.$$

Clearly, $\phi_f^2(x;\cdot)$ is continuous on $\mathbb{R}_+$. Under the assumption made on the function f, we get $\phi_f^2(x,\cdot)\in C_b(\mathbb{R}_+)\subset\mathcal{F}(\mathbb{R}_+)$, and in harmony with Remark 2.3, we have $\lim\limits_n\left(T_{n,q_n}\phi_f^2(x;\cdot)\right)(x)=\phi_f^2(x;x)=0$. On the other hand, since (3.7) holds, Lemma 3.4 guarantees that a constant $k(x)$ independent of n exists, such that $\lambda_n^2\left(T_{n,q_n}\varphi_x^4\right)(x)\le k(x)$ for each $n\in\mathbb{N}$.

Consequently, $\lim\limits_n\lambda_n\left|T_{n,q_n}\left(\varphi_x^2\phi_f(x;\cdot)\right)(x)\right|=0$. Returning at (4.2), for n tending to infinity, on the basis of (3.8) and (3.6) with $k=1,2$, we obtain the desired pointwise convergence.

□

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