Applied Mathematics Letters 24 (2011) 359–363

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

# Poincaré inequalities in reflexive cones

Petru Jebelean [a,*], Radu Precup [b]

[a] West University of Timişoara, Department of Mathematics, Blvd. V. Pârvan no. 4, 300223 Timişoara, Romania

[b] Babeş–Bolyai University, Department of Applied Mathematics, Str. M. Kogălniceanu no. 1, 400084 Cluj, Romania

ARTICLE INFO

*Article history:*
Received 4 May 2010
Accepted 19 October 2010

*Keywords:*
Reflexive cone
Compact embedding
Trace operator
Boundary value problem
A priori bounds

ABSTRACT

We present an abstract result concerning Poincaré inequalities in cones. Some examples in Sobolev spaces are provided. We also discuss an application to a priori bounds of solutions for a general boundary value problem involving the vector $p$-Laplacian operator.



## 1. Introduction

Given a bounded open domain $\Omega \subset \mathbb{R}^N$ and $p > 1$, the classical Poincaré (others call it Friedrichs–Poincaré) inequality says that there exists a positive constant $k = k(p, \Omega)$ such that

$$\|u\|_{L^p} \le k\|\nabla u\|_{L^p} \quad \forall u \in W, \tag{1.1}$$

where $W$ is either $W_0^{1,p}(\Omega)$ or $\tilde{W} = \{u \in W^{1,p}(\Omega) : \int_\Omega u = 0\}$; for $N = 1$ and $W = \tilde{W}$ inequality (1.1) is also known as the Wirtinger inequality [1]. More generally, the following result is known (see e.g. [2, Theorem 2.5.19]).

**Theorem A.** *If $W$ is a closed subspace of $W^{1,p}(\Omega)$ which contains no nonzero constant functions, then there exists a constant $k > 0$ such that (1.1) holds true.*

The aim of this *note* is to give an abstract unifying approach to a class of Poincaré type inequalities. As the examples provided show, this approach can be applied, among others to cones from $W^{1,p}(\Omega)$. The application that we give suggests the applicability of these inequalities for inferring a priori bounds on the solutions for a class of boundary value problems involving the vector $p$-Laplacian operator.

## 2. An abstract result

Below, $X$ will be a subspace of a fixed normed linear space $(Y, \| \, \|)$ and $| \, | : X \to \mathbb{R}$ will be a seminorm on $X$. For $K \subset X$ a given nonzero (pointed) cone ($\alpha K \subset K, \forall \alpha \ge 0$), we introduce the constant

$$\mu_K := \inf\left\{\frac{|u|}{\|u\|} : u \in K \setminus \{0\}\right\}. \tag{2.1}$$

* Corresponding author.
*E-mail address:* jebelean@math.uvt.ro (P. Jebelean).

0893-9659/$ – see front matter 
doi:10.1016/j.aml.2010.10.024

Clearly, as

$$\left\{ \frac{|u|}{\|u\|} : u \in K \setminus \{0\} \right\} = \{|u| : u \in K, \|u\| = 1\},$$

one has that

$$\mu_K = \inf \{|u| : u \in K, \|u\| = 1\}. \tag{2.2}$$

The space $X$ will be endowed with the norm

$$[u] = \left(\|u\|^p + |u|^p\right)^{\frac{1}{p}}, \quad \forall u \in X$$

and the topological properties of $K$ will be understood with respect to $(X, [\ ])$. We say that the cone $K$ is *reflexive* if every bounded sequence $\{u_n\} \subset K$ has a subsequence which is weakly convergent to some $u \in K$. Defining

$$N := \{u \in X : |x| = 0\},$$

we have the following:

**Theorem 2.1.** *Assume that:*

(i) *the cone $K$ is reflexive,*
(ii) *the embedding $K \subset Y$ is compact.*

*Then the infimum in (2.1) is attained: there exists some $e \in K \setminus \{0\}$ such that $\mu_K = |e|/\|e\|$; if, in addition $N \cap K = \{0\}$, one has that $\mu_K > 0$.*

**Proof.** It suffices to prove that the infimum is attained in (2.2). Let $\{u_n\} \subset K$ be such that

$$\|u_n\| = 1 \quad \text{and} \quad \mu_K \le |u_n| \le \mu_K + \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

As $[u_n]^p = 1 + |u_n|^p$, the sequence $\{u_n\}$ is bounded in $X$. From (i) there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, with $u_n \to u \in K$ weakly in $X$. Using (ii) we infer that $u_n \to u$ strongly in $Y$. This implies that $\|u\| = 1$, and so $\mu_K \le |u|$. Then, by the weak lower semicontinuity of the norm [ ], we get

$$\left(1 + |u|^p\right)^{\frac{1}{p}} = [u] \le \liminf_{n\to\infty} [u_n] = \left(1 + \mu_K^p\right)^{\frac{1}{p}},$$

which gives $|u| \le \mu_K$. We infer that $\mu_K = |u|$ and the proof is complete. □

**Corollary 2.2.** *If $(X, [\ ])$ is reflexive, the cone $K \subset X$ is weakly closed, the embedding $K \subset Y$ is compact and $N \cap K = \{0\}$, then $\mu_K > 0$.*

**Corollary 2.3.** *If $(X, [\ ])$ is reflexive, the embedding $X \subset Y$ is compact and $N = \{0\}$, then $\mu_X > 0$ and*

$$\|u\| \le \frac{1}{\mu_X} |u| \quad \forall u \in X.$$

*In addition, $|\ |$ is a norm on $X$ which is equivalent to [ ].*

**Proof.** We have

$$|u| \le [u] \le \left(\frac{1}{\mu_X^p} |u|^p + |u|^p\right)^{\frac{1}{p}} \le \left(2 \max\left\{\frac{1}{\mu_X^p}, 1\right\}\right)^{\frac{1}{p}} |u| \quad \forall u \in X. \quad \square$$

**Remark 2.4.** It is clear that Theorem A is an immediate consequence of Corollary 2.3.

## 3. Some examples

**Example 1.** Let $C_1 \subset \mathbb{R}^N \times \mathbb{R}^N$ be a closed cone and $d = \{(x, x) : x \in \mathbb{R}^N\}$. We set

$$K_1 := \left\{u \in W^{1,p}([0, T]; \mathbb{R}^N) : (u(0), u(T)) \in C_1\right\}.$$

Note that since $W^{1,p}([0, T]; \mathbb{R}^N) \subset C([0, T]; \mathbb{R}^N)$, the evaluations of $u(0)$ and $u(T)$ in the definition of $K_1$ make sense.

**Theorem 3.1.** *If $C_1 \cap d = \{(0, 0)\}$ then*

$$\mu_{K_1} = \inf\left\{\frac{\|u'\|_{L^p}}{\|u\|_{L^p}} : u \in K_1 \setminus \{0\}\right\} > 0.$$

**Proof.** We apply Corollary 2.2 with the following choices:

$$(Y, \| \ \|) = (L^p([0,T];\mathbb{R}^N), \| \ \|_{L^p}),$$
$$X = W^{1,p}([0,T];\mathbb{R}^N), \qquad |u| = \|u'\|_{L^p} \quad \forall u \in W^{1,p}([0,T];\mathbb{R}^N)$$

and $K = K_1$. The space $W^{1,p}([0,T];\mathbb{R}^N)$ endowed with the norm

$$[u] := [u]_{1,p} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{\frac{1}{p}} \tag{3.1}$$

is uniformly convex (see [3, Theorem 2.1] or [4, Corollary 3.8]) and hence it is reflexive by the Milman–Pettis theorem. On the other hand, $C_1$ being closed, by the compactness of the embedding $W^{1,p}([0,T];\mathbb{R}^N) \subset C([0,T];\mathbb{R}^N)$ we deduce that $K_1$ is weakly closed. Also, as $W^{1,p}([0,T];\mathbb{R}^N) \subset L^p([0,T];\mathbb{R}^N)$ is compact, the embedding $K_1 \subset L^p([0,T];\mathbb{R}^N)$ is compact. From $C_1 \cap d = \{(0,0)\}$ we get $\{u \in K_1 : \|u'\|_{L^p} = 0\} = \{0\}$ and the proof is complete. □

**Corollary 3.2.** *If $C_1 \cap d = \{(0,0)\}$ then there is a constant $k > 0$ such that*

$$\|u\|_{L^p} \le k\|u'\|_{L^p} \quad \forall u \in K_1 \cup (-K_1).$$

**Example 2.** Next, let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ and $C_2 \subset L^p(\partial\Omega)$ be a closed cone. We denote by $\mathcal{R}$ the set of all constant functions defined on $\partial\Omega$. Putting

$$K_2 := \{u \in W^{1,p}(\Omega) : u|_{\partial\Omega} \in C_2\},$$

we have the following result.

**Theorem 3.3.** *If $C_2 \cap \mathcal{R} = \{0\}$ then*

$$\mu_{K_2} = \inf\left\{\frac{\|\nabla u\|_{L^p}}{\|u\|_{L^p}} : u \in K_2 \setminus \{0\}\right\} > 0.$$

**Proof.** Again, we apply Corollary 2.2 with the following choices:

$$(Y, \| \ \|) = (L^p(\Omega), \| \ \|_{L^p}),$$
$$X = W^{1,p}(\Omega), \qquad |u| = \|\nabla u\|_{L^p} \quad \forall u \in W^{1,p}(\Omega)$$

and $K = K_2$. It is standard that the space $W^{1,p}(\Omega)$ endowed with its usual norm

$$[u] := \|u\|_{1,p} = \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{\frac{1}{p}}$$

is reflexive. Since $C_2$ is closed in $L^p(\partial\Omega)$ and the trace mapping $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ $(\gamma(u) = u|_{\partial\Omega})$ is compact, we infer that $K_2$ is weakly closed in $W^{1,p}(\Omega)$. By the compactness of the inclusion $W^{1,p}(\Omega) \subset L^p(\Omega)$ it follows that the embedding $K_2 \subset L^p(\Omega)$ is also compact. From $C_2 \cap \mathcal{R} = \{0\}$ we obtain $\{u \in K_2 : \|\nabla u\|_{L^p} = 0\} = \{0\}$, which completes the proof. □

**Corollary 3.4.** *If $C_2 \cap \mathcal{R} = \{0\}$ then there is a constant $k > 0$ such that*

$$\|u\|_{L^p} \le k\|\nabla u\|_{L^p} \quad \forall u \in K_2 \cup (-K_2).$$

**Example 3.** We conclude this section with a Poincaré inequality on a set of type

$$K_3 := \left\{u \in W^{1,p}(\Omega) : \int_\Omega |u|^{q-2}u = 0\right\}.$$

Let $p^*$ be the Sobolev conjugate of $p$. Recall that this is

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ \infty & \text{if } p \ge N. \end{cases}$$

**Theorem 3.5.** *If $q \in (1, p^*)$ then there is a constant $k > 0$ such that*

$$\|u\|_{L^p} \le k\|\nabla u\|_{L^p} \quad \forall u \in K_3.$$

**Proof.** *Claim*: The cone $K_3$ is weakly closed in the space $(W^{1,p}(\Omega), \| \ \|_{1,p})$. To prove this we can argue as follows. Denoting by $N_f$ the Nemytskii operator associated with $f(s) = |s|^{q-2}s$, $s \in \mathbb{R}$, we know that $N_f$ is continuous from $L^q(\Omega)$ into $L^{q'}(\Omega)$

$(1/q + 1/q' = 1)$ (see [5]). On the other hand, the linear functional $L : L^{q'}(\Omega) \to \mathbb{R}$ defined by

$$\langle L, u\rangle = \int_\Omega u$$

is continuous. Therefore, the mapping $L \circ N_f : L^q(\Omega) \to \mathbb{R}$ is also continuous. But

$$L \circ N_f(u) = \int_\Omega |u|^{q-2}u \quad \forall u \in L^q(\Omega)$$

and the *claim* follows by the compactness of the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$.

Then, like in the proof of Theorem 3.3 we have that

$$\mu_{K_3} = \inf\left\{\frac{\|\nabla u\|_{L^p}}{\|u\|_{L^p}} : u \in K_3 \setminus \{0\}\right\} > 0,$$

which yields the conclusion. □

**Remark 3.6.** It should be noticed that Corollary 3.4 and Theorem 3.5 cannot be inferred by means of Theorem A.

## 4. An application

In this section we obtain the boundedness of the set of solutions for a two-parameter boundary value problem involving the vector $p$-Laplacian operator. The approach that we provide appears to be of interest when techniques such as Schaefer's theorem are employed in order to derive existence results (see e.g. [6, Theorem 1.1], [7, Theorem 2.1]).

Below we shall denote by $|\cdot|$ the Euclidean norm and $(\cdot|\cdot)$ will stand for the usual inner product on $\mathbb{R}^N$.

Let $h_p : \mathbb{R}^N \to \mathbb{R}^N$ be the homeomorphism defined by $h_p(x) = |x|^{p-2}x$ $(x \in \mathbb{R}^N)$ and $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ be a *Carathéodory* function—this means:

- for almost every $t \in [0,T]$ the function $f(t,\cdot)$ is continuous,
- for each $x \in \mathbb{R}^N$ the function $f(\cdot,x)$ is measurable,
- for each $\rho > 0$ there is $\alpha_\rho \in L^1([0,T];\mathbb{R}_+)$ such that, for almost every $t \in [0,T]$ and every $x \in \mathbb{R}^N$ with $|x| \le \rho$, one has

$$|f(t,x)| \le \alpha_\rho(t). \tag{4.1}$$

By a *weak solution* of the equation

$$-[h_p(u')]' = f(t,u) \quad \text{in } [0,T] \tag{4.2}$$

we will understand a function $u \in W^{1,p}([0,T];\mathbb{R}^N)$ which satisfies

$$\int_0^T (h_p(u')|\varphi') = \int_0^T (f(t,u)|\varphi) \quad \forall \varphi \in C_0^\infty((0,T);\mathbb{R}^N). \tag{4.3}$$

**Proposition 4.1.** *If $u$ is a weak solution of the Eq. (4.2) then $u \in C^1([0,T];\mathbb{R}^N)$, $h_p(u')$ is absolutely continuous and (4.2) is satisfied almost everywhere in $[0,T]$.*

**Proof.** Since $u \in W^{1,p}([0,T];\mathbb{R}^N)$, it follows that $h_p(u') \in L^{p'}([0,T];\mathbb{R}^N)$ $(1/p + 1/p' = 1)$. Also, the embedding $W^{1,p}([0,T];\mathbb{R}^N) \subset C([0,T];\mathbb{R}^N)$ and (4.1) imply $f(\cdot,u) \in L^1([0,T];\mathbb{R}^N)$. Using (4.3) we deduce that $h_p(u') \in W^{1,1}([0,T];\mathbb{R}^N)$ and (4.2) is satisfied almost everywhere in $[0,T]$. As $h_p$ is a homeomorphism, we get $u' \in C([0,T];\mathbb{R}^N)$. □

Let $f_\lambda : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ $(\lambda \in \Lambda)$ be a family of Carathéodory functions and $\mathcal{M} : D(\mathcal{M}) \subset \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ be a monotone mapping. Given a set $\Sigma \subset (0,\infty)$, we consider the two-parameter boundary value problem

$$-[h_p(u')]' = f_\lambda(t,u') \quad \text{in } [0,T], \tag{4.4$_\lambda$}$$

$$(h_p(su')(0), -h_p(su')(T)) \in \mathcal{M}(su(0), su(T)), \tag{4.4$_s$}$$

with $\lambda \in \Lambda$ and $s \in \Sigma$. Note that on account of Proposition 4.1 the boundary condition in (4.4$_s$) makes sense. It is worth pointing out that (4.4$_s$) recovers the classical boundary conditions as well as other ones of special interest [8,3,9].

Denoting by $\mathcal{S}$ the set of those $u \in W^{1,p}([0,T];\mathbb{R}^N)$ for which there is a pair $(\lambda,s) \in \Lambda \times \Sigma$ such that $u$ is a weak solution of (4.4$_\lambda$), (4.4$_s$), we are interested in the boundedness of $\mathcal{S}$ in the space $(W^{1,p}([0,T];\mathbb{R}^N), [\ ]_{1,p})$ (see (3.1)).

We introduce the constant

$$\mu_{\mathcal{M}} := \inf\left\{\frac{\|u'\|_{L^p}}{\|u\|_{L^p}} : u \in W^{1,p}([0,T];\mathbb{R}^N) \setminus \{0\},\ (u(0),u(T)) \in D(\mathcal{M})\right\}.$$

**Theorem 4.2.** *Assume that* $(0, 0) \in \mathcal{M}(0, 0)$. *If there are constants* $a < \mu_{\mathcal{M}}^p$ *and* $b \in \mathbb{R}_+$ *such that, for all* $\lambda \in \Lambda$, *the Carathéodory function* $f_\lambda$ *satisfies*

$$(f_\lambda(t, x)|x) \le a|x|^p + b \quad \text{for a.e. } t \in [0, T],\ \forall x \in \mathbb{R}^N, \tag{4.5}$$

*then the set* $\mathcal{S}$ *is bounded in* $(W^{1,p}([0, T]; \mathbb{R}^N), [\ ]_{1,p})$.

**Proof.** Let $u \in W^{1,p}([0, T]; \mathbb{R}^N)$ be a weak solution of $(4.4_\lambda)$, $(4.4_s)$ for some $(\lambda, s) \in \Lambda \times \Sigma$. Multiplying $(4.4_\lambda)$ by $u$, integrating over $[0, T]$ and using the integration by parts formula, we get

$$-(h_p(u')(T)|u(T)) + (h_p(u')(0)|u(0)) + \int_0^T |u'|^p = \int_0^T (f_\lambda(t, u)|u).$$

The monotonicity of $\mathcal{M}$, $(4.4_s)$ and the hypothesis $(0, 0) \in \mathcal{M}(0, 0)$ give

$$-(h_p(u')(T)|u(T)) + (h_p(u')(0)|u(0)) \ge 0.$$

Then, using (4.5) we obtain

$$\|u'\|_{L^p}^p \le a\|u\|_{L^p}^p + bT. \tag{4.6}$$

If $\mu_{\mathcal{M}}^p = 0$ then $a < 0$ and from (4.6) it follows that

$$[u]_{1,p}^p \le \frac{bT}{\min\{1, -a\}}.$$

If $\mu_{\mathcal{M}}^p > 0$ then, since $(su(0), su(T)) \in D(\mathcal{M})$, by the definition of $\mu_{\mathcal{M}}$ and (4.6) we infer

$$\mu_{\mathcal{M}}^p\|u\|_{L^p}^p \le \|u'\|_{L^p}^p \le a\|u\|_{L^p}^p + bT,$$

which gives

$$\|u\|_{L^p}^p \le \frac{bT}{\mu_{\mathcal{M}}^p - a}. \tag{4.7}$$

Then, using (4.6) we infer

$$\|u'\|_{L^p}^p \le \frac{\mu_{\mathcal{M}}^p}{\mu_{\mathcal{M}}^p - a}. \tag{4.8}$$

The conclusion follows from (4.7) and (4.8). □

Notice that according to Theorem 3.1, a sufficient condition ensuring that $\mu_{\mathcal{M}} > 0$ (equivalently, *the Poincaré inequality holds true*) is that $D(\mathcal{M})$ be a closed cone and $D(\mathcal{M}) \cap d = \{(0, 0)\}$. The case $\mu_{\mathcal{M}} > 0$ is of particular interest since the a priori boundedness of the set $\mathcal{S}$ can be established for positive values of the constant $a$ in (4.5), or more precisely for $a \in [0, \mu_{\mathcal{M}}^p)$.

## References


[1] C. Bandle, M. Flucher, Table of inequalities in elliptic boundary value problems, in: G.V. Milovanovic (Ed.), Recent Progress in Inequalities, Niš, 1996, in: Math. Appl., vol. 430, Kluwer Acad. Publ., Dordrecht, 1998, pp. 97–125.
[2] L. Gasinski, N.S. Papageorgiou, Nonlinear Analysis, Chapman & Hall/CRC, Boca Raton, 2006.
[3] P. Jebelean, Variational methods for ordinary $p$-Laplacian systems with potential boundary conditions, Adv. Differential Equations 13 (2008) 273–322.
[4] D. O'Regan, R. Precup, Theorems of Leray–Schauder Type and Applications, Gordon and Breach, Amsterdam, 2001.
[5] D.G. de Figueiredo, Lectures on the Ekeland Variational Principle with Applications and Detours, Tata Institute of Fundamental Research, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1989.
[6] G. Dincă, P. Jebelean, A priori estimates for the vector $p$-Laplacian with potential boundary conditions, Arch. Math. 90 (2008) 60–69.
[7] P. Jebelean, R. Precup, Solvability of $p, q$-Laplacian systems with potential boundary conditions, Appl. Anal. 89 (2) (2010) 221–228.
[8] L. Gasinski, N.S. Papageorgiou, Nonlinear second-order multivalued boundary value problems, Proc. Indian Acad. Sci. Math. Sci. 113 (3) (2003) 293–319.
[9] E.H. Papageorgiou, N.S. Papageorgiou, Nonlinear boundary value problems involving $p$-Laplacian and $p$-Laplacian-like operators, Z. Anal. Anwend. 24 (4) (2005) 691–707.