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Two positive nontrivial solutions for a class of semilinear elliptic variational systems

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ABSTRACT

We obtain existence and localization results of positive nontrivial solutions for a class of semilinear elliptic variational systems. The proof is based on variants of Schechter's localized critical point theorems for Hilbert spaces not identified to their duals and on the technique of inverse-positive matrices. The Leray–Schauder boundary condition is also involved.

1. Introduction

Schechter's localized critical point theorems [14] allow us to establish existence and localization of critical points of a C^1-functional, in a ball of a Hilbert space identified to its dual by using the Leray–Schauder condition on the ball boundary. Recently, in [9], a version of Schechter's results was obtained for Hilbert spaces which are not identified to their duals making possible an easier application to nonlinear equations. Also in [10] (see additionally [6,7,11,12]) we dealt with the vector method for the treatment of operator systems based on the use of inverse-positive matrices. The aim of this paper is to combine these two approaches in order to prove the existence and localization of positive nontrivial solutions for semilinear elliptic variational systems. Our new approach enriches the range of methods in the field (see e.g. [1–5,15]) and the results extend those for equations established in [9].

Firstly we present the variants of Schechter's localized critical point theorems for a Hilbert space not identified to its dual. Consider two real Hilbert spaces, X with inner product and norm $(.,.)$, $|.|$, and H with inner product and norm $\langle .,. \rangle$, $\|.\|$, and assume that $X \subset H$, X is dense in H, the injection being continuous. Denote by γ_0 the best embedding constant with

$$\|u\| \leqslant \gamma_0 |u| \quad \text{for all } u \in X, \tag{1.1}$$

that is $\gamma_0 = \sup\{\|u\|:\ u \in X,\ |u| = 1\}$. We identify H to its dual H', thanks to the Riesz representation theorem and we obtain

$$X \subset H \equiv H' \subset X'$$

where each space is dense in the following one, the injections being continuous. By $\langle .,. \rangle$ we also denote de natural duality between X and X', that is $\langle x^*, x \rangle = x^*(x)$ for $x \in X$ and $x^* \in X'$. When $x^* \in H$, one has that $\langle x^*, x \rangle$ is exactly the scalar

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product in H of x and x^*. Let L be the linear continuous operator from X to X' (the canonical isomorphism of X onto X'), given by

$$(u, v) = \langle Lu, v\rangle \quad \text{for all } u, v \in X,$$

and let J from X' into X be the inverse of L. Then

$$(Ju, v) = \langle u, v\rangle \quad \text{for all } u \in X',\ v \in X.$$

We note that, in particular, when $X = H$, $\|.\| = |.|$, one has $J = I$ and $\gamma_0 = 1$.

By a wedge of X we shall understand a convex closed nonempty set $K \subset X$, $K \neq \{0\}$, with $\lambda u \in K$ for every $u \in K$ and $\lambda \geqslant 0$. Thus K has not necessarily be a cone (when $K \cap (-K) = \{0\}$) and, in particular, K might be the whole space X.

In what follows we shall assume that J is "positive" with respect to K, i.e.,

$$Ju \in K \quad \text{for every } u \in K.$$

For a number $R > 0$, we denote by K_R the set $\{u \in K\colon\ |u| \leqslant R\}$ and by ∂K_R the set $\{u \in K\colon\ |u| = R\}$. We consider a C^1 real functional E defined on X and we are interested to solve the equation $E'(u) = 0$ in K_R.

We shall say that E satisfies the *Schechter–Palais–Smale condition*, *SPS condition* for short, in K_R provided that any sequence of elements $u_k \in K_R \setminus \{0\}$ for which

$$E(u_k) \to \mu, \qquad JE'(u_k) - \frac{(JE'(u_k), u_k)}{|u_k|^2} u_k \to 0, \qquad \big(JE'(u_k), u_k\big) \to \nu \leqslant 0$$

as $k \to \infty$, has a convergent subsequence.

We say that E has the *mountain pass property* in K_R if there are elements $v_0, v_1 \in K_R$ and number r such that $|v_0| < r < |v_1|$ and

$$\max\big\{E(v_0), E(v_1)\big\} < \inf\big\{E(u)\colon\ u \in X,\ |u| = r\big\}.$$

We denote by Γ_R the set of all continuous paths connecting v_0 and v_1 which do not leave K_R, i.e., $\Gamma_R = \{\gamma \in C([0,1]; K_R)\colon\ \gamma(0) = v_0,\ \gamma(1) = v_1\}$, and by ξ_R and m_R the following levels of energy

$$\xi_R = \inf_{\gamma \in \Gamma_R} \max_{t \in [0,1]} E\big(\gamma(t)\big), \qquad m_R = \inf_{u \in K_R} E(u).$$

Also, we say that E is *bounded from below* in K_R, if $m_R > -\infty$.

The main abstract results in [9] are as follows:

Theorem 1.1. *Assume that $(JE'(u), u) \geqslant -\nu_0$ for all $u \in K$ with $|u| = R$ and some $\nu_0 > 0$, E satisfies the SPS condition in K_R and the Leray–Schauder boundary condition*

$$JE'(u) + \mu u \neq 0 \quad \text{for all } u \in K \quad \text{with } |u| = R \text{ and } \mu > 0. \tag{1.2}$$

If E has the mountain pass property in K_R, then E has at least one critical point $u_\xi \in K_R \setminus \{v_0, v_1\}$ with $E(u_\xi) = \xi_R$.

Theorem 1.2. *Assume that $(JE'(u), u) \geqslant -\nu_0$ for all $u \in K$ with $|u| = R$ and some $\nu_0 > 0$, E satisfies the SPS condition in K_R and the Leray–Schauder boundary condition* (1.2)*. If E is bounded from below in K_R, then E has at least one critical point $u_m \in K_R$ with $E(u_m) = m_R$.*

Remark 1.1.

1^0 Let $N(u) := u - JE'(u)$. Then condition (1.2) can be written under the form

$$u \neq \lambda N u \quad \text{for all } u \in K \quad \text{with } |u| = R \text{ and } \lambda \in (0,1). \tag{1.3}$$

2^0 If N is a compact map in K_R, then E satisfies the SPS condition in K_R.

3^0 If all the assumptions of Theorems 1.1 and 1.2 hold, then the two critical points u_ξ, u_m are different. If in addition

$$E(v_1) \leqslant E(v_0), \tag{1.4}$$

then $u_m \neq v_0$.

Secondly we recall that a square matrix M of real numbers is said to be *inverse-positive* if its inverse M^{-1} has all the elements nonnegative. An invertible matrix M for which $I - M$ has all the elements nonnegative is inverse-positive if and only if $(I - M)^k \to 0$ as $k \to \infty$, or equivalently, if the spectral radius of $I - M$ is less than 1 (see [8,10,13]).

2. Main results

Consider the semilinear elliptic variational system

$$\begin{cases} -\Delta u_1 = F_1(u_1, u_2) & \text{in } \Omega, \\ -\Delta u_2 = F_2(u_1, u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded open set in $\mathbf{R}^n$ $(n \geqslant 1)$, $F \in C^1(\mathbf{R}_+^2, \mathbf{R}_+)$ with $F(0,0) = 0$, F_i stands for the partial derivative of $F(\tau_1, \tau_2)$ with respect to τ_i $(i = 1, 2)$, and Δ is the Laplacian operator. In case that $n \geqslant 2$ we also assume that

$$\begin{aligned} &F_1(\tau_1, \tau_2) \leqslant a_1 \tau_1^{p_1 - 1} + b_1 \tau_2^{q_1 - 1} + c_1, \\ &F_2(\tau_1, \tau_2) \leqslant a_2 \tau_1^{p_2 - 1} + b_2 \tau_2^{q_2 - 1} + c_2 \end{aligned} \tag{2.2}$$

for all $\tau_1, \tau_2 \in \mathbf{R}_+$, some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbf{R}_+$ and some $p_1, p_2, q_1, q_2 \in [1, 2^*)$. Here $2^* = 2n/(n-2)$ if $n > 2$ and $2^* = \infty$ if $n = 2$.

Let $X = H_0^1(\Omega, \mathbf{R}^2)$ with inner product and norm

$$(u, v) = \int_\Omega (\nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2)\, dx, \qquad |u| = \left(\int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dx \right)^{1/2}$$

and let $H = L^2(\Omega, \mathbf{R}^2)$ with inner product and norm

$$\langle u, v \rangle = \int_\Omega (u_1 v_1 + u_2 v_2)\, dx, \qquad \|u\| = \left(\int_\Omega \left(u_1^2 + u_2^2\right) dx \right)^{1/2}.$$

We also denote by $|.|_e$ the Euclidean norm in $\mathbf{R}^2$ and by $|u|_q$ the norm in $L^q(\Omega, \mathbf{R}^2)$ $(1 \leqslant q \leqslant \infty)$.

Here $E : H_0^1(\Omega, \mathbf{R}^2) \to \mathbf{R}$ is given by

$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u_1|^2 + \frac{1}{2}|\nabla u_2|^2 - F(u) \right) dx, \quad u \in H_0^1\left(\Omega, \mathbf{R}^2\right).$$

One has that $E'(u) = (-\Delta u_1 - F_1(u), -\Delta u_2 - F_2(u))$ in $H^{-1}(\Omega, \mathbf{R}^2)$,

$$(Jv, w) = \langle v, w \rangle \quad \text{for all } v \in H^{-1}\left(\Omega, \mathbf{R}^2\right),\ w \in H_0^1\left(\Omega . \mathbf{R}^2\right),$$

and $Jv = (-\Delta)^{-1} v$, for $v \in H^{-1}(\Omega, \mathbf{R}^2)$. Also $N(u) := u - JE'(u) = Jf(u)$, where $f(u) = (F_1(u), F_2(u))$, and

$$Jf(u) = (-\Delta)^{-1} f(u). \tag{2.3}$$

We note that the symbols $|.|_q$, $|.|$, $(.,.)$, $\langle .,. \rangle$ and J will also be used for scalar functions, i.e. in respect to the spaces $L^q(\Omega, \mathbf{R})$, $H_0^1(\Omega, \mathbf{R})$, $L^2(\Omega, \mathbf{R})$ and $H^{-1}(\Omega, \mathbf{R})$.

Assume that there are Banach spaces $(X_i, |.|_{X_i})$, $(Y_i, |.|_{Y_i})$ and $(Z_i, |.|_{Z_i})$ $(i = 1, 2)$ and continuous functions ψ_1, ψ_2: $\mathbf{R}_+^2 \to \mathbf{R}_+$ nondecreasing in both variables such that:

$$H_0^1(\Omega) \subset X_i \quad \text{and} \quad H_0^1(\Omega) \subset Y_i \quad \text{compactly}; \qquad Z_i \subset H^{-1}(\Omega) \quad \text{continuously}; \tag{2.4}$$

$$\text{the map } u \mapsto F_i(u) \text{ is continuous from } X_i \times Y_i \text{ to } Z_i; \quad \text{and} \quad \left|F_i(u)\right|_{Z_i} \leqslant \psi_i\left(|u_1|_{X_i}, |u_2|_{Y_i}\right) \tag{2.5}$$

for $i = 1, 2$. Then N is completely continuous from $H_0^1(\Omega, \mathbf{R}^2)$ to itself.

Denote by γ_{X_i}, γ_{Y_i}, γ_{Z_i} the embedding constants corresponding to (2.4), i.e.,

$$|u|_{X_i} \leqslant \gamma_{X_i} |u|, \qquad |u|_{Y_i} \leqslant \gamma_{Y_i} |u| \quad \text{for all } u \in H_0^1(\Omega) \quad \text{and}$$
$$|u|_{H^{-1}(\Omega)} \leqslant \gamma_{Z_i} |u|_{Zi} \quad \text{for all } u \in Z_i.$$

Now we consider the cone K in $H_0^1(\Omega, \mathbf{R}^2)$, defined by

$$K = \left\{u \in H_0^1\left(\Omega, \mathbf{R}^2\right):\ u \geqslant 0 \text{ in } \Omega\right\}.$$

Here by $u \geqslant 0$ we mean that $u_1 \geqslant 0$ and $u_2 \geqslant 0$. If $u \geqslant 0$ in Ω, then $f(u) \geqslant 0$ in Ω since $f(\mathbf{R}_+^2) \subset \mathbf{R}_+^2$ and so, according to (2.3), $Jf(u) \in K$. Consequently,

$$u - JE'(u) \in K \quad \text{for every } u \in K.$$

Notice that, in our case, (1.1) holds for $\gamma_0 = \frac{1}{\sqrt{\lambda_1}}$, and denote by ϕ the positive eigenfunction corresponding to the first eigenvalue λ_1, i.e.,

$$\Delta\phi + \lambda_1\phi = 0 \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \partial\Omega,$$

$\phi \geqslant 0$ and $|\phi| = 1$.

Our assumptions are as follows:

(H1) There exists $R > 0$ such that

$$\psi_1(\gamma_{X_1} R, 0) \leqslant \frac{1}{\gamma_{Z_1}} R, \tag{2.6}$$

$$\psi_2(0, \gamma_{Y_2} R) \leqslant \frac{1}{\gamma_{Z_2}} R, \tag{2.7}$$

$$|\tau|_e = R,\ \tau_1 \neq 0,\ \tau_2 \neq 0 \quad \Rightarrow \quad \begin{cases} \psi_i(\gamma_{X_i}\tau_1, \gamma_{Y_i}\tau_2) \leqslant \frac{1}{\gamma_{Z_i}}\tau_i \quad \text{for} \\ \quad i = 1 \text{ or } i = 2. \end{cases} \tag{2.8}$$

(H2) E has the mountain pass property in K_R, i.e., there exist $v_0, v_1 \in K_R$, and $r > 0$ such that $|v_0| < r < |v_1|$ and

$$\max\{E(v_0), E(v_1)\} < \inf_{\substack{u \in K \\ |u| = r}} E(u). \tag{2.9}$$

Theorem 2.1. *Assume that* (H1) *and* (H2) *hold. Then* (2.1) *has at least two distinct nontrivial solutions in* K_R.

Proof. We shall apply Theorems 1.1 and 1.2. First we show that (1.3) holds. Assume the contrary. Then $u = \lambda N u$ for some $u = (u_1, u_2) \in K$ with $|u| = R$ and $\lambda \in (0, 1)$. If $u_1 \neq 0$ and $u_2 \neq 0$, then

$$\begin{aligned} |u_1|^2 &= \lambda\big(N_1(u), u_1\big) = \lambda\big(JF_1(u), u_1\big) = \lambda\big\langle F_1(u), u_1\big\rangle \\ &< \big|F_1(u)\big|_{H^{-1}(\Omega)}|u_1| \leqslant \gamma_{Z_1}\big|F_1(u)\big|_{Z_1}|u_1| \\ &\leqslant \gamma_{Z_1}\psi_1\big(|u_1|_{X_1}, |u_2|_{Y_1}\big)|u_1| \\ &\leqslant \gamma_{Z_1}\psi_1\big(\gamma_{X_1}|u_1|, \gamma_{Y_1}|u_2|\big)|u_1|, \end{aligned} \tag{2.10}$$

and so

$$\psi_1\big(\gamma_{X_1}|u_1|, \gamma_{Y_1}|u_2|\big) > \frac{1}{\gamma_{Z_1}}|u_1|. \tag{2.11}$$

Similarly

$$\psi_2\big(\gamma_{X_2}|u_1|, \gamma_{Y_2}|u_2|\big) > \frac{1}{\gamma_{Z_2}}|u_2|. \tag{2.12}$$

Clearly (2.11) and (2.12) contradict (2.8). Similarly, using (2.6) and (2.7) we derive a contradiction if $u_2 = 0$ and $u_1 = 0$, respectively. Therefore (1.3) holds.

Next we check that E is bounded from below in K_R. First assume that $n \geqslant 2$. From (2.2) we have

$$\begin{aligned} F(\tau_1, \tau_2) &= \int_0^{\tau_1} F_1(s_1, \tau_2)\, ds_1 + F(0, \tau_2) \\ &= \int_0^{\tau_1} F_1(s_1, \tau_2)\, ds_1 + \int_0^{\tau_2} F_2(0, s_2)\, ds_2 \\ &\leqslant \frac{a_1}{p_1}\tau_1^{p_1} + b_1\tau_2^{q_1 - 1}\tau_1 + c_1\tau_1 + \frac{b_2}{q_2}\tau_2^{q_2} + c_2\tau_2. \end{aligned} \tag{2.13}$$

Let γ_p be the embedding constant for $H_0^1(\Omega) \subset L^p(\Omega)$, i.e., $|u|_p \leqslant \gamma_p|u|$ for every $u \in H_0^1(\Omega)$. Then using (2.13), if $u \in K_R$, we obtain

$$\begin{aligned}E(u) &= \int_\Omega \left(\frac{1}{2}|\nabla u_1|^2 + \frac{1}{2}|\nabla u_2|^2 - F(u)\right) dx \\ &\geqslant -\int_\Omega F(u)\,dx \\ &\geqslant -\int_\Omega \left(\frac{a_1}{p_1}u_1^{p_1} + b_1 u_2^{q_1-1}u_1 + c_1 u_1 + \frac{b_2}{q_2}u_2^{q_2} + c_2 u_2\right) dx \\ &\geqslant -\frac{a_1}{p_1}\gamma_{p_1}^{p_1}R^{p_1} - b_1\gamma_{q_1}^{q_1}R^{q_1} - c_1\gamma_1 R - \frac{b_2}{q_2}\gamma_{q_2}^{q_2}R^{q_2} - c_2\gamma_1 R \\ &> -\infty.\end{aligned}$$

Now assume that $n = 1$. Let γ_∞ be an embedding constant of $H_0^1(\Omega) \subset C(\overline{\Omega})$. Then, for every $u \in K_R$, we have

$$E(u) \geqslant -\int_\Omega F(u)\,dx \geqslant -\max_{\tau_1,\tau_2 \in [0, c_\infty R]} F(\tau_1,\tau_2)\,mes(\Omega) > -\infty.$$

Hence in any case $\inf_{K_R} E(u) > -\infty$. Thus Theorems 1.1 and 1.2 apply. □

The next results are concerning with some examples for which both conditions (H1) and (H2) are satisfied. We shall first assume that $n \geqslant 2$. In this case, in (2.4), we take

$$X_i = L^{p_i}(\Omega), \qquad Y_i = L^{q_i}(\Omega) \quad \text{and} \quad Z_i = L^{s_i}(\Omega)$$

with p_i, q_i like in (2.2) and $s_i = \min\{\frac{p_i}{p_i-1}, \frac{q_i}{q_i-1}\}$. Also, in (2.5), we take

$$\psi_i(\tau_1,\tau_2) = \widetilde{a_i}\tau_1^{p_i-1} + \widetilde{b_i}\tau_2^{q_i-1} + \widetilde{c_i},$$

where

$$\widetilde{a_i} = a_i\,mes(\Omega)^{\frac{p_i(1-s_i)+s_i}{p_i s_i}}, \qquad \widetilde{b_i} = b_i\,mes(\Omega)^{\frac{q_i(1-s_i)+s_i}{q_i s_i}}, \qquad \widetilde{c_i} = c_i\,mes(\Omega)^{\frac{1}{s_i}}.$$

Theorem 2.2. *Let $n \geqslant 2$. Assume that $F \in C^1(\mathbf{R}_+^2, \mathbf{R}_+)$, $F(0,0) = 0$, and (2.2) holds for some $p_1, q_2 \in [1,2)$ and $p_2, q_1 \in [1, 2^*)$ satisfying*

$$(q_1 - 1)(p_2 - 1) < 1. \tag{2.14}$$

In addition assume that

$$F(\tau) \geqslant c|\tau|_e^{\alpha+1} \quad \text{for all } \tau \in \mathbf{R}_+^2 \quad \text{with } 0 \leqslant |\tau|_e \leqslant \tau_0, \tag{2.15}$$

where $c, \tau_0 > 0$ and $\alpha > 1$,

$$\limsup_{|\tau|_e \to 0^+} \frac{F(\tau)}{|\tau|_e^2} < \frac{\lambda_1}{2} \tag{2.16}$$

and

$$\frac{1}{2} - c\tau_0^{\alpha-1} \int_{(\phi \leqslant 1)} \phi(x)^{\alpha+1}\,dx \leqslant 0. \tag{2.17}$$

Then problem (2.1) has at least two distinct positive nontrivial solutions.

Proof. Fix any number β with $\max\{2, p_1, p_2, q_1, q_2\} < \beta \leqslant 2^*$ and choose a d with

$$\limsup_{|\tau|_e \to 0^+} \frac{F(\tau)}{|\tau|_e^2} < d < \frac{\lambda_1}{2}.$$

From (2.13) and (2.16) we find that there exists a constant $c_d > 0$ with

$$F(\tau) \leqslant d|\tau|_e^2 + c_d|\tau|_e^\beta \quad \text{for all } \tau \in \mathbf{R}_+^2. \tag{2.18}$$

Then, for every $u \in K$, we have

$$\begin{aligned} E(u) &= \frac{|u|^2}{2} - \int_\Omega F(u)\,dx \\ &\geqslant \frac{|u|^2}{2} - \int_\Omega \left(d|u|_e^2 + c_d|u|_e^\beta\right) dx \\ &\geqslant \frac{|u|^2}{2} - d\frac{1}{\lambda_1}|u|^2 - c_d\gamma_\beta^\beta|u|^\beta \\ &= |u|^2\left(\frac{1}{2} - d\frac{1}{\lambda_1} - c_d\gamma_\beta^\beta|u|^{\beta-2}\right). \end{aligned}$$

Here γ_β is the embedding constant for $H_0^1(\Omega, \mathbf{R}^2) \subset L^\beta(\Omega, \mathbf{R}^2)$. Since $\frac{1}{2} - d\frac{1}{\lambda_1} > 0$ and $\beta > 2$, we can find a small enough number $r \in (0, \tau_0)$, such that

$$E(u) \geqslant \eta > 0$$

for all $u \in K$ with $|u| = r$ and some $\eta > 0$.

Let $v_0 = 0$ and $v_1 = \frac{\tau_0}{\sqrt{2}}(\phi, \phi)$. Clearly $|v_0| = 0 < r < \tau_0 = |v_1|$. Also $E(0) = 0$. Next (2.15) and (2.17) give

$$E(v_1) = \frac{\tau_0^2}{2} - \int_\Omega F(v_1)\,dx \leqslant \frac{\tau_0^2}{2} - c\tau_0^{\alpha+1} \int_{(\phi\leqslant 1)} \phi(x)^{\alpha+1}\,dx \leqslant 0.$$

Hence (1.4) and (2.9) hold.

Since $p_1, q_2 < 2$ one can see that (2.6), (2.7) hold for large enough R. Also, one of the following quantities

$$\frac{\widetilde{a_1}\tau_1^{p_1-1} + \widetilde{b_1}\tau_2^{q_1-1} + \widetilde{c_1}}{\tau_1}, \qquad \frac{\widetilde{a_2}\tau_1^{p_2-1} + \widetilde{b_2}\tau_2^{q_2-1} + \widetilde{c_2}}{\tau_2}$$

converges to zero as $|\tau|_e \to \infty$, proving (2.8). Indeed, three cases are possible if $|\tau|_e \to \infty$:

(a) $\tau_1 \to \infty$ and τ_2 is bounded: then the first ratio tends to zero obviously;

(b) τ_1 is bounded and $\tau_2 \to \infty$: the second ratio goes to zero;

(c) both τ_1, τ_2 tend to infinity: if $\frac{\tau_1^{p_2-1}}{\tau_2} \to 0$, then the second ratio goes to zero; if not, i.e. if $\frac{\tau_1^{p_2-1}}{\tau_2}$ is bounded from below by a positive number, then

$$\frac{\tau_1^{\frac{1}{q_1-1}}}{\tau_2} = \frac{\tau_1^{p_2-1}}{\tau_2}\tau_1^{\frac{1-(q_1-1)(p_2-1)}{(q_1-1)(p_2-1)}} \to \infty$$

and so

$$\frac{\tau_2^{q_1-1}}{\tau_1} = \left(\frac{\tau_2}{\tau_1^{\frac{1}{q_1-1}}}\right)^{q_1-1} \to 0.$$

Therefore both conditions (H1), (H2) are satisfied. The conclusion now follows from Theorem 2.1. □

Remark 2.1. According to Theorem 2.1 functions F_1 and F_2 are assumed to be sublinear in τ_1 and τ_2, respectively. However, by (2.14), one of them can be superlinear in the other variable.

Example 2.1. Let $F : \mathbf{R}_+^2 \to \mathbf{R}_+$ be defined by

$$F(\tau) = \begin{cases} c|\tau|_e^3 & \text{for } 0 \leqslant |\tau|_e \leqslant \tau_0, \\ a(|\tau|_e - \tau_0)^{3/2} + 3c\tau_0^2|\tau|_e - 2c\tau_0^3 & \text{for } |\tau|_e > \tau_0 \end{cases}$$

for some $a, c, \tau_0 > 0$. If $c\tau_0$ is sufficiently large, then (2.1) has at least two distinct positive nontrivial solutions. Indeed, conditions (2.2), (2.14), (2.15) and (2.16) can be immediately checked with $p_i = q_i = \frac{3}{2}$ for $i = 1, 2$ and $\alpha = 2$, while (2.17) holds provided that $c\tau_0$ is sufficiently large.

The next result deals with the case where both F_1 and F_2 have a linear growth in τ_1 and τ_2, i.e. $p_i = q_i = 2$ $(i = 1, 2)$ in (2.2). Then a non-resonance condition in terms of coefficients a_1, a_2, b_1, b_2 is necessary.

Theorem 2.3. *Assume all the assumptions of Theorem* 2.1 *are satisfied except* (2.2) *which holds with* $p_i = q_i = 2$ *for* $i = 1, 2$. *In addition assume that matrix*

$$M := I - \frac{1}{\lambda_1}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

is inverse-positive. Then problem (2.1) *has at least two distinct positive nontrivial solutions* $u = (u_1, u_2)$ *with* $|u_1| \leqslant R_1$ *and* $|u_2| \leqslant R_2$, *where*

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \frac{1}{\sqrt{\lambda_1}} M^{-1} \begin{pmatrix} \widetilde{c_1} \\ \widetilde{c_2} \end{pmatrix}.$$

Proof. First note that for $p_i = q_i = 2$, $\gamma_{X_i} = \gamma_{Y_i} = \gamma_{Z_i} = \gamma_2 = \frac{1}{\sqrt{\lambda_1}}$. We show that condition (1.3) holds without passing through (H1). Let $u = (u_1, u_2)$ be any solution in K of $u = \lambda N(u)$ for some $\lambda \in (0, 1)$. According to (2.10) we have

$$|u_1| \leqslant \frac{1}{\sqrt{\lambda_1}}\left(a_1 \frac{1}{\sqrt{\lambda_1}}|u_1| + b_1 \frac{1}{\sqrt{\lambda_1}}|u_2| + \widetilde{c_1}\right),$$
$$|u_2| \leqslant \frac{1}{\sqrt{\lambda_1}}\left(a_2 \frac{1}{\sqrt{\lambda_1}}|u_1| + b_2 \frac{1}{\sqrt{\lambda_1}}|u_2| + \widetilde{c_2}\right).$$

These can be written in the form

$$\begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} \leqslant \frac{1}{\lambda_1}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} + \frac{1}{\sqrt{\lambda_1}}\begin{pmatrix} \widetilde{c_1} \\ \widetilde{c_2} \end{pmatrix}.$$

Then

$$\begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} \leqslant \frac{1}{\sqrt{\lambda_1}} M^{-1}\begin{pmatrix} \widetilde{c_1} \\ \widetilde{c_2} \end{pmatrix},$$

that is $|u_1| \leqslant R_1$, $|u_2| \leqslant R_2$. Clearly (1.3) is satisfied for every $R > \sqrt{R_1^2 + R_2^2}$. □

A better result than Theorem 2.2 can be established for $n = 1$. In this case, in (2.4) and (2.5), we take

$$X_i = Y_i = Z_i = C(\overline{\Omega}) \quad \text{and} \quad \psi_i(\tau_1, \tau_2) = \max_{\substack{s_1 \in [0, \tau_1] \\ s_2 \in [0, \tau_2]}} F_i(s_1, s_2).$$

Theorem 2.4. *Let* $n = 1$ *and* $\Omega = (0, 1)$. *Assume that* $F \in C^1(\mathbf{R}_+^2, \mathbf{R}_+)$, $F(0, 0) = 0$ *and conditions* (2.15) *and* (2.17) *hold for some* $c, \tau_0 > 0$ *and* $\alpha > 1$. *In addition assume that*

$$\lim_{|\tau|_e \to 0^+} \frac{F(\tau)}{|\tau|_e^2} < \frac{1}{2} \tag{2.19}$$

and that there exists $R \geqslant \tau_0$ *such that*

$$\max_{\tau_1 \in [0, R]} F_1(\tau_1, 0) \leqslant \pi R,$$
$$\max_{\tau_2 \in [0, R]} F_2(0, \tau_2) \leqslant \pi R, \tag{2.20}$$

$$|\tau|_e = R,\ \tau_1 \neq 0,\ \tau_2 \neq 0 \quad \Rightarrow \quad \begin{cases} \max_{\substack{s_1 \in [0, \tau_1] \\ s_2 \in [0, \tau_2]}} F_i(s_1, s_2) \leqslant \pi \tau_i \\ \text{for } i = 1 \text{ or } i = 2. \end{cases} \tag{2.21}$$

Then problem (2.1) *has at least two distinct positive nontrivial solutions* $u = (u_1, u_2)$ *with* $|u| \leqslant R$.

Proof. In this case we may take $\gamma_{X_i} = \gamma_{Y_i} = 1$ and $\gamma_{Z_i} = \frac{1}{\sqrt{\lambda_1}}$. Indeed, if $u \in H_0^1(0, 1)$, then

$$|u(x)| = \left| \int_0^x u'(s)\, ds \right| \leqslant \int_0^1 |u'(s)|\, ds \leqslant \left(\int_0^1 u'^2\, ds \right)^{\frac{1}{2}} = |u|.$$

Hence $|u|_\infty \leqslant |u|$ and so $\gamma_{X_i} = \gamma_{Y_i} = 1$. Also, if $u \in L^2(0, 1)$, then

$$|u|_{H^{-1}(0,1)} = \sup_{\substack{v\in H_0^1(0,1)\\ v\neq 0}} \frac{|\langle u,v\rangle|}{|v|} \leqslant \frac{|u|_2|v|_2}{|v|} \leqslant \frac{1}{\sqrt{\lambda_1}}|u|_2.$$

If in addition $u \in Z_i = C[0,1]$, then $|u|_2 \leqslant |u|_\infty$ and so $|u|_{H^{-1}(0,1)} \leqslant \frac{1}{\sqrt{\lambda_1}}|u|_\infty$. Hence $\gamma_{Z_i} = \frac{1}{\sqrt{\lambda_1}}$.

From (2.19) it follows that there exists $r \in (0, \tau_0)$ with

$$F(\tau) \leqslant r^2\left(\frac{1}{2} - \varepsilon\right) \quad \text{for } |\tau|_e \leqslant r.$$

Let $u \in K$ with $|u| = r$. Since $|u|_\infty \leqslant |u| = r$, we have $|u(x)|_e \leqslant r$ for all $x \in [0,1]$. It follows that

$$E(u) = \frac{r^2}{2} - \int_0^1 F(u)\,dx \geqslant \frac{r^2}{2} - r^2\left(\frac{1}{2} - \varepsilon\right) = \varepsilon r^2 > 0.$$

This together with $E(0) = 0$ and $E(\tau_0\phi, \tau_0\phi) \leqslant 0$ (as follows from (2.15), (2.17)) guarantees (1.4) and (2.9). Finally, (2.20) and (2.21) guarantees (2.6)–(2.8) since $\gamma_{Z_i} = \frac{1}{\sqrt{\lambda_1}} = \frac{1}{\pi}$. □

Example 2.2. Let $F : \mathbf{R}_+^2 \to \mathbf{R}_+$ be defined by

$$F(\tau) = \begin{cases} c|\tau|_e^m & \text{for } 0 \leqslant |\tau|_e \leqslant \tau_0, \\ mc\tau_0^{m-1}|\tau|_e - (m-1)c\tau_0^m & \text{for } |\tau|_e > \tau_0 \end{cases}$$

for some $c, \tau_0 > 0$ and $m > 2$. If $c\tau_0^{m-2}$ is sufficiently large, then (2.1) has at least two distinct positive nontrivial solutions $u = (u_1, u_2)$ with $|u| \leqslant \frac{mc\tau_0^{m-1}}{\pi}$. Indeed, conditions (2.19) and (2.15) are obviously satisfied with $\alpha = m-1$, while (2.17) holds provided that $c\tau_0^{m-2}$ is sufficiently large. Also, since

$$F_i(\tau) = \begin{cases} mc|\tau|_e^{m-2}\tau_i & \text{for } 0 \leqslant |\tau|_e \leqslant \tau_0, \\ \frac{mc\tau_0^{m-1}\tau_i}{|\tau|_e} & \text{for } |\tau|_e > \tau_0 \end{cases}$$

we can immediately see that conditions (2.20), (2.21) are fulfilled provided that $mc\tau_0^{m-1} \leqslant \pi R$. Thus, if we assume that $c\tau_0^{m-2} \geqslant \frac{\pi}{m}$ (in order to have $R \geqslant \tau_0$), we may choose $R = \frac{mc\tau_0^{m-1}}{\pi}$.

Example 2.3. Let $F : \mathbf{R}_+^2 \to \mathbf{R}_+$ be defined by

$$F(\tau) = \begin{cases} c|\tau|_e^m & \text{for } 0 \leqslant |\tau|_e \leqslant \tau_0, \\ a(|\tau|_e - \tau_0)^p|\tau|_e^q + mc\tau_0^{m-1}|\tau|_e - (m-1)c\tau_0^m & \text{for } |\tau|_e > \tau_0 \end{cases}$$

for some $a, c, \tau_0 > 0$; $1 < p < 2$, $0 \leqslant q < 2 - p$ and $m > 2$. If $c\tau_0^{m-2}$ is sufficiently large, then (2.1) has at least two distinct positive nontrivial solutions. Indeed, one has

$$F_i(\tau) = \begin{cases} mc|\tau|_e^{m-2}\tau_i & \text{for } 0 \leqslant |\tau|_e \leqslant \tau_0, \\ pa(|\tau|_e - \tau_0)^{p-1}|\tau|_e^q + qa(|\tau|_e - \tau_0)^p|\tau|_e^{q-1}\frac{\tau_i}{|\tau|_e} + mc\tau_0^{m-1}\frac{\tau_i}{|\tau|_e} & \text{for } |\tau|_e > \tau_0. \end{cases}$$

Then

$$\max_{\tau_1 \in [0,R]} F_1(\tau_1, 0) = pa(R - \tau_0)^{p-1}R^q + qa(R - \tau_0)^pR^{q-1} + mc\tau_0^{m-1}.$$

Since $p + q < 2$, we have that

$$pa(R - \tau_0)^{p-1}R^q + qa(R - \tau_0)^pR^{q-1} + mc\tau_0^{m-1} \leqslant \pi R$$

for large enough R. Similarly one can guarantee the second condition in (2.20). To check (2.21) take any $\tau \in \mathbf{R}_+^2$ with $|\tau|_e = R$ and $\tau_1, \tau_2 \neq 0$. From $|\tau|_e = R$, it follows that $\tau_1 \geqslant \frac{R}{\sqrt{2}}$ or $\tau_2 \geqslant \frac{R}{\sqrt{2}}$. Assume that $\tau_1 \geqslant \frac{R}{\sqrt{2}}$. Then condition (2.21) is fulfilled for $i = 1$ provided that

$$pa(R - \tau_0)^{p-1}R^q + qa(R - \tau_0)^pR^{q-1} + mc\tau_0^{m-1} \leqslant \pi\frac{R}{\sqrt{2}}$$

which again holds for large enough R. Similarly, if $\tau_2 \geqslant \frac{R}{\sqrt{2}}$, then (2.21) is fulfilled for $i = 2$.

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