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# Positive solutions of functional-differential systems via the vector version of Krasnoselskii's fixed point theorem in cones

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ABSTRACT. We study the existence of positive solutions of the functional-differential system

$$\begin{cases} u_1''(t) + a_1(t)f_1(u_1(g(t)), u_2(g(t))) = 0, \\ u_2''(t) + a_2(t)f_2(u_1(g(t)), u_2(g(t))) = 0 \end{cases}$$

$(0 < t < 1)$, subject to linear boundary conditions. We prove the existence of at least one positive solution by using the vector version of Krasnoselskii's fixed point theorem in cones.

## 1. INTRODUCTION

This paper deals with the second-order functional-differential system

$$\begin{cases} u_1''(t) + a_1(t)f_1(u_1(g(t)), u_2(g(t))) = 0, \\ u_2''(t) + a_2(t)f_2(u_1(g(t)), u_2(g(t))) = 0 \end{cases} \tag{1.1}$$

$(0 < t < 1)$, under the boundary conditions

$$\begin{cases} \alpha_i u_i(0) - \beta_i u_i'(0) = 0, \\ \gamma_i u_i(1) + \delta_i u_i'(1) = 0, \\ u_i(t) = k_i \quad \text{for } -\theta \le t < 0 \ (i = 1, 2). \end{cases} \tag{1.2}$$

Here $\theta > 0$ and $g : [0,1] \to [-\theta, 1]$. We seek positive solutions to (1.1)-(1.2), that is a couple $u = (u_1, u_2)$ with $u_i(t) > 0$ for $0 < t < 1$ and $i = 1, 2$.

We shall assume that the following conditions are satisfied for $i \in \{1, 2\}$ :

(A1) $f_i \in C([0,\infty)^2, [0,\infty))$ and $g \in C([0,1], [-\theta, 1])$;

(A2) $a_i \in C([0,1], [0,\infty))$ and $a_i(t)$ is not identically zero on any proper subinterval of $[0,1]$;

(A3) $\alpha_i, \beta_i, \gamma_i, \delta_i, k_i \ge 0$, $\rho_i := \gamma_i\beta_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$.

Our existence result is based on the vector version of Krasnoselskii's fixed point theorem in cones, due to the second author [5] (see also [6]-[8]) and extends to systems the main result from [1]. To present the vector version of Krasnoselskii's theorem, we need to introduce some notations and notions. Let $(X, \|.\|)$ be a normed linear space, let $K_1$, $K_2$ be two cones of $X$ and let $K := K_1 \times K_2$. We shall use the same symbol $\preceq$ to denote the partial order relation induced by $K$ in $X^2$ and by $K_1$, $K_2$ in $X$. Similarly, the same symbol $\prec$ will be used to denote the strict order relation induced by $K_1$ and $K_2$ in $X$. Also, in $X^2$, the symbol $\prec$ will have the following meaning: $u \prec v$ $(u, v \in X^2)$ if

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$u_i \prec v_i$ for $i=1,2$. For $r,R \in \mathbf{R}_+^2$, $r=(r_1,r_2)$, $R=(R_1,R_2)$, we write $0<r<R$ if $0<r_1<R_1$ and $0<r_2<R_2$ and we use the notations:

$$\begin{aligned} (K_i)_{r_i,R_i} &: = \{v \in K_i : r_i \leq \|v\| \leq R_i\} \quad (i=1,2) \\ K_{r,R} &: = \{u=(u_1,u_2) \in K : r_i \leq \|u_i\| \leq R_i \text{ for } i=1,2\}. \end{aligned}$$

Clearly, $K_{r,R} = (K_1)_{r_1,R_1} \times (K_2)_{r_2,R_2}$.

Now we are ready to present the vector version of Krasnoselskii's fixed point theorem in cones.

**Theorem 1.1 ([6]).** *Let $(X, \|.\|)$ be a normed linear space; $K_1, K_2 \subset X$ two cones; $K := K_1 \times K_2$; $r,R \in \mathbf{R}_+^2$ with $0<r<R$, and let $N : K_{r,R} \to K$, $N=(N_1,N_2)$ be a compact map. Assume that for each $i \in \{1,2\}$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) *$N_i(u) \not\prec u_i$ if $\|u_i\| = r_i$, and $N_i(u) \not\succ u_i$ if $\|u_i\| = R_i$;*

(b) *$N_i(u) \not\succ u_i$ if $\|u_i\| = r_i$, and $N_i(u) \not\prec u_i$ if $\|u_i\| = R_i$.*

*Then $N$ has a fixed point $u$ in $K$ with $r_i \leq \|u_i\| \leq R_i$ for $i \in \{1,2\}$.*

**Remark 1.1.** In Theorem 1.1 four cases are posible for $u \in K_{r,R}$:

(c1) $N_1(u) \not\prec u_1$ if $\|u_1\| = r_1$, and $N_1(u) \not\succ u_1$ if $\|u_1\| = R_1$,
$N_2(u) \not\prec u_2$ if $\|u_2\| = r_2$, and $N_2(u) \not\succ u_2$ if $\|u_2\| = R_2$;

(c2) $N_1(u) \not\prec u_1$ if $\|u_1\| = r_1$, and $N_1(u) \not\succ u_1$ if $\|u_1\| = R_1$,
$N_2(u) \not\succ u_2$ if $\|u_2\| = r_2$, and $N_2(u) \not\prec u_2$ if $\|u_2\| = R_2$;

(c3) $N_1(u) \not\succ u_1$ if $\|u_1\| = r_1$, and $N_1(u) \not\prec u_1$ if $\|u_1\| = R_1$,
$N_2(u) \not\prec u_2$ if $\|u_2\| = r_2$, and $N_2(u) \not\succ u_2$ if $\|u_2\| = R_2$;

(c4) $N_1(u) \not\succ u_1$ if $\|u_1\| = r_1$, and $N_1(u) \not\prec u_1$ if $\|u_1\| = R_1$,
$N_2(u) \not\succ u_2$ if $\|u_2\| = r_2$, and $N_2(u) \not\prec u_2$ if $\|u_2\| = R_2$.

Related results obtained by means of Krasnoselskii's fixed point theorem in cones [4] can be found in [2], [3], [9] - [11].

## 2. The main result

Let $G_i$ be the Green function of the problem

$$\begin{cases} u''=0, \\ \alpha_i u(0) - \beta_i u'(0) = 0, \\ \gamma_i u(1) + \delta_i u'(1) = 0. \end{cases}$$

One has

$$G_i(t,s) = \begin{cases} \dfrac{1}{\rho_i}\varphi_i(t)\psi_i(s), & 0 \leq s \leq t \leq 1 \\ \dfrac{1}{\rho_i}\varphi_i(s)\psi_i(t), & 0 \leq t \leq s \leq 1, \end{cases}$$

where $\varphi_i(t) := \gamma_i + \delta_i - \gamma_i t$, $\psi_i(t) := \beta_i + \alpha_i t$ $(0 \leq t \leq 1)$.
Notice that

$$G_i(t,s) \leq \frac{1}{\rho_i}\varphi_i(s)\psi_i(s) = G_i(s,s) \quad \text{for } 0 \leq t,s \leq 1. \tag{2.1}$$

In addition the following result (Lemma 2.1 from [1]) holds:

**Lemma 2.1.** *For every $n > \frac{4}{3}$ and $i \in \{1,2\}$, there exists $M_i(n) > 0$ such that*

$$\frac{G_i(t,s)}{G_i(s,s)} \geq M_i(n) \quad \textit{for } \frac{1}{n} \leq t \leq \frac{3}{4} \textit{ and } 0 \leq s \leq 1. \tag{2.2}$$

In what follows by $\|v\|$, where $v \in C[0,1]$, we mean the norm $\|v\| = \max_{t\in[0,1]} |v(t)|$. Also we make the following notations:

$$(f_i)_0(x_j) := \lim_{x_i \to 0} \frac{f_i(x_1,x_2)}{x_i}, \quad (f_i)_\infty(x_j) := \lim_{x_i \to \infty} \frac{f_i(x_1,x_2)}{x_i}$$

for $i,j \in \{1,2\}$ with $i+j=3$.

The main result of this paper is the following existence theorem.

**Theorem 2.2.** *Assume that conditions* (A1)-(A3) *hold. In addition assume that* $g \in C^1([0,1],[-\theta,1])$, $g' > 0$, $g(0) \le 0$ *and* $g(1) > 0$. *Then problem* (1.1)-(1.2) *has at least one positive solution* $u := (u_1,u_2)$ *in each of the following four cases:*

$$\text{(2.3)} \quad \begin{cases} (f_1)_0(x_2) = 0 & \textit{uniformly for all } x_2 \ge 0, \\ (f_1)_\infty(x_2) = \infty & \textit{uniformly for all } x_2 \ge 0, \\ (f_2)_0(x_1) = 0 & \textit{uniformly for all } x_1 \ge 0, \\ (f_2)_\infty(x_1) = \infty & \textit{uniformly for all } x_1 \ge 0; \end{cases}$$

$$\text{(2.4)} \quad \begin{cases} (f_1)_0(x_2) = \infty & \textit{uniformly for all } x_2 \ge 0, \\ (f_1)_\infty(x_2) = 0 & \textit{uniformly for all } x_2 \ge 0, \textit{ if } f_1 \textit{ is unbounded,} \\ (f_2)_0(x_1) = \infty & \textit{uniformly for all } x_1 \ge 0, \\ (f_2)_\infty(x_1) = 0 & \textit{uniformly for all } x_1 \ge 0, \textit{ if } f_2 \textit{ is unbounded;} \end{cases}$$

$$\text{(2.5)} \quad \begin{cases} (f_1)_0(x_2) = \infty & \textit{uniformly for all } x_2 \ge 0, \\ (f_1)_\infty(x_2) = 0 & \textit{uniformly for all } x_2 \ge 0, \textit{ if } f_1 \textit{ is unbounded,} \\ (f_2)_0(x_1) = 0 & \textit{uniformly for all } x_1 \ge 0, \\ (f_2)_\infty(x_1) = \infty & \textit{uniformly for all } x_1 \ge 0; \end{cases}$$

$$\text{(2.6)} \quad \begin{cases} (f_1)_0(x_2) = 0 & \textit{uniformly for all } x_2 \ge 0, \\ (f_1)_\infty(x_2) = \infty & \textit{uniformly for all } x_2 \ge 0, \\ (f_2)_0(x_1) = \infty & \textit{uniformly for all } x_1 \ge 0, \\ (f_2)_\infty(x_1) = 0 & \textit{uniformly for all } x_1 \ge 0, \textit{ if } f_2 \textit{ is unbounded.} \end{cases}$$

*Proof.* First note that $u = (u_1,u_2) \in C([0,1],R_+^2)$ is a solution of (1.1)-(1.2) if and only if $u$ solves the operator system

$$u_i(t) = \int_0^1 G_i(t,s)a_i(s)f_i(u_1(g(s)), u_2(g(s)))ds := N_i u(t), \ i = 1,2.$$

Let $K_i$ be the cone in $C[0,1]$ given by

$$K_i = \{v \in C[0,1] : v \ge 0, \min_{a\le t\le b} v(t) \ge M_i \|v\|\},$$

where $a = \frac{1}{n}$, $b = \min\left\{g(1), \frac{3}{4}\right\}$ and $M_i$ stands for $M_i(n)$ given by Lemma 2.1. Then, for $u \in K = K_1 \times K_2$, using (2.1) and (2.2) we obtain

$$\begin{aligned} \min_{a\le t\le b} N_i u(t) &= \min_{a\le t\le b} \int_0^1 G_i(t,s)a_i(s)f_i(u(g(s)))ds \\ &\ge M_i \int_0^1 G_i(s,s)a_i(s)f_i(u(g(s)))ds \\ &\ge M_i \int_0^1 G_i(t',s)a_i(s)f_i(u(g(s)))ds = M_i N_i u(t') \end{aligned}$$

for every $t' \in [0,1]$. Consequently

$$\min_{a\le t\le b} N_i u\,(t) \ge M_i \left\| N_i u \right\|.$$

Therefore, $N_i(K) \subset K_i$, for $i = 1,2$. Moreover, it is easy to see that $N_i : K \to K_i$ is completely continuous. Hence $N : K_{r,R} \to K$, $N = (N_1, N_2)$ is well defined and compact for every $r, R \in \mathbf{R}^2_+$ with $0 < r < R$.

Assume now that (2.3) holds. We will prove that $N$ satisfies condition (c4) from Remark 1.1. The first relation from (2.3) guarantees that there exists $H_1 > 0$ so that $f_1(u_1,u_2) \le \eta u_1$, for $0 < u_1 \le H_1$ and $u_2 \ge 0$, where $\eta > 0$ may be chosen conveniently. We choose $0 < r_1 \le H_1$. Suppose that $N_1(u) \succ u_1$ for $\|u_1\| = r_1$. From $\|u_1\| = r_1 \le H_1$, we have that $0 < u_1 \le H_1$, so

$$\begin{aligned} \text{(2.7)} \qquad \|N_1 u\| &\le \int_0^1 G_1(s,s)a_1(s)f_1(u\,(g\,(s)))ds \\ &\le \eta \int_0^1 G_1(s,s)a_1(s)u_1\,(g\,(s))\,ds \\ &= \eta \int_{g(0)}^{g(1)} G_1(g^{-1}(y),g^{-1}(y))\frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))}u_1(y)dy \\ &\le \eta r_1 \int_{g(0)}^{g(1)} G_1(g^{-1}(y),g^{-1}(y))\frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))}dy, \end{aligned}$$

using that $u_1(y) \le \|u_1\| = r_1$. We choose $\eta > 0$ so that

$$\text{(2.8)} \qquad \eta r_1 \int_{g(0)}^{g(1)} G_1(g^{-1}(y),g^{-1}(y))\frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))}dy \le M_1 r_1.$$

From $u_1 \in K_1$ we obtain $r_1 M_1 = \|u_1\|\, M_1 \le \min_{a\le t\le b} u_1(t)$, so (2.7) and (2.8) imply

$$\|N_1 u\| \le \min_{a\le t\le b} u_1(t),$$

and from $N_1 u \le \|N_1 u\|$ it follows

$$N_1 u(t) \le \min_{a\le t\le b} u_1(t), \ \text{ for } t \in [0,1],$$

so $N_1 u(t) \le \min_{a\le t\le b} u_1(t) \le u_1(t)$, for $t \in [a,b]$, a contradiction with the assumption $N_1 u \succ u_1$. So $N_1 u \not\succ u_1$ if $\|u_1\| = r_1$. From (2.3) we have that there exists $\bar{H}_2 > 0$ so that $f_1(u_1,u_2) \ge \mu u_1$, for $u_1 \ge \bar{H}_2$ and $u_2 \ge 0$, where $\mu > 0$ may be chosen conveniently. Let $R_1 := \max\left\{2H_1, \dfrac{\bar{H}_2}{M_1}\right\}$. We suppose that $N_1(u) \prec u_1$ if $\|u_1\| = R_1$. Using the hypothesis $g(1) > 0$, $g(0) < 0$, $g' > 0$, we obtain for $a \le t_0 \le b$, $u_1 \in K_1$, $\|u_1\| = R_1$, that

$$\min_{a\le t\le b} u_1(t) \ge M_1 \left\|u_1\right\| \ge \bar{H}_2,$$

so

$$
\begin{aligned}
N_1 u(t_0) &= \int_{g(0)}^{g(1)} G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} f_1(u(y))dy \\
&\geq \int_a^b G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} f_1(u(y))dy \\
&\geq \mu \int_a^b G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} u_1(y)dy \\
&\geq \mu M_1 \left[ \int_a^b G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} dy \right] \|u_1\| \\
&\geq \|u_1\| \geq u_1(t), \quad \text{for } t \in [0,1],
\end{aligned}
$$

if we choose $\mu > 0$ so that

$$
\mu M_1 \left[ \int_a^b G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} dy \right] \geq 1.
$$

Here we used the inclusion $[a, b] \subset [g(0), g(1)]$ which is true for the above chosen $b$ and some $n$ with $a = 1/n < b$. So $N_1(u)(t_0) \geq u_1(t_0)$, a contradiction with the assumption $N_1(u) \prec u_1$ if $\|u_1\| = R_1$. So, we obtain $N_1(u) \not\prec u_1$ if $\|u_1\| = R_1$.

Similarly, from (2.3) we obtain that

$$
N_2(u) \not\succ u_2 \text{ if } \|u_2\| = r_2 \text{ and } N_2(u) \not\prec u_2 \text{ if } \|u_2\| = R_2.
$$

Thus condition (c4) from Remark 1.1 holds.

Assume now that (2.4) holds. We will prove that $N$ satisfies condition (c1) from Remark 1.1. From (2.4) we have that there exists $H_1 > 0$ so that $f_1(u) \geq \bar{\eta} u_1$, for $0 < u_1 \leq H_1$ and all $u_2 \geq 0$, where $\bar{\eta}$ may be chosen conveniently. We choose $M_1$ from the definition of cone $K_1$. We also choose $r_1$ so that $0 < r_1 \leq H_1$. We suppose that $N_1(u) \prec u_1$ if $\|u_1\| = r_1$. Then, for $a \leq t_0 \leq b, u_1 \in K_1$ and $\|u_1\| = r_1 \leq H_1$, we have

$$
\begin{aligned}
N_1(u)(t_0) &= \int_{g(0)}^{g(1)} G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} f_1(u(y))dy \\
&\geq \int_a^b G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} f_1(u(y))dy \\
&\geq \bar{\eta} \int_a^b G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} u_1(y)dy \\
&\geq \left[ \bar{\eta} M_1 \int_a^b G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} dy \right] \|u_1\| \geq \|u_1\|,
\end{aligned}
$$

if we choose $\bar{\eta} > 0$ so that

$$
\bar{\eta} M_1 \int_a^b G_1(t_0, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g'(g^{-1}(y))} dy \geq 1.
$$

We obtain $N_1(u)(t_0) \geq \|u_1\| \geq u_1(t)$, for all $t \in [0,1]$, and we have that $N_1(u)(t_0) \geq u_1(t_0)$, a contradiction with the assumption that we made. It follows that $N_1(u) \not\prec u_1$ if $\|u_1\| = r_1$.

Part 1. We assume that $f_1$ is unbounded. From (2.4) we have that there exists $Q_1 > 0$ so that $f_1(u) \leq \lambda u_1$, for $u_1 \geq Q_1$, for all $u_2 \geq 0$, where $\lambda > 0$ may be chosen conveniently. There exists $H_2 = (H_2^{'}, H_2^{''})$ with $H_2^{'} > \max\{Q_1, r_1\}$ so that $f_1(u) \leq f_1(H_2)$, for $0 < u \leq H_2$ (we may do this because $f_1$ is unbounded). We choose $R_1 := H_2^{'}$ and we suppose that $N_1(u) \succ u_1$ if $\|u_1\| = R_1$. Then, for all $u_2$ with $\|u_2\| \leq H_2^{''}$, we have:

$$\begin{aligned} N_1(u)(t) &= \int_{g(0)}^{g(1)} G_1(t, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g^{'}(g^{-1}(y))} f_1(u(y))dy \\ &\leq \int_{g(0)}^{g(1)} G_1(g^{-1}(y), g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g^{'}(g^{-1}(y))} f_1(u(y))dy \\ &\leq \int_{g(0)}^{g(1)} G_1(g^{-1}(y), g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g^{'}(g^{-1}(y))} f_1(H_2)dy \\ &\leq \lambda H_2^{'} \int_{g(0)}^{g(1)} G_1(g^{-1}(y), g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g^{'}(g^{-1}(y))} dy < \frac{H_2^{'}}{2} = \frac{R_1}{2}, \end{aligned}$$

if we choose $\lambda > 0$ so that

$$\lambda \int_{g(0)}^{g(1)} G_1(g^{-1}(y), g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g^{'}(g^{-1}(y))} dy < \frac{1}{2}.$$

So $N_1(u)(t) \leq \frac{R_1}{2}$, for all $t \in [0,1]$. But, by our assumption, $N_1(u) \succ u_1$, we obtain $\|N_1(u)\| \geq \|u_1\| = R_1$, so $R_1 \leq \frac{R_1}{2}$, a contradiction. It follows that $N_1(u) \not\succ u_1$, for $\|u_1\| = R_1$.

Part 2. We assume that $f_1$ is bounded, so $\sup\limits_{u \in (0,\infty)^2} f_1(u) = M^{'} < \infty$. We choose $R_1$ so that

$$R_1 > 2M^{'} \left( \int_{g(0)}^{g(1)} G_1(g^{-1}(y), g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g^{'}(g^{-1}(y))} dy \right).$$

We suppose that $N_1(u) \succ u_1$ for $\|u_1\| = R_1$, so

$$\begin{aligned} N_1(u)(t) &= \int_{g(0)}^{g(1)} G_1(t, g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g^{'}(g^{-1}(y))} f_1(u(y))dy \leq \\ &\leq M^{'} \int_{g(0)}^{g(1)} G_1(g^{-1}(y), g^{-1}(y)) \frac{a_1(g^{-1}(y))}{g^{'}(g^{-1}(y))} dy < \frac{R_1}{2}. \end{aligned}$$

It follows that $\|N_1 u\| \leq \frac{R_1}{2}$. But, from the assumption that we made, we obtain $\|N_1 u\| \geq \|u_1\| = R_1$, a contradiction. It follows that $N_1(u) \not\succ u_1$ if $\|u_1\| = R_1$. Similarly, we obtain, for $f_2$, that $N_2(u) \not\prec u_2$ if $\|u_2\| = r_2$ and $N_2(u) \not\succ u_2$ if $\|u_2\| = R_2$, so condition (c1) from Remark 1.1 holds.

Assume now that (2.5) holds. Using the same arguments for $f_1$, like in the case (2.4), we obtain:

$$N_1(u) \not\prec u_1 \text{ if } \|u_1\| = r_1 \text{ and } N_1(u) \not\succ u_1 \text{ if } \|u_1\| = R_1.$$

Using the same arguments for $f_2$, like in the case (2.3), we obtain:

$$N_2(u) \not\succ u_2 \text{ if } \|u_2\| = r_2 \text{ and } N_2(u) \not\prec u_2 \text{ if } \|u_2\| = R_2.$$

It follows that condition (c2) from Remark 1.1 is satisfied.

Assume now that (2.6) holds. Using the same arguments for $f_1$, like in the case (2.3), we obtain:

$$N_1(u) \not\geq u_1 \text{ if } \|u_1\| = r_1 \text{ and } N_1(u) \not\leq u_1 \text{ if } \|u_1\| = R_1.$$

Using the same arguments for $f_2$, like in the case (2.4), we obtain:

$$N_2(u) \not\leq u_2 \text{ if } \|u_2\| = r_2 \text{ and } N_2(u) \not\geq u_2 \text{ if } \|u_2\| = R_2.$$

Thus condition (c3) from Remark 1.1 holds. □

**Remark 2.2.** (1) An example of functions like in (2.3):

$$\begin{aligned} f_1(x_1,x_2) &= \frac{x_1^p(x_2+1)}{x_1+x_2+1}, \\ f_2(x_1,x_2) &= x_1^p\left[\frac{1}{(x_2+1)^q}+1\right]+x_1^r\left[\frac{1}{(x_2+1)^s}+1\right], \end{aligned}$$

where $r > 1$; $p > 2$; $q, s > 0$.

Indeed,

$$\left|\frac{f_1(x_1,x_2)}{x_1}\right| = \frac{x_1^{p-1}(x_2+1)}{x_1+x_2+1} = \frac{x_1^{p-1}}{1+\frac{x_1}{x_2+1}} \leq x_1^{p-1} \to 0$$

as $x_1 \to 0$, for $p > 2$, so $(f_1)_0(x_2) = 0$ uniformly for all $x_2 \geq 0$, and

$$\left|\frac{f_1(x_1,x_2)}{x_1}\right| = \frac{x_1^{p-1}}{1+\frac{x_1}{x_2+1}} \geq \frac{x_1^{p-1}}{1+x_1} \to \infty$$

as $x_1 \to \infty$, for $p > 2$, so $(f_1)_\infty(x_2) = \infty$ uniformly for all $x_2 \geq 0$. Also

$$\begin{aligned} \left|\frac{f_2(x_1,x_2)}{x_1}\right| &= x_1^{p-1}\left[\frac{1}{(x_2+1)^q}+1\right]+x_1^{r-1}\left[\frac{1}{(x_2+1)^s}+1\right] \\ &\leq 2\,x_1^{p-1}+2x_1^{r-1} \to 0 \end{aligned}$$

as $x_1 \to 0$, for $r, p > 1$, so $(f_2)_0(x_2) = 0$ uniformly for all $x_2 \geq 0$, and

$$\begin{aligned} \left|\frac{f_2(x_1,x_2)}{x_1}\right| &= x_1^{p-1}\left[\frac{1}{(x_2+1)^q}+1\right]+x_1^{r-1}\left[\frac{1}{(x_2+1)^s}+1\right] \\ &\geq x_1^{p-1}+x_1^{r-1} \to \infty \end{aligned}$$

as $x_1 \to \infty$, for $r, p > 1$, so $(f_2)_\infty(x_2) = \infty$ uniformly for all $x_2 \geq 0$.

(2) An example of functions like in (2.4):

$$\begin{aligned} f_1(x_1,x_2) &= x_1^p\left[\frac{1}{(x_2+1)^q}+1\right]+x_1^r\left[\frac{1}{(x_2+1)^s}+1\right], \\ f_2(x_1,x_2) &= x_1^p\left[\frac{qx_2}{x_2+1}+1\right]+x_1^r\left[\frac{sx_2}{x_2+1}+1\right], \end{aligned}$$

where $0 < r$, $p < 1; q$, $s > 0$.

Indeed, $f_1$ and $f_2$ are unbounded and one has

$$\begin{aligned} \left|\frac{f_1(x_1,x_2)}{x_1}\right| &= x_1^{p-1}\left[\frac{1}{(x_2+1)^q}+1\right]+x_1^{r-1}\left[\frac{1}{(x_2+1)^s}+1\right] \\ &\geq x_1^{p-1}+x_1^{r-1} \to \infty \end{aligned}$$

as $x_1 \to 0$, for $0 < r, p < 1$, so $(f_1)_0(x_2) = \infty$ uniformly for all $x_2 \geq 0$, and

$$\begin{aligned} \left|\frac{f_1(x_1,x_2)}{x_1}\right| &= x_1^{p-1}\left[\frac{1}{(x_2+1)^q}+1\right]+x_1^{r-1}\left[\frac{1}{(x_2+1)^s}+1\right] \\ &\leq 2\,x_1^{p-1}+2x_1^{r-1} \to 0 \end{aligned}$$

as $x_1 \to \infty$, for $0 < r, p < 1$, so $(f_1)_\infty(x_2) = 0$ uniformly for all $x_2 \geq 0$. Also

$$\begin{aligned} \left|\frac{f_2(x_1,x_2)}{x_1}\right| &= x_1^{p-1}\left[\frac{qx_2}{x_2+1}+1\right]+x_1^{r-1}\left[\frac{sx_2}{x_2+1}+1\right] \\ &\geq x_1^{p-1}+x_1^{r-1} \to \infty \end{aligned}$$

as $x_1 \to 0$, for $0 < r, p < 1$, so $(f_2)_0(x_2) = \infty$ uniformly for all $x_2 \geq 0$, and

$$\begin{aligned} \left|\frac{f_2(x_1,x_2)}{x_1}\right| &= x_1^{p-1}\left[\frac{qx_2}{x_2+1}+1\right]+x_1^{r-1}\left[\frac{sx_2}{x_2+1}+1\right] \\ &\leq (q+1)x_1^{p-1}+(s+1)x_1^{r-1} \to 0 \end{aligned}$$

as $x_1 \to \infty$, for $0 < r, p < 1$, so $(f_2)_\infty(x_2) = 0$ uniformly for all $x_2 \geq 0$.

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