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Approximation of bivariate functions by truncated classes of operators

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Abstract

Starting from a general class of positive approximation processes of discrete type expressed by series, we indicate a way to modify the operators into finite sums. The new operators are suitable to be generated by software. Examples are delivered.

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1 Introduction

It is known that Approximation Theory, an old field of mathematical research, has a great potential for applications to a wide variety of problems. The study of the linear methods of approximation, which are given by sequence of linear and positive operators, became a firmly entrenched part of Approximation Theory. Usually, two types of positive approximation processes are used – the discrete respectively continuous form. In the first case, multiple classes of linear positive operators are expressed by series. We recall two classical examples of such operators used to approximate functions defined on unbounded intervals. We refer to Mirakjan-Szász operators S_n, $n \in \mathbb{N}$, and Baskakov operators V_n, $n \in \mathbb{N}$, respectively. They are defined as follows

$$\begin{aligned} (S_n f)(x) &= \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad s_{n,k}(x) = \frac{(nx)^k}{k!} e^{-nx}, \quad x \geq 0, \\ (V_n f)(x) &= \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right), \quad v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \quad x \geq 0, \end{aligned} \tag{1}$$

where f belongs to the space $C_2(\mathbb{R}_+)$, $\mathbb{R}_+ := [0, \infty)$,

$$C_2(\mathbb{R}_+) = \{f \in C(\mathbb{R}_+) : \lim_{x\to\infty} (1+x^2)^{-1} f(x) \text{ is finite}\},$$

endowed with the norm $\|\cdot\|$, $\|f\| = \sup_{x\geq 0}(1+x^2)^{-1}|f(x)|$.

As can be seen, the construction of such operators requires an estimation of infinite sums and this fact restricts the operators usefulness from the computational point of view. A question arises: how can we modify the operators to became usable for generating software programmes for approximation of functions. In this respect it is useful to consider partial sums which have only finite terms depending upon n and x. For the above mentioned operators this approach has already been made. For example,

J. Grof [5] examined the operator $(S_{n,N}f)(x) = \sum_{k=0}^{N(n)} s_{n,k}(x)f(k/n)$ establishing that if $(N(n))_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty}(N(n)/n) = \infty$, then $(S_{n,N}f)$ converges pointwise to f. Also the following modified operators of Mirakjan-Szász respectively Baskakov-type were investigated

$$(S_{n,\delta}f)(x) = \sum_{k=0}^{[n(x+\delta)]} s_{n,k}(x)f\left(\frac{k}{n}\right), \quad (V_{n,\delta}f)(x) = \sum_{k=0}^{[n(x+\delta)]} v_{n,k}(x)f\left(\frac{k}{n}\right), \quad x \geq 0. \tag{2}$$

Here $[\alpha]$ indicates the largest integer not exceeding α. The first class was studied by Heinz-Gerd Lehnhoff [6] and the second has approached by J. Wang and S. Zhou [9]. In (2) the number of terms considered in sum depends on the function argument. Roughly speaking, the initial operators are truncated losing their "tails". Following this route, a one dimensional general case is investigated in [1].

The aim of this note is to present similar constructions for bivariate classes of discrete operators. Instead of a double series we consider a finite sum. This way the use of computers in approximating functions is possible with lesser effort. The focus of the paper is on presenting different examples comparing the approximations generated by the series and by corresponding "amputated" series.

2 The operators and their truncated variants

Following [2], we investigate operators useful to approximate functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$. Therefore, on this domain we define for every $(m,n) \in \mathbb{N}\times\mathbb{N}$ a net of form $\Delta_{m,n} = \Delta_{1,m}\times\Delta_{2,n}$, where

$$\Delta_{1,m}(0 = x_{m,0} < x_{m,1} < \ldots) \quad \text{and} \quad \Delta_{2,n}(0 = y_{n,0} < y_{n,1} < \ldots).$$

Set $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$. Products of parametric extensions of two univariate operators are appropriate tools to approximate functions of two variables. For this reason, the starting point is given by the following one-dimensional operators

$$(A_m f)(x) = \sum_{i=0}^{\infty} a_{m,i}(x)f(x_{m,i}), \quad (B_n f)(y) = \sum_{j=0}^{\infty} b_{n,j}(y)f(y_{n,j}), \tag{3}$$

where $a_{m,i}$, $b_{n,j}$ are non-negative functions belonging to $C(\mathbb{R}_+)$, $(i,j)\in\mathbb{N}_0\times\mathbb{N}_0$, such that the following identities

$$\sum_{i=0}^{\infty} a_{m,i}(t) = \sum_{j=0}^{\infty} b_{n,j}(t) = 1, \quad t \geq 0, \tag{4}$$

take place.

In the above $f \in \mathcal{F}_1(\mathbb{R}_+)$ where $\mathcal{F}_1(\mathbb{R}_+)$ stands for the domain of L_n containing the set of all continuous functions on $\mathbb{R}_+$ for which the series in (3) is convergent.

Starting from (3), for each $(m,n)\in\mathbb{N}\times\mathbb{N}$ we introduce a linear positive operator as follows

$$(L_{m,n}f)(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{m,i}(x)b_{n,j}(y)f(x_{m,i},y_{n,j}), \quad (x,y)\in\mathbb{R}_+^2, \tag{5}$$

where $f\in\mathcal{F}_2(\mathbb{R}_+\times\mathbb{R}_+)$, the space of all continuous functions on $\mathbb{R}_+\times\mathbb{R}_+$ for which the double series in (5) is convergent. We notice if the function f can be decomposed in the following manner $f(x,y) = f_1(x)f_2(y)$, $(x,y)\in\mathbb{R}_+^2$, then one has

$$(L_{m,n}f)(x,y) = (A_m f_1)(x)(B_n f_2)(y). \tag{6}$$

Actually, the method of using the product of parametric extensions of univariate operators is a classic one. It was first used in the context of multivariate polynomial interpolation. For example in [4] can be found many historical information on this topic.

Further on we indicate a truncated variant of operators defined at (5). Let $u = (u_s)_{s\geq 1}$, $v = (v_s)_{s\geq 1}$ be sequences of positive numbers such that

$$\lim_{s\to\infty} \sqrt{s}u_s = \lim_{s\to\infty} \sqrt{s}v_s = \infty. \tag{7}$$

Taking in view the net $\Delta_{1,m}$, we divide the set $\mathbb{N}_0$ into two parts

$$I(x,u_m) = \{i \in \mathbb{N}_0 : \ x_{m,i} \leq x + u_m\} \quad \text{and} \quad \overline{I}(x,u_m) = \mathbb{N}_0 \setminus I(x,u_m).$$

Similarly, via the network $\Delta_{2,n}$, we introduce $J(y,v_n)$ and $\overline{J}(y,v_n)$.

For each $(m,n) \in \mathbb{N}\times\mathbb{N}$ and any $f \in \mathcal{F}_2(\mathbb{R}_+\times\mathbb{R}_+)$ in [2], we defined the linear positive operators

$$\begin{aligned}(L^*_{m,n}f)(x,y;u,v) &\equiv (L^*_{m,n}f)(x,y;u_n,v_n)\\ &= \sum_{i\in I(x,u_m)}\sum_{j\in J(y,v_n)} a_{m,i}(x)b_{n,j}(y)f(x_{m,i},y_{n,j}), \quad (x,y)\in\mathbb{R}^2_+.\end{aligned} \tag{8}$$

The approach made above represents the general framework. In particular we can consider $a_{n,i} = b_{n,i}$, $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$. Moreover, the network applied to the set $\mathbb{R}_+\times\mathbb{R}_+$ is usually of the form $(i/m, j/n)$, $(i,j) \in \mathbb{N}_0\times\mathbb{N}_0$. In this case the operators defined by (5) turn into the following operators

$$(L_{m,n}f)(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{m,i}(x)a_{n,j}(y)f\left(\frac{i}{m},\frac{j}{n}\right), \quad (x,y)\in\mathbb{R}^2_+. \tag{9}$$

Their truncated version given at (8) becomes

$$(L^*_{m,n}f)(x,y;u,v) = \sum_{i=0}^{[m(x+u_m)]}\sum_{j=0}^{[n(y+v_n)]} a_{m,i}(x)a_{n,j}(y)f\left(\frac{i}{m},\frac{j}{n}\right), \quad (x,y)\in\mathbb{R}^2_+. \tag{10}$$

We mention that in the particular case $a_{n,k} = v_{n,k}$, $k \in \mathbb{N}_0$, see (1), the above sequence turns into the truncated version of bidimensional Baskakov operators studied by Walczak [8].

We discuss how the sequences defined by (10) are becoming approximation processes. Let $(\Lambda_n)_{n\geq 1}$ be a sequence of positive linear operators defined on the Banach space $C(K)$, $K \subset \mathbb{R}$, a compact interval.

The classical theorem of Bohman-Korovkin states: if $(\Lambda_n e_k)_{k\geq 1}$ converges to e_k uniformly on K, $k \in \{0,1,2\}$, for the test functions $e_0(x) = 1$, $e_1(x) = x$, $e_2(x) = x^2$, then $(\Lambda_n f)_{n\geq 1}$ converges to f uniformly on K for each $f \in C(K)$. The requirement (4) ensures the identity $A_n e_0 = e_0$. If we assume

$$\lim_n A_n e_j = e_j, \quad j \in \{1,2\}, \tag{11}$$

then $(A_n f)_{n\geq 1}$ converges to f uniformly on any compact $K \subset \mathbb{R}_+$.

Setting $e_{i,j}(x,y) = x^i y^j$, $i \in \mathbb{N}_0$, $j \in \mathbb{N}_0$, $i + j \leq 2$, according to a result of Volkov [7] the test functions corresponding to the bidimensional case are the following four: $e_{0,0}$, $e_{1,0}$, $e_{0,1}$, $e_{2,0} + e_{0,2}$.

Since $L_{m,n}e_{i,j} = (A_m e_i)(A_n e_j)$, see (6), relation (4) and our hypotheses (11) guarantee that the sequence $(L_{m,n})$ is an approximation process,. Taking in view (7) and following a similar route as in [1, Theorem 2] we can assert that $(L^*_{m,n}f)$ is also an approximation process. The advantage of using $L^*_{m,n}$ is that we work with finite sums, software enabling fast construction of operators.

3 Examples and graphs

We illustrate the effectiveness of construction given in (10) by choosing $a_{n,k} = s_{n,k}, k \in \mathbb{N}_0$, see (1). Consider the following functions

$$\begin{aligned}&f_i : [0,\infty)\times[0,\infty) \to \mathbb{R}, \qquad i = 1,2,3\\ &f_1(x,y) = e^{-x-y},\\ &f_2(x,y) = e^{x+y},\\ &f_3(x,y) = \sin x \sin y.\end{aligned}$$

If we apply the operator $L_{m,n}$ to our functions we obtain

$$\begin{aligned}\left(L_{m,n}f_1\right)(x,y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\exp\left(-mx-ny\right)(mx)^i\,(nx)^j\exp\left(-\frac{i}{m}-\frac{j}{n}\right)}{i!j!}\\ &= \frac{1}{\exp\left(mx+ny-\frac{ny}{\exp(-1/n)}-\frac{mx}{\exp(-1/m)}\right)};\end{aligned}$$

$$\begin{aligned}\left(L_{m,n}f_2\right)(x,y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\exp\left(mx+ny\right)(mx)^i\,(nx)^j\exp\left(\frac{i}{m}+\frac{j}{n}\right)}{i!j!}\\ &= \exp\left(-mx-ny+ny\exp\left(1/n\right)+mx\exp\left(1/m\right)\right).\end{aligned}$$

$$\begin{aligned}\left(L_{m,n}f_3\right)(x,y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\exp\left(-mx-ny\right)(mx)^i\,(nx)^j\sin\frac{i}{m}\sin\frac{j}{n}}{i!j!}\\ &= -\frac{1}{2}\left(\cos\left(mx\sin\frac{1}{m}+ny\sin\frac{1}{n}\right)-\cos\left(mx\sin\frac{1}{m}-ny\sin\frac{1}{n}\right)\right)\cdot\\ &\exp\left(mx\cos\frac{1}{m}+ny\cos\frac{1}{n}-mx-ny\right)\end{aligned}$$

For $L^*_{m,n}$ we consider successively the following sequences

(i) $u^{(1)},v^{(1)}$, where $u^{(1)}_m=m$, $v^{(1)}_n=n$;

(ii) $u^{(2)},v^{(2)}$, where $u^{(2)}_m=\sqrt[3]{m}$, $v^{(2)}_(n)=\sqrt[3]{n}$.

In the sequel, for each function f_i, $i=1,2,3$, we give the following graphs

- $L_{10,10}f_i$
- $(L^*_{10,10}f_i)(.,.,u^{(j)},v^{(j)})$, $j=1,2$;
- $\left|L^*_{10,10}f_i)(.,.,u^{(j)},v^{(j)})-L_{10,10}f_i\right|$, $j=1,2$.

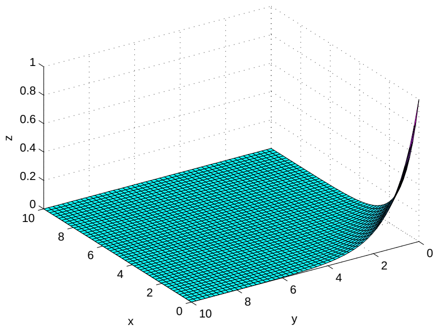
See Figures 1, 2, and 3.

Finally, we consider an example for which $L_{m,n}$ cannot be computed exactly. Let f_4 be given as follows

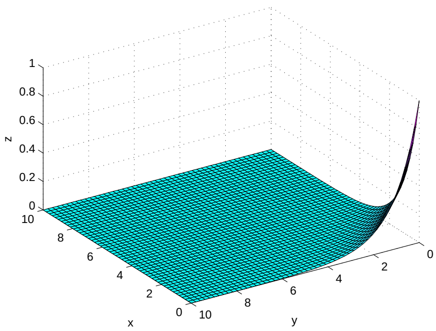
$$f_4:[0,\infty)\times[0,\infty)\to\mathbb{R},\qquad f_4(x,y)=\sin\sqrt{x^2+y^2}.$$

Figure 4 gives the graphs as above, excepting $L_{10,10}f_4$.

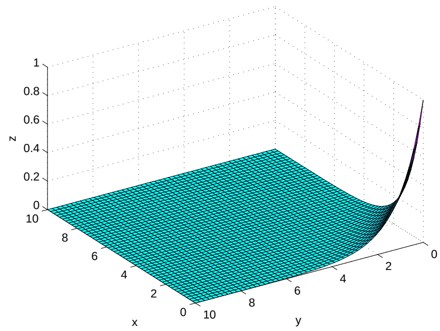
For graphical treatment of other types of bivariate operators see [3].



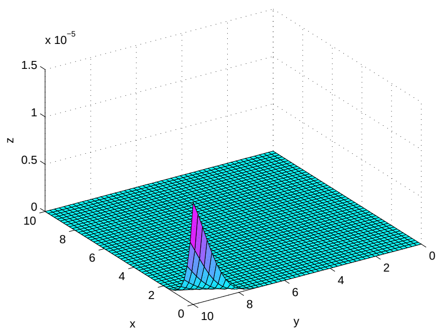
(a) $L_{10,10}f_1$



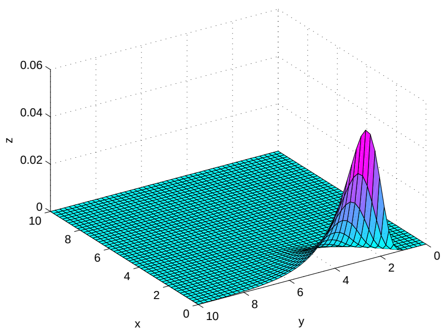
(b) $\left(L^*_{10,10}f_1\right)(\cdot,\cdot,u^{(1)},v^{(1)})$



(c) $\left(L^*_{10,10}f_1\right)(\cdot,\cdot,u^{(2)},v^{(2)})$

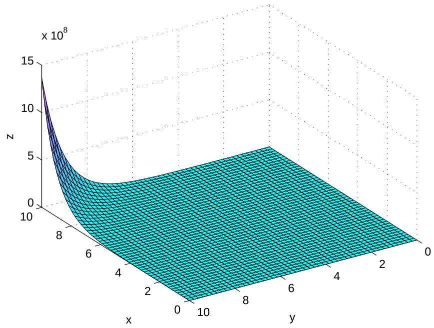


(d) $\left|\left(L^*_{10,10}f_1\right)(\cdot,\cdot,u^{(1)},v^{(1)}) - (L_{10,10}f_1)\right|$

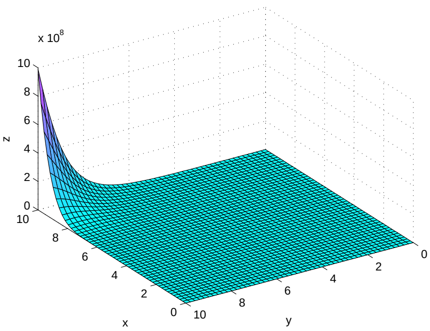


(e) $\left|\left(L^*_{10,10}f_1\right)(\cdot,\cdot,u^{(2)},v^{(2)}) - (L_{10,10}f_1)\right|$

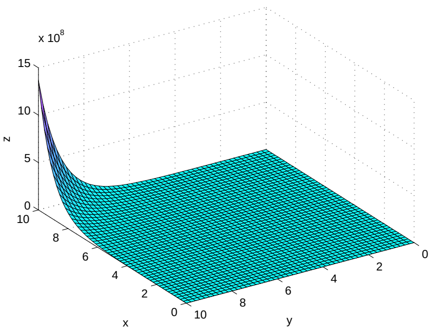
Figure 1: The graphs corresponding to f_1



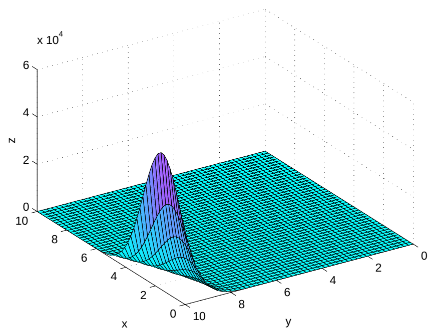
(a) $L_{10,10}f_2$



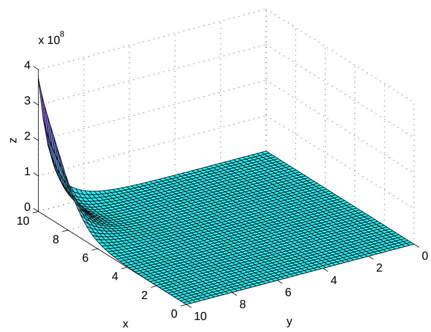
(b) $\left(L^*_{10,10}f_2\right)(\cdot,\cdot,u^{(1)},v^{(1)})$



(c) $\left(L^*_{10,10}f_2\right)(\cdot,\cdot,u^{(2)},v^{(2)})$

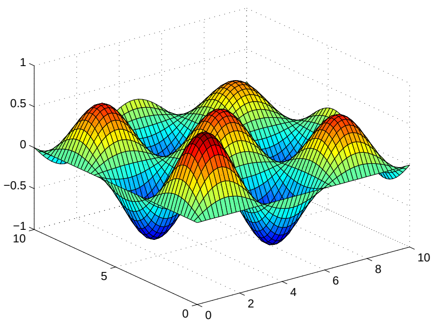


(d) $\left|\left(L^*_{10,10}f_2\right)(\cdot,\cdot,u^{(1)},v^{(1)})-(L_{10,10}f_2)\right|$

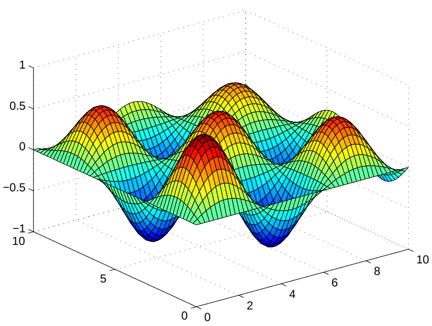


(e) $\left|\left(L^*_{10,10}f_2\right)(\cdot,\cdot,u^{(2)},v^{(2)})-(L_{10,10}f_2)\right|$

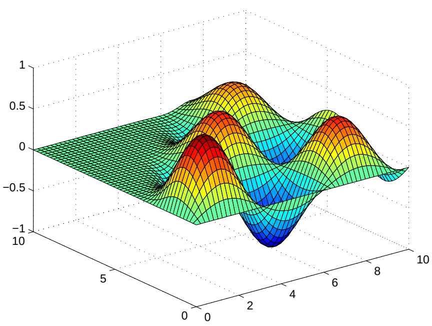
Figure 2: The graphs corresponding to f_2



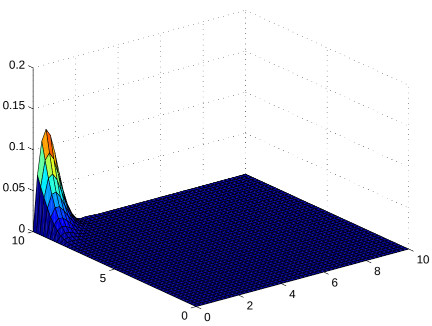
(a) $L_{10,10}f_3$



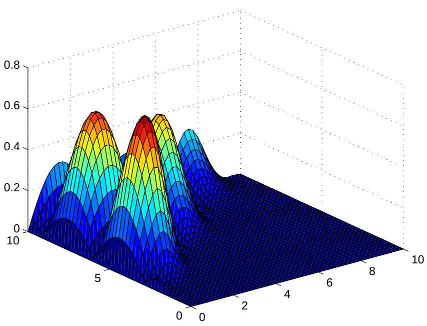
(b) $\left(L^*_{10,10}f_3\right)(\cdot,\cdot,u^{(1)},v^{(1)})$



(c) $\left(L^*_{10,10}f_3\right)(\cdot,\cdot,u^{(2)},v^{(2)})$

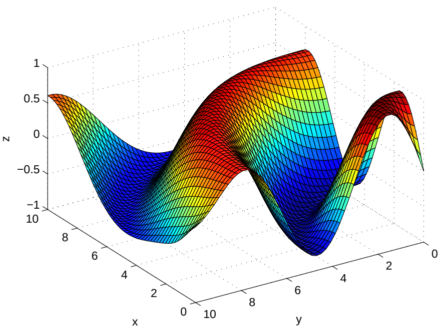


(d) $\left|\left(L^*_{10,10}f_3\right)(\cdot,\cdot,u^{(1)},v^{(1)}) - (L_{10,10}f_3)\right|$

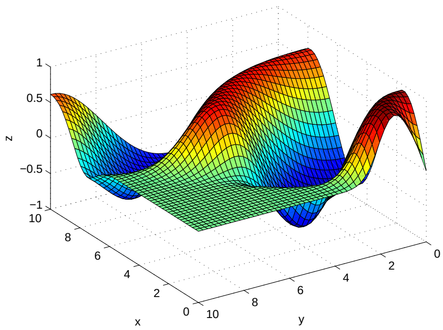


(e) $\left|\left(L^*_{10,10}f_3\right)(\cdot,\cdot,u^{(2)},v^{(2)}) - (L_{10,10}f_3)\right|$

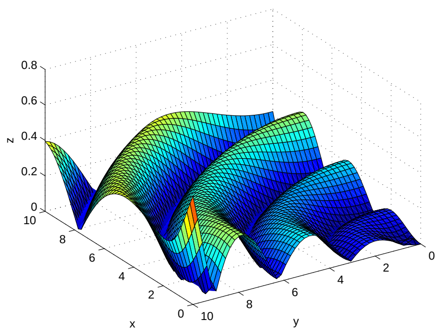
Figure 3: The graphs corresponding to f_3



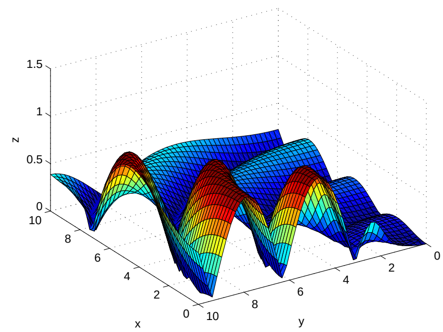
(a) $\left(L^*_{10,10}f_4\right)(\cdot,\cdot,u^{(1)},v^{(1)})$



(b) $\left(L^*_{10,10}f_4\right)(\cdot,\cdot,u^{(2)},v^{(2)})$



(c) $\left|\left(L^*_{10,10}f_4\right)(\cdot,\cdot,u^{(1)},v^{(1)})-f_4\right|$



(d) $\left|\left(L^*_{10,10}f_3\right)(\cdot,\cdot,u^{(2)},v^{(2)})-f_4\right|$

Figure 4: The graphs corresponding to f_4

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