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On the Durrmeyer-Type Variant and Generalizations of Lototsky–Bernstein Operators

Ulrich Abel [1] and **Octavian Agratini** [2,3,*]

[1] Fachbereich MND, Technische Hochschule Mittelhessen, 61169 Friedberg, Germany; Ulrich.Abel@mnd.thm.de
[2] Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
[3] Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy, 400320 Cluj-Napoca, Romania
* Correspondence: agratini@math.ubbcluj.ro

Abstract: The starting points of the paper are the classic Lototsky–Bernstein operators. We present an integral Durrmeyer-type extension and investigate some approximation properties of this new class. The evaluation of the convergence speed is performed both with moduli of smoothness and with K-functionals of the Peetre-type. In a distinct section we indicate a generalization of these operators that is useful in approximating vector functions with real values defined on the hypercube $[0,1]^q$, $q > 1$. The study involves achieving a parallelism between different classes of linear and positive operators, which will highlight a symmetry between these approximation processes.

Keywords: Lototsky operator; Korovkin theorem; modulus of smoothness; K-functional; Durrmeyer extension

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1. Introduction

It is widely acknowledged that the most studied linear positive operators are Bernstein operators, which have known innumerable generalizations over time.

Bernstein operators $B_n : C([0,1]) \to C([0,1])$ are defined by

$$(B_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \; x \in [0,1],$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, 0 \le k \le n$, represent the $n+1$ Bernstein basis polynomials of degree n. As usual, $C([0,1])$ denotes the real Banach space of all continuous functions $f : [0,1] \to \mathbb{R}$ endowed with the sup-norm $\|\cdot\|$, $\|f\| = \sup_{x\in[0,1]} |f(x)|$. With the same norm we endow $B([0,1])$, the space of bounded real-valued functions defined on $[0,1]$.

Based on the generalized Lototsky matrix, an extension of these operators was given by King [1]. We present this in the following. For each $j \in \mathbb{N}$, let $h_j : [0,1] \to [0,1]$ be a continuous function. Further, for each $n \in \mathbb{N}$, a system of functions $(a_{n,k})_{k=\overline{0,n}}$ on $[0,1]$ is defined by the relation

$$\prod_{j=1}^{n} (h_j(x) y + 1 - h_j(x)) = \sum_{k=0}^{n} a_{n,k}(x) y^k, \; y \in \mathbb{R}. \tag{1}$$

From the above identity we immediately obtain the coefficient of y^k, $k = \overline{0,n}$, that is

$$a_{n,k}(x) = \sum_{\substack{J \cup \overline{J} = \mathbb{N}_n \\ Card(J) = k}} \prod_{i \in \overline{J}} (1 - h_i(x)) \prod_{j \in J} h_j(x), \tag{2}$$

where $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and $\bar{J} = \mathbb{N}_n \setminus J$. For each real-valued function f defined on $[0,1]$, the n-th Lototsky–Bernstein operator is defined as follows

$$(L_n f)(x) = \sum_{k=0}^{n} a_{n,k}(x) f\left(\frac{k}{n}\right), \tag{3}$$

see [1], Equation (4). L_n operators are linear. Since $h_j([0,1]) \subseteq [0,1]$ for all $j \in \mathbb{N}$, they are also positive. It is clear that in the special case $h_j(x) = x$, $j \in \mathbb{N}$, the functions $(a_{n,k})_{0 \leq k \leq n}$, $n \in \mathbb{N}$ become Bernstein bases $(p_{n,k})_{0 \leq k \leq n}$, $n \in \mathbb{N}$; consequently the L_n operator turns into a B_n operator.

King has established the sufficient condition on the sequence $(h_j)_{j \in \mathbb{N}}$ to ensure that $(L_n)_{n \geq 1}$ is an approximation process on $C([0,1])$. His result can be written as follows. If

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} h_i(x) = x \text{ uniformly with respect to } x \text{ on } [0,1], \tag{4}$$

then

$$\lim_{n \to \infty} (L_n f)(x) = f(x) \text{ uniformly with respect to } x \text{ on } [0,1] \tag{5}$$

for every $f \in C([0,1])$.

In recent years the study of these operators has been deepened; see, for example, the papers of Ron Goldman, Xiao-Wei Xu, and Xiao-Ming Zeng [2–4].

The purpose of this paper is to define and to establish approximation properties for a Durmmeyer-type extension of L_n, $n \in \mathbb{N}$, operators. We mention that, using elements of probability theory, a Kantorovich-type extension was achieved in 2020 by Popa [5]. A second goal of this paper is to extend the univariate operators for vector functions with real values.

A third purpose of this paper is to present a construction of discrete Lototsky operators in q-dimensional space. The approach follows a symmetrical path with the construction of other approximation processes, the main instrument of investigation being the multidimensional Korovkin theorem.

2. D_n Operators

Set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and denote by e_j, $j \in \mathbb{N}_0$ the monomials of degree j, $e_0(x) = 1$, $e_j(x) = x^j$, $j \in \mathbb{N}$. The statement (5) is motivated by the Bohman–Korovkin criterion, which says: If a sequence of linear positive operators $(\Lambda_n)_{n \geq 1}$ defined on $C([a,b])$ has the property that $(\Lambda_n e_k)_{n \geq 1}$ converges to e_k uniformly on $[a,b]$, $k \in \{0,1,2\}$, then $(\Lambda_n f)_{n \geq 1}$ converges to f uniformly on $[a,b]$ for each f belonging to $C([a,b])$. In relation (1) let us denote by $P_n(x;y)$ the polynomial of degree n in y and with the parameter $x \in [0,1]$. Following [6] we obtain

$$(L_n e_0)(x) = P_n(x;1) = 1, \tag{6}$$

$$(L_n e_1)(x) = \frac{1}{n} \frac{\partial P_n(x;1)}{\partial y} = \frac{1}{n} \sum_{i=1}^{n} h_i(x), \tag{7}$$

$$\begin{aligned} (L_n e_2)(x) &= \frac{1}{n^2} \left(\frac{\partial^2 P_n(x;1)}{\partial y^2} + \frac{\partial P_n(x;1)}{\partial y} \right) \\ &= \left(\frac{1}{n} \sum_{i=1}^{n} h_i(x) \right)^2 + \frac{1}{n^2} \sum_{i=1}^{n} h_i(x)(1 - h_i(x)). \end{aligned} \tag{8}$$

Taking in view the mentioned criterion, the proof of (5) is completed.

Moreover, for any $k \geq 1$,

$$(L_n e_k)(x) = \frac{1}{n^k} \sum_{j=1}^{k} s(k,j) \frac{\partial^j P_n(x;1)}{\partial y^j}, \tag{9}$$

see [6], Equation (2.6). In the above $s(k,j)$ denotes a Stirling number of the second kind. For $1 \leq j \leq k$, its closed form is given as follows

$$s(k,j) = \frac{1}{j!} \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{j}{\nu} \nu^k,$$

see, e.g., [7], p. 824. For $k = 1$ and $k = 2$, from (9) we reobtain the identities (7) and (8).

Usually, in the papers that approached Lototsky operators, in order to obtain significant results, the authors imposed additional conditions on the functions h_j, $j \in \mathbb{N}$. For a similar reason, we define a particular Durrmeyer-type construction that involves both Lototsky–Bernstein and classical Bernstein bases. $L_1([0,1])$ stands for the Banach space of all real-valued integrable functions on $[0,1]$ endowed with the norm $\|\cdot\|_1$, $\|f\|_1 = \int_0^1 |f(x)|dx$. Define $D_n : L_1([0,1]) \to C([0,1])$ by formula

$$(D_n f)(x) = (n+1) \sum_{k=0}^{n} a_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \ x \in [0,1]. \tag{10}$$

Remark 1. *(a) The operators keep the properties of linearity and positivity.*

(b) By using a bivariate kernel we can write $D_n f$ in a more compact form as follows

$$(D_n f)(x) = \int_0^1 k_n^*(x,t) f(t) dt, \ x \in [0,1],$$

where

$$k_n^*(x,t) = (n+1) \sum_{k=0}^{n} a_{n,k}(x) p_{n,k}(t), \ (x,t) \in [0,1] \times [0,1].$$

(c) If $f \in C([0,1])$, then $\|D_n f\| \leq \|f\|$, $n \in \mathbb{N}$; consequently the operators are non-expansive in the space $C([0,1])$.

(d) For the particular case $h_j = e_1$, $j \in \mathbb{N}$, D_n operators turn into the classical Durrmeyer operators [8].

Lemma 1. *Let D_n, $n \in \mathbb{N}$, be the operators defined by (10). The following identities*

$$D_n e_0 = e_0, \tag{11}$$

$$D_n e_1 = \frac{n}{n+2} L_n e_1 + \frac{1}{n+2}, \tag{12}$$

$$D_n e_2 = \frac{n^2}{(n+2)(n+3)} L_n e_2 + \frac{3n}{(n+2)(n+3)} L_n e_1 + \frac{2}{(n+2)(n+3)} \tag{13}$$

take place for each $n \in \mathbb{N}$.

Proof. Using the Beta function, for any $p \in \mathbb{N}_0$ we deduce

$$\int_0^1 p_{n,k}(t) t^p dt = \frac{(k+p)!}{k!} \frac{n!}{(n+p+1)!}, \ k = \overline{0,n}.$$

By a straightforward calculation, the definition of D_n operators as well as the relation (3) lead us to the enunciated identities. $\square$

At this point we introduce the j-th central moment of D_n operators, $j \in \mathbb{N}$, i.e., $D_n\varphi_x^j$, where

$$\varphi_x(t) = t - x,\ (t,x) \in [0,1] \times [0,1].$$

Lemma 2. *The second-order central moments of the operators D_n, $n \in \mathbb{N}$ operators satisfy the following relations*

$$(D_n\varphi_x^2)(x) = \frac{n^2}{(n+2)(n+3)}\left(\alpha_n^2(x) + \frac{1}{n^2}\sum_{i=1}^n h_i(x)(1-h_i(x))\right) + \frac{3n\alpha_n(x)+2}{(n+2)(n+3)}$$
$$- 2x\alpha_n(x) + x^2 \le (\alpha_n(x)-x)^2 + \frac{4}{n}, \tag{14}$$

where

$$\alpha_n(x) = \frac{1}{n}\sum_{i=1}^n h_i(x),\ x \in [0,1]. \tag{15}$$

Proof. The identity is a direct consequence of the relations (11)–(13) and (6)–(8). Further, since $0 \le h_i(x) \le 1$, $x \in [0,1]$, $i = \overline{1,n}$, we conclude that $\alpha_n(x) \le 1$ and

$$\frac{1}{n^2}\sum_{i=1}^n h_i(x)(1-h_i(x)) \le \frac{1}{4n}.$$

Using simple increases in relation to n, we reach the postulated inequality. □

3. Approximation Properties

Theorem 1. *Let D_n, $n \in \mathbb{N}$, be the operators defined by (10). If relation (4) takes place, then*

$$\lim_{n\to\infty}(D_nf)(x) = f(x) \text{ uniformly with respect to } x \text{ on } [0,1]$$

for every $f \in C([0,1])$.

Proof. The motivation is based on the Bohman–Korovkin criterion, on the property of the Lototsky operators mentioned in relations (4) and (5) and on the identities (11)–(13). The conclusion follows immediately. □

For estimating the approximation error, we will use the first modulus of smoothness $\omega(f;\cdot)$ associated to any bounded real-valued function defined in our case on $[0,1]$ and expressed by the formula

$$\omega(f;\delta) = \sup_{0\le h\le\delta}\ \sup_{x,x+h\in[0,1]} |f(x+h)-f(x)|,\ \delta \ge 0.$$

Theorem 2. *Let D_n, $n \in \mathbb{N}$, be the operators defined by (10). If $f \in B([0,1]) \cap L_1([0,1])$, then*

$$|(D_nf)(x) - f(x)| \le 2\omega(f;\delta_n(x)),\ x \in [0,1], \tag{16}$$

where $\delta_n(x) = ((\alpha_n(x)-x)^2 + 4/n)^{1/2}$.

Proof. We appeal to a classic result due to Shisha and Mond [9] and which can be formulated as follows: if Λ is a linear positive operator, then

$$|(\Lambda f)(x) - f(x)| \le |f(x)|\,|(\Lambda e_0)(x) - 1|$$
$$+ \left((\Lambda e_0)(x) + \frac{1}{\delta}\sqrt{(\Lambda e_0)(x)(\Lambda\varphi_x^2)(x)}\right)\omega(f;\delta),\ \delta > 0,$$

holds for every bounded function belonging to the operator's domain. We take in view (11) and pick

$$\delta := \sqrt{(D_n \varphi_x^2)(x)}.$$

Based on (14) and knowing that $\omega(f;\cdot)$ is an increasing function we arrive at (16). □

Remark 2. *If $f \in C([0,1])$, then f is uniformly continuous on $[0,1]$ and $\omega(f;\cdot)$ satisfies $\lim\limits_{\delta\to 0^+} \omega(f;\delta) = 0$, see, e.g., [10], p. 40. Assuming that relation (4) takes place, from (16) we obtain again the conclusion of Theorem 1*

$$\lim_{n\to\infty} \|D_n f - f\| = 0, \; f \in C([0,1]).$$

In the following, for a given $M > 0$, set

$$Lip_M 1 = \{f : [0,1] \to \mathbb{R} \mid \; |f(x) - f(y)| \le M|x-y| \text{ for every } x,y \in [0,1]\}$$

and consider the space $Lip([0,1]) = \bigcup\limits_{M>0} Lip_M 1$ endowed with the seminorm $|\cdot|_{Lip}$, where

$$|f|_{Lip} = \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \left| \frac{f(x)-f(y)}{x-y} \right|.$$

Further we introduce the least concave majorant of a function $f \in C([0,1])$, which is defined by

$$\widetilde{\omega}(f;\delta) = \sup\left\{ \frac{(\delta - x)\omega(f;y) + (y-\delta)\omega(f;x)}{y-x} \mid 0 \le x \le \delta \le y \le 1, \; x \neq y \right\},$$

if $0 \le \delta \le 1$. For $\delta > 1$, $\widetilde{\omega}(f;\delta)$ is the constant $\omega(f;1)$.

Obviously

$$\omega(f;\delta) \le \widetilde{\omega}(f;\delta), \; \delta \ge 0, \tag{17}$$

and this inequality follows from the definition of $\widetilde{\omega}$ as indicated above.

However, it is known [11] that

$$\widetilde{\omega}(f;2\delta) = 2K(f,\delta;C([0,1]),Lip([0,1])), \tag{18}$$

where K is the Peetre K-functional of $f \in C([0,1])$ with respect to the space $Lip([0,1])$ defined as follows

$$K(f,\delta) \equiv K(f,\delta;C([0,1]),Lip([0,1])) = \inf_{g\in Lip([0,1])} (\|f-g\| + \delta|g|_{Lip}), \; \delta \ge 0. \tag{19}$$

Starting from estimate (16), relations (17) and (18) lead us to the following result.

Theorem 3. *Let D_n, $n \in \mathbb{N}$, be the operators defined by (10). If $f \in C([0,1])$, then*

$$|(D_n f)(x) - f(x)| \le 4K\left(f, \frac{1}{2}\delta_n(x); C([0,1]), Lip([0,1])\right), \; x \in [0,1],$$

where $\delta_n(x)$ is specified in Theorem 2.

Remark 3. *$K(f,\delta)$ expresses some approximation properties of $f \in C([0,1])$. More precisely, the inequality $K(f,\delta) < \varepsilon$ for $\delta > 0$ implies that f can be approximated with the error $\|f-g\| < \varepsilon$ in $C([0,1])$ by a function $g \in Lip([0,1])$ whose seminorm is not too large, namely $|g|_{Lip} < \varepsilon\delta^{-1}$.*

For the evaluation of the approximation error we can also involve the second modulus of smoothness $\omega_2(f;\cdot)$ given by the formula

$$\omega_2(f;\delta) = \sup\{|f(x+2h) - 2f(x+h) + f(x)| : 0 \le h \le \delta,\ x, x+2h \in [0,1]\},\ \delta \ge 0,$$

$f \in B([0,1])$.

In (19) we can replace the space $Lip([0,1])$ by

$$C^2([0,1]) = \{g \in C([0,1]) : g \text{ is twice continuously differentiable in } [0,1]\}$$

endowed with the seminorm $|g|_{C^2([0,1])} = \|g''\|$ and $\|\cdot\|$ is the sup-norm. In this case, between ω_2 and K-functional the following relations hold: the positive constants c_1 and c_2 independent of f exist such that

$$c_1\omega_2(f;\delta) \le K(f,\delta^2; C([0,1]), C^2([0,1])) \le c_2\omega_2(f;\delta),\ \delta > 0, \tag{20}$$

see [12], Proposition 6.1.

Theorem 4. *Let D_n, $n \in \mathbb{N}$, be the operators defined by (10). If $f \in C([0,1])$, then*

$$|(D_n f)(x) - f(x)| \le c\omega_2\left(f; \frac{1}{2}\gamma_n(x)\right) + \omega(f;\beta_n(x)) \tag{21}$$

where c is a constant independent of f, $\beta_n(x) = \left|\alpha_n(x) - x - \dfrac{2x-1}{n}\right|$ and

$$\gamma_n(x) = \left((\alpha_n(x) - x)^2 + \frac{4}{n} + \beta_n^2(x)\right)^{1/2}. \tag{22}$$

Proof. First we define the operators E_n, $n \ge 1$ as follows

$$(E_n f)(x) = (D_n f)(x) - f((D_n e_1)(x)) + f(x),\ x \in [0,1]. \tag{23}$$

Based on relations (12) and (7), $0 < (D_n e_1)(x) < 1$ takes place and the construction is correct. Moreover, $E_n e_k = e_k$ for $k \in \{0,1\}$ and, in this way, the first central moment $E_n\varphi_x$ is null. Remark 1(c) together with (1) implies for any function $f \in C([0,1])$ the relation

$$|(E_n f)(x)| \le 3\|f\|,\ x \in [0,1]. \tag{24}$$

Let $g \in C^2([0,1])$ be arbitrarily chosen. Taylor's expansion with integral expression of the remainder allows us to write

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du,\ (t,x) \in [0,1]\times[0,1].$$

Applying in the above the linear operator E_n and using (23), we can write successively

$$\begin{aligned}(E_n g)(x) - g(x) &= g'(x)(E_n\varphi_x)(x) + E_n\left(\int_x^{e_1}(e_1 - u)g''(u)du; x\right)\\ &= D_n\left(\int_x^{e_1}(e_1-u)g''(u)du; x\right) - \int_x^{(D_n e_1)(x)}((D_n e_1)(x) - u)g''(u)du.\end{aligned} \tag{25}$$

Further, we obtain

$$D_n\left(\left|\int_x^{e_1}(e_1-u)g''(u)du\right|; x\right) \le D_n\left(\left|\int_x^{e_1}|e_1 - u|\,|g''(u)|du\right|\right) \le \|g''\|(D_n\varphi_x^2)(x).$$

At the same time, from (12) and (15) we deduce

$$\begin{aligned}\left|\int_x^{(D_ne_1)(x)}((D_ne_1)(x)-u)g''(u)du\right| &\le \|g''\|((D_ne_1)(x)-x)^2\\ &= \|g''\|\left(\frac{n}{n+2}\alpha_n(x)+\frac{1}{n+2}-x\right)^2\\ &\le \|g''\|\beta_n^2(x).\end{aligned}$$

Consequently, from relation (25) we obtain

$$|(E_ng)(x)-g(x)| \le \|g''\|((D_n\varphi_x^2)(x)+\beta_n^2(x)) \le \left((\alpha_n(x)-x)^2+\frac{4}{n}+\beta_n^2(x)\right)\|g''\|, \tag{26}$$

see Lemma 2.

Returning to (23) and taking into account (26), (24) and the definition of the modulus of smoothness $\omega(f;\cdot)$, we decompose the absolute error of approximation into several terms, thus

$$\begin{aligned}&|(D_nf)(x)-f(x)|\\ &\le |E_n(f-g;x)|+|(E_ng)(x)-g(x)|+|g(x)-f(x)|+|f((D_ne_1)(x))-f(x)|\\ &\le 4\|f-g\|+\left((\alpha_n(x)-x)^2+\frac{4}{n}+\beta_n^2(x)\right)\|g''\|+\omega\left(f;\left|\frac{n}{n+2}\alpha_n(x)+\frac{1}{n+2}-x\right|\right)\\ &\le 4\left(\|f-g\|+\frac{1}{4}\gamma_n^2(x)\|g''\|\right)+\omega(f;\beta_n(x)),\end{aligned}$$

see (22). Taking the infimum with respect to all $g\in C^2([0,1])$ and using (20), we obtain (21). The proof is completed. □

The construction carried out in (10) allowed us to obtain some notable results in terms of approximation of functions. A complete generalization of the Lototsky operators would be as follows:

$$(\widetilde{D}_nf)(x)=\sum_{k=0}^{n}c_{n,k}a_{n,k}(x)\int_0^1 a_{n,k}(t)f(t)dt,\ x\in[0,1],\ f\in L_1([0,1]), \tag{27}$$

where $c_{n,k}^{-1}=\int_0^1 a_{n,k}(t)dt$, all assumptions about the functions $a_{n,k}$, $k=\overline{0,n}$, $n\in\mathbb{N}$, remaining unchanged.

This form implies $\widetilde{D}_ne_0=e_0$. For this version we can prove the following.

Theorem 5. *Let $\widetilde{D}_n$, $n\in\mathbb{N}$, be the operators defined by (27). If $f,g\in L_1([0,1])$, then*

$$\int_0^1(\widetilde{D}_nf)(x)g(x)dx=\int_0^1 f(t)(\widetilde{D}_ng)(t)dt,\ n\in\mathbb{N}. \tag{28}$$

Proof. $\displaystyle\int_0^1(\widetilde{D}_nf)(x)g(x)dx=\int_0^1\sum_{k=0}^{n}c_{n,k}a_{n,k}(x)\left(\int_0^1 a_{n,k}(t)f(t)dt\right)g(x)dx$

$$=\int_0^1\sum_{k=0}^{n}c_{n,k}a_{n,k}(t)\left(\int_0^1 a_{n,k}(x)g(x)dx\right)f(t)dt=\int_0^1 f(t)(\widetilde{D}_ng)(t)dt.$$

Consequently, the stated identity is valid. □

Remark 4. *(a)* *In the particular case $f, g \in L_2([0,1])$, the identity (28) says*

$$\langle \widetilde{D}_n f, g\rangle = \langle f, \widetilde{D}_n g\rangle,$$

where $\langle\cdot,\cdot\rangle$ represents the inner product of the Hilbert space $L_2([0,1])$.

(b) *Since the monomial e_0 is a fixed point for all three classes of operators $(L_n, D_n, \widetilde{D}_n, n \in \mathbb{N})$, they are of the Markov type.*

4. The Case of the Multidimensional Lototsky–Bernstein Operators

We aim to present a q-dimensional generalization of discrete-type operators defined by (3). For a positive integer $q \geq 2$ we consider the hypercube $H_q = [0,1]^q$. The following notations will be used throughout this section: $\mathbf{x} = (x_i)_{1\leq i\leq q} \in H_q$, $\mathbf{n} = (n_\nu)_{1\leq \nu\leq q} \in \mathbb{N}^q$. For each $i = \overline{1,q}$ and $j \in \mathbb{N}$ let $h_{i,j} : [0,1] \to [0,1]$ be a continuous function on its domain. With the help of these functions we define the systems $(a_{n_\nu,s})_{0\leq s\leq n_\nu}$, $n_\nu \in \mathbb{N}$, similar to those of (2), by the following relations

$$\prod_{j=1}^{n_\nu}(h_{i,j}(x_i)y + 1 - h_{i,j}(x_i)) = \sum_{s=0}^{n_\nu} a_{n_\nu,s}(x_i)y^s := P_{n_\nu}(x_i; y),\ i = \overline{1,q}. \tag{29}$$

The announced multidimensional operators will be written as follows:

$$(L_{\mathbf{n}} f)(\mathbf{x}) = \sum_{k_1=0}^{n_1} \dots \sum_{k_q=0}^{n_q} a_{n_1,k_1}(x_1)\dots a_{n_q,k_q}(x_q) f\left(\frac{k_1}{n_1}, \dots, \frac{k_q}{n_q}\right), \tag{30}$$

where f is a real-valued function defined on H_q.

We focus on highlighting certain approximation properties of this general family of operators. As a first step we introduce $q+2$ test functions connected with the multivariate Korovkin theorem. The constant function on H_q of the constant value 1 is denoted by $\mathbf{1}$. For each $i \in \{1, 2, \dots, q\}$ we denote by $pr_i : H_q \to \mathbb{R}$ the i-th canonical projection, which is given by

$$pr_i(\mathbf{x}) = x_i \text{ for every } \mathbf{x} \in H_q.$$

Setting $\pi_q = \sum_{i=1}^{q} pr_i^2$, it is clear that $\pi_q(\mathbf{x})$ represents the Euclidean inner product in H_q of $\mathbf{x}$ with itself. We specify that for any vector $\mathbf{v} \in \mathbb{R}^q$, its i-th component will also be denoted by $pr_i(\mathbf{v})$.

Further, $C(H_q)$ stands for the space of all real-valued continuous functions on H_q endowed with the usual norm of the uniform convergence $\|\cdot\|$, $\|f\| = \sup_{\mathbf{x}\in H_q} |f(\mathbf{x})|$.

Theorem 6. *Let $L_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^q$, be the operators defined by (30). Set*

$$\alpha_{\mathbf{n},i}(\mathbf{x}) = \frac{1}{pr_i(\mathbf{n})} \sum_{j=1}^{pr_i(\mathbf{n})} h_{i,j}(pr_i(\mathbf{x})) - pr_i(\mathbf{x}),\ 1 \leq i \leq q. \tag{31}$$

If

$$\lim_{\mathbf{n}\to\infty} \|\alpha_{\mathbf{n},i}\| = 0,\ 1 \leq i \leq q, \tag{32}$$

then, for any $f \in C(H_q)$,

$$\lim_{\mathbf{n}\to\infty} \|L_{\mathbf{n}} f - f\| = 0 \tag{33}$$

takes place.

Proof. Based on the relations (29), we deduce that the defined operators are linear and positive. For proving (33) we resort to the multivariate Korovkin theorem. In this frame we are reminded that

$$\mathcal{K}_q = \{\mathbf{1}, pr_1, \dots, pr_q, \pi_q\} \tag{34}$$

is a Korovkin system of the test functions in $C(H_q)$, see, e.g., [13], Theorem 4.1. Mentioning that the q-dimensional Korovkin criterion was first reached by Šaškin [14], we have to prove the following relations

$$\lim_{\mathbf{n}\to\infty} \|L_{\mathbf{n}}\mathbf{1} - \mathbf{1}\| = 0, \tag{35}$$

$$\lim_{\mathbf{n}\to\infty} \|L_{\mathbf{n}} pr_i - pr_i\| = 0 \text{ for } i = \overline{1,q}, \tag{36}$$

$$\lim_{\mathbf{n}\to\infty} \|L_{\mathbf{n}}\pi_q - \pi_q\| = 0. \tag{37}$$

Since for any $n_\nu \in \mathbb{N}$, based on (29), we obtain $P_{n_\nu}(x_i;1) = 1$, $i = \overline{1,q}$, it follows that $L_{\mathbf{n}}\mathbf{1} = \mathbf{1}$ and (35) is fulfilled.

Further, taking in view the definition of $L_{\mathbf{n}}$ operators and inspired by (7), for each $i = 1, 2, \dots, q$, and $\mathbf{x} \in H_q$, we obtain

$$(L_{\mathbf{n}} pr_i)(\mathbf{x}) = \sum_{k_i=0}^{n_i} a_{n_i,k_i}(x_i)\frac{k_i}{n_i} = \frac{1}{n_i}\frac{\partial P_{n_i}(x_i;y)}{\partial y}\Big|_{y=1} = \frac{1}{pr_i(\mathbf{n})}\sum_{j=1}^{pr_i(\mathbf{n})} h_{i,j}(pr_i(\mathbf{x})).$$

Consequently, using the notation from (31), we obtain

$$|L_{\mathbf{n}} pr_i(\mathbf{x}) - pr_i(\mathbf{x})| = |\alpha_{\mathbf{n},i}(\mathbf{x})| \le \|\alpha_{\mathbf{n},i}\|,\ 1 \le i \le q, \tag{38}$$

and relation (32) leads us to (36).

Further, we prove the fulfilling of the relation (37). Based on (8) we have for each $i = \overline{1,q}$

$$(L_{\mathbf{n}} pr_i^2)(\mathbf{x}) - pr_i^2(\mathbf{x}) = \alpha_{\mathbf{n},i}^2(\mathbf{x}) + \frac{1}{pr_i^2(\mathbf{n})}\sum_{j=1}^{pr_i(\mathbf{n})} h_{i,j}(x_i)(1 - h_{i,j}(x_i))$$
$$+ 2pr_i(\mathbf{x})\alpha_{\mathbf{n},i}(\mathbf{x}),\ \mathbf{x} \in H_q.$$

Since $h_{i,j}([0,1]) \subseteq [0,1]$, $j \in \mathbb{N}$, $i = \overline{1,q}$, we can write

$$|(L_{\mathbf{n}} pr_i^2)(\mathbf{x}) - pr_i^2(\mathbf{x})| \le \alpha_{\mathbf{n},i}^2(\mathbf{x}) + \frac{1}{n_i} + 2\alpha_{\mathbf{n},i}(\mathbf{x}),\ \mathbf{x} \in H_q.$$

Consequently,

$$|(L_{\mathbf{n}}\pi_q)(\mathbf{x}) - \pi_q(\mathbf{x})| \le \sum_{i=1}^{q}(\|\alpha_{\mathbf{n},i}\|^2 + 2\|\alpha_{\mathbf{n},i}\|) + \|\tau_{\mathbf{n}}\|_{l_1}, \tag{39}$$

where $\tau_{\mathbf{n}} = \left(\frac{1}{n_1}, \dots, \frac{1}{n_q}\right)$ and $\|\cdot\|_{l_1}$ stands for the l_1 norm (or so-called taxicab norm) defined as the simple sum of the absolute values of the $\tau_{\mathbf{n}}$ vector components. The assumption (32) guarantees the validity of the relation (37) and the proof is ended. □

In order to complete the study of this sequence of multidimensional operators, we must establish the error of approximation. We achieve this task by involving the test functions from the set $\mathcal{K}_q$, as defined in (34). For the q-dimensional case, Censor [15] obtained in a general case such an estimation with the help of the following modulus

$$\omega(f;\delta) = \max_{\substack{\mathbf{x},\mathbf{y}\in X \\ d(\mathbf{x},\mathbf{y})\le\delta}} |f(\mathbf{x}) - f(\mathbf{y})|,\ \delta \ge 0, \tag{40}$$

where $X \subseteq \mathbb{R}^q$ is a convex compact, $d(\cdot,\cdot)$ stands for the Euclidean metric in $\mathbb{R}^q$ and f is a real-valued function continuous on X.

Theorem 7. *Let $L_{\mathbf{n}}$, $n \in \mathbb{N}^q$, be the operators defined by (30). The following relation*

$$\|L_{\mathbf{n}} f - f\| \le 2\omega(f;\delta_{\mathbf{n}})$$

holds, where ω is given by (40), $\alpha_{\mathbf{n},i}$, $i = \overline{1,q}$, are defined by (31) and

$$\delta_{\mathbf{n}}^2 = \sum_{i=1}^{q}(\|\alpha_{\mathbf{n},i}\|^2 + 4\|\alpha_{\mathbf{n},i}\|) + \|\tau_{\mathbf{n}}\|_{l_1}. \tag{41}$$

Proof. The statement is based on [15], Theorem 1, which says: if Λ_n is a positive linear operator on $C(X)$ such that $\Lambda_n \mathbf{1} = \mathbf{1}$, then

$$\|\Lambda_n f - f\| \le 2\omega(f;\mu_n)$$

holds, where $\mu_n^2 = \left\|\Lambda_n\left(\sum_{k=1}^{q}(\xi_k - x_k)^2; x_1,\ldots,x_q\right)\right\|$. Here Λ_n operates on a function of $\xi_1,\ldots,\xi_q$ and the resulting function is evaluated at the point $(x_1,\ldots,x_q)$. Above we used the notations and the explanations given in [15]. Taking in view our notations, for the operators $L_{\mathbf{n}}$ we obtain

$$\begin{aligned} L_{\mathbf{n}}\left(\sum_{i=1}^{q}(pr_i - x_i)^2;\mathbf{x}\right) &= (L_{\mathbf{n}}\pi_q)(\mathbf{x}) - 2\sum_{i=1}^{q} x_i (L_{\mathbf{n}} pr_i)(\mathbf{x}) + \sum_{i=1}^{q} x_i^2 \\ &= (L_{\mathbf{n}}\pi_q)(\mathbf{x}) - \pi_q(\mathbf{x}) - 2\sum_{i=1}^{q} x_i (L_{\mathbf{n}} pr_i)(\mathbf{x}) + \sum_{i=1}^{q} x_i^2. \end{aligned}$$

Consequently, appealing also to relations (39) and (38), we deduce

$$\begin{aligned} \left\|L_{\mathbf{n}}\left(\sum_{i=1}^{q}(pr_i - x_i)^2;\cdot\right)\right\| &\le \|L_{\mathbf{n}}\pi_q - \pi_q\| + 2\sum_{i=1}^{q}\|L_{\mathbf{n}} pr_i - pr_i\| \\ &\le \sum_{i=1}^{q}(\|\alpha_{\mathbf{n},i}\|^2 + 4\|\alpha_{\mathbf{n},i}\|) + \|\tau_{\mathbf{n}}\|_{l_1}. \end{aligned}$$

Defining $\delta_{\mathbf{n}}^2$ as in (41), the conclusion follows. □

Remark 5. *If condition (32) is valid and noting the definition of the vector $\tau_{\mathbf{n}}$, we deduce $\lim_{\mathbf{n}\to\infty} \delta_{\mathbf{n}} = 0$ in Theorem 7.*

5. Conclusions

Starting from a brief presentation of the Lototsky–Bernstein discrete operators, this paper has proposed two targets.

An extension of these operators on a hypercube included in $\mathbb{R}^q$ was achieved and the study of convergence with the determination of an upper limit of the error of approximation was obtained. At the same time, an integral Durrmeyer-type variant of one-dimensional operators was introduced. The new class is linear and positive. The approximation properties presented and proved in this paper apply to functions belonging to the space $C([0,1])$. The error of approximation was estimated by using both K-functionals and moduli of smoothness of first- and second-order. The least concave majorant of the modulus of smoothness $\omega(f;\cdot)$ was also involved.

Our main results are concentrated in seven theorems stated and proved in Sections 3 and 4. We specify that the constructions of integral operators of the Durrmeyer-type have caught the attention of many researchers in the field of Approximation Theory. Their study involves different spaces of functions and generalizations in various directions, revealing some of their essential properties. We support this statement by indicating some significant papers, randomly selected, published in the years 2020–2021, see [16–20]. Regarding the extension in multidimensional Euclidean space, what we have achieved in Section 4 can be found for many other linear positive operators. For the sake of edification, we recall paper [21] published this year, which targets Landau operators.

A weak point of the paper is that the construction of these integral operators was based on an unspecified system of functions $h_n \in C([0,1])$, $n \in \mathbb{N}$. This involves formulating the results at a more general level, without specific issues. By identifying the above-mentioned functions by particular laws, stronger results can be obtained but, following this route, the general properties of the Durrmeyer–Lototsky operators are not well highlighted. For example, in [5], after systematically studying general properties of Kantorovich–Lototsky operators, the authors considered the following functions as two particular examples:

$$h_n(x) = \frac{n^\alpha x}{n^\alpha + x} \quad \text{and} \quad h_n(x) = \frac{(\ln n)^\alpha x}{(\ln n)^\alpha + x}, \quad n \in \mathbb{N},\ x \in [0,1],$$

where α is a positive parameter.

Finally we mention that we wanted the presentation to be self-contained such that it is accessible to a wide audience.

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References

1. King, J.P. The Lototsky transform and Bernstein polynomials. *Canad. J. Math.* **1966**, *18*, 89–91. [CrossRef]
2. Xu, X.-W.; Goldman, R. On Lototsky-Bernstein operators and Lototsky-Bernstein bases. *Comput. Aided Geom. Design* **2019**, *68*, 48–59. [CrossRef]
3. Goldman, R.; Xu, X.-W.; Zeng, X.-M. Applications of the Shorgin identity to Bernstein type operators. *Results Math.* **2018**, *73*, 2. [CrossRef]
4. Xu, X.-W.; Zeng, X.-M.; Goldman, R. Shape preserving properties of univariate Lototsky-Bernstein operators. *J. Approx. Theory* **2017**, *224*, 13–42. [CrossRef]
5. Popa, D. Intermediate Voronovskaja type results for the Lototsky-Bernstein type operators. *RACSAM* **2020**, *114*, 12. [CrossRef]
6. Eisenberg, S.; Wood, B. Approximation of analytic functions by Bernstein-type operators. *J. Approx. Theory* **1972**, *6*, 242–248. [CrossRef]
7. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*; Series 55; National Bureau of Standards Applied Mathematics: Washington, DC, USA, 1964. Available online: https://eric.ed.gov/?id=ED250164 (accessed on 30 July 2021)
8. Derriennic, M.M. Sur l'approximation de fonctions intégrables sur $[0,1]$ par des polynômes de Bernstein modifies. *J. Approx. Theory* **1981**, *31*, 325–343. [CrossRef]
9. Shisha, O.; Mond, B. The degree of convergence of linear positive operators. *Proc. Natl. Acad. Sci. USA* **1968**, *60*, 1196–1200. [CrossRef] [PubMed]
10. DeVore, R.A.; Lorentz, G.G. *Constructive Approximation*; Grundlehren der Mathematischen Wissenschaften; Springer: Berlin, Germany, 1993; Volume 303.
11. Mitjagin, B.S.; Semenov, E.M. Lack of interpolation of linear operators in spaces of smooth functions. *Math. USSR. Izv.* **1977**, *11*, 1229–1266. [CrossRef]
12. Johnen, H. Inequalities connected with the moduli of smoothness. *Mat. Vesnik* **1972**, *9*, 289–305.
13. Altomare, F. Korovkin-type theorems and approximation by positive linear operators. *Surv. Approx. Theory* **2010**, *5*, 92–164.

14. Šaškin, Y.A. Korovkin systems in spaces of continuous functions. *Izv. Akad. Nauk SSSR Ser. Mat.* **1962**, *26*, 495–512; translated in *Amer. Math. Soc. Transl.* **1966**, *54*, 125–144. (In Russian)
15. Censor, E. Quantitative results for positive linear approximation operators. *J. Approx. Theory* **1971**, *4*, 442–450. [CrossRef]
16. Deo, N.; Kumar, S. Durrmeyer variant of Apostol-Genocchi-Baskakov operators. *Quaes. Math.* **2020**. [CrossRef]
17. Garg, T.; Ispir, N.; Agrawal, P.N. Bivariate q-Bernstein-Chlodowsky-Durrmeyer type operators and associated GBS operators. *Asian-Eur. J. Math.* **2020**, *13*, 2050091. [CrossRef]
18. Kajla, A.; Mursaleen, M.; Acar, T. Durrmeyer type generalization of parametric Bernstein operators. *Symmetry* **2020**, *12*, 1141. [CrossRef]
19. Neer, T.; Acu, A.M.; Agrawal, P.N. Baskakov-Durrmeyer type operators involving generalized Appell polynomials. *Math. Methods Appl. Sci.* **2020**, *43*, 2911–2923. [CrossRef]
20. Abel, U.; Leviatan, D.; Rasa, I. On the q-monotonicity preservation of Durrmeyer-type operators. *Mediterr. J. Math.* **2021**, *18*, 173. [CrossRef]
21. Agratini, O.; Aral, A. Approximation of some classes of functions by Landau type operators. *Results Math.* **2021**, *76*, 12. [CrossRef]