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# Simultaneous Approximation by Gauss–Weierstrass–Wachnicki Operators

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**Abstract.** In this note, we spotlight a generalization of Weierstrass integral operators introduced by Eugeniusz Wachnicki. The construction involves modified Bessel functions. The operators are correlated with diffusion equation. Our main result consists in obtaining the asymptotic expansion of derivatives of any order of Wachnicki's operators. All coefficients are explicitly calculated, and distinct expressions are provided for analytical functions.

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**Keywords.** Gauss–Weierstrass operator, Bessel function, Gamma function, hypergeometric function, asymptotic expansion.

## 1. Introduction

The starting point of this note is the following generalized Gauss–Weierstrass transform:

$$\begin{aligned} W(f;x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4t}\right) f(y)dy \\ &= \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(x-y) \exp\left(-\frac{y^2}{4t}\right) dy, \end{aligned} \tag{1}$$

where $t > 0$ is a parameter, $x \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is chosen such that the integral exists and is finite. Actually, (1) represents the convolution of $f$ with the density of the normal distribution (also called Gaussian distribution) having the expectation null and the variance $2t$. $W(f;\cdot,t)$ is a smoothed out version of $f$, and physically $W(\cdot;\cdot,t) \equiv W_t$ is correlated with a heat or diffusion equation for $t$ time units. It is additive

$$W_{t_1} \circ W_{t_2} = W_{t_1+t_2},\ (t_1,t_2) \in (0,\infty) \times (0,\infty),$$

this being read as follows: diffusion for $t_1$ time units and then $t_2$ time units is equivalent to diffusion for $t_1 + t_2$ time units.

Ulrich Abel and Octavian Agratini equal contributed to this work.

We sketch a formal proof of this relation:

$$\begin{aligned}
&\left(\left(W_{t_1} \circ W_{t_2}\right) f\right)(x) \\
&= \frac{1}{2\sqrt{\pi t_1}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-z)^2}{4t_1}\right) \left(\frac{1}{2\sqrt{\pi t_2}} \int_{-\infty}^{+\infty} \right. \\
&\left. \exp\left(-\frac{(z-y)^2}{4t_2}\right) f(y)\, dy\right) dz \\
&= \frac{1}{4\pi\sqrt{t_1 t_2}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \right. \\
&\left. \exp\left(-\frac{(x-z)^2}{4t_1}\right) \exp\left(-\frac{(z-y)^2}{4t_2}\right) dz\right) f(y)\, dy.
\end{aligned}$$

Direct calculation confirms that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-z)^2}{4t_1}\right) \exp\left(-\frac{(z-y)^2}{4t_2}\right) dz \\
&= \frac{2\sqrt{\pi}}{\sqrt{\frac{1}{t_1}+\frac{1}{t_2}}} \exp\left(-\frac{(x-y)^2}{4(t_1+t_2)}\right),
\end{aligned}$$

and we obtain

$$\begin{aligned}
\left(\left(W_{t_1} \circ W_{t_2}\right) f\right)(x) = \frac{1}{2\sqrt{\pi(t_1+t_2)}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4(t_1+t_2)}\right) \\
f(y)\, dy = \left(W_{t_1+t_2} f\right)(x).
\end{aligned}$$

Examining relation (1) shows that $W_t$ is translation invariant, meaning that the transform of $f(x+a)$ is $W(f; x+a, t)$, for $a \in \mathbb{R}$.

We mention that the $W_t$ transform can extend to $t = 0$ by setting $W_0$ to be the convolution with the Dirac delta function. This case does not come to our attention.

The special case $t = 1$ can be interpreted this way. Taking in view the formula:

$$e^{u^2} = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-uy} e^{-y^2/4} dy,\ u \in \mathbb{R},$$

if we replace $u$ with the formal differential operator $D = \dfrac{d}{dx}$ and utilize the Lagrange shift operator

$$e^{-yD} f(x) = f(x-y),\ x \in \mathbb{R},$$

then we get

$$e^{D^2} f(x) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-yD} f(x) e^{-y^2/4} dy = W(f; x, 1),$$

which allows us to get the following formal expression for this particular transform:

$$W_1 = e^{D^2}.$$

The operator $e^{D^2}$ is to be understood as acting on the signal $f$ as

$$e^{D^2} f(x) = \sum_{k=0}^{\infty} \frac{D^{2k} f(x)}{k!},\ x \in \mathbb{R}.$$

For more documentation on $W_t$ transform, see Zayed's book [6, *Chapter 18: The Weierstrass Transform*].

In 2000, Eugeniusz Wachnicki [5] defined and studied an integral operator representing a generalization of $W_t$ which involves modified Bessel functions. Our paper focuses on bringing to light new properties of Gauss–Weierstrass–Wachnicki integral operators. In the next section, we present these operators pointing out some already established properties based on which we will highlight their noteworthy features. The main results are set out in Sect. 3.

## 2. The Operators

In the beginning, we recall the modified Bessel function of the first kind and fractional order $\alpha > -1$, see [2, *Chapter 10*]. Using the traditional notation $I_\alpha$, it is described by the series

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\alpha+1)} \left(\frac{z}{2}\right)^{2k+\alpha}, \tag{2}$$

where $\Gamma$ is the Gamma function. $I_\alpha$ forms a class of particular solutions of the ordinary linear differential equation

$$z^2 w''(z) + z w'(z) - (\alpha^2 + z^2) w(z) = 0.$$

In the motivation of our results, we will also use the low-order differentiation with respect to $z$ described as follows:

$$\frac{d}{dz}(z^{-\alpha} I_\alpha(z)) = z^{-\alpha} I_{\alpha+1}(z); \tag{3}$$

see [2, *Formula (9.6.28)*].

Set $\mathbb{R}_+ = [0, \infty)$. For a fixed constant $K \geq 0$, we consider the space

$$E_K = \{f : \mathbb{R}_+ \to \mathbb{R} \mid f \text{ is locally integrable and exists } M_f \geq 0,$$
$$|f(s)| \leq M_f \exp(Ks^2),\ s > 0\}.$$

The space can be endowed with the norm $\|\cdot\|_K$ as such

$$\|f\|_K = \sup_{s \in \mathbb{R}_+} |f(s)| \exp(-Ks^2).$$

We consider the operator $W_\alpha$ defined on $E_K$ by the following relation:

$$W_\alpha(f; r, t) = \frac{1}{2t} \int_0^\infty r^{-\alpha} s^{\alpha+1} \exp\left(-\frac{r^2+s^2}{4t}\right) I_\alpha\left(\frac{rs}{2t}\right) f(s) ds, \tag{4}$$

where $\alpha \geq -1/2$, $(r, t) \in (0, \infty) \times (0, \infty)$ and $I_\alpha$ is given at (2).

This operator was introduced in [5, *Eq. (1)*] with a minor modification of the domain $E_K$ in which the author inserted $f \in C(\mathbb{R}_+)$, the space of all real-valued continuous functions defined on $\mathbb{R}_+$. Since $W_\alpha f$, $f \in E_K$, is well defined for any $K > 0$ , we can consider the domain of $W_\alpha$ as

$$E = \bigcup_{K>0} E_K.$$

Wachnicki [5, *Theorem 4*] showed the convergence

$$\lim_{t\to 0^+} W_\alpha(f;r,t) = f(r),\ \ f \in E \cap C(\mathbb{R}_+),$$

uniformly on compact subintervals of $(0,\infty)$.

Also, in [5], the author specified that for $\alpha = -1/2$, the operator defined by (4) turns out to be the authentic Gauss–Weierstrass operators. Because this statement was not accompanied by a proof, we insert it as a detail in our paper. More precisely, we prove

$$W_{-1/2} f = W\widehat{f},\ \ f \in E_K, \tag{5}$$

see (1), where $\widehat{f}(s) = f((\operatorname{sgn} s)s)$, $s \in \mathbb{R}$.

By using the hyperbolic cosine, the identity

$$I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cosh(z)$$

takes place; see [2, *page 443*]. Further, for any $(r,t) \in (0,\infty)\times(0,\infty)$, we can write successively

$$\begin{aligned}
&W_{-1/2}(f;r,t)\\
&= \frac{1}{2t}\sqrt{\frac{4t}{\pi}}\int_0^\infty \exp\left(-\frac{r^2+s^2}{4t}\right)\cosh\left(\frac{rs}{2t}\right) f(s)ds\\
&= \frac{1}{2\sqrt{\pi t}}\int_0^\infty \exp\left(-\frac{r^2+s^2}{4t}\right)\left(\exp\left(\frac{rs}{2t}\right)+\exp\left(-\frac{rs}{2t}\right)\right) f(s)ds\\
&= \frac{1}{2\sqrt{\pi t}}\int_0^\infty \left(\exp\left(-\frac{(r-s)^2}{4t}\right)+\exp\left(-\frac{(r+s)^2}{4t}\right)\right) f(s)ds\\
&= \frac{1}{2\sqrt{\pi t}}\left(\int_0^\infty \exp\left(-\frac{(r-s)^2}{4t}\right) f(s)ds + \int_{-\infty}^0 \exp\left(-\frac{(r-s)^2}{4t}\right)\right) f(-s)ds\\
&= W(\widehat{f};r,t)
\end{aligned}$$

and statement (5) is completed.

$W_\alpha f$ is intimately connected to a generalized heat equation having the expression; see [3, *Eq. (1.3)*],

$$\frac{\partial u(r,t)}{\partial t} = \Delta_\mu u(r,t), \tag{6}$$

where $\mu = 2(\alpha+1)$, $\alpha > -1/2$, and the operator

$$\Delta_\mu = \frac{\partial^2}{\partial r^2} + \frac{\mu-1}{r}\frac{\partial}{\partial r}$$

is the Laplacian in radial coordinates when $\mu = n \in \mathbb{N}$. As already mentioned, it is usual to refer to $t$ as time. If $f \in E_K \cap C(\mathbb{R})$, then $W_\alpha f$ with $\alpha = n/2-1$

is a caloric function, which means it is a solution of equation (6) on a certain domain $D$,

$$D=\left\{(r,t):\ r>0,\ 0<t<\frac{1}{4K}\right\}.$$

For detailed proof, see [3, *pages 254–255*].

Recently, these operators have come back to the attention of some authors. For example, in [4], an extension of $W_\alpha f$ was achieved for continuous functions defined on the domain $(0,\infty)\times\mathbb{R}$ and bounded by certain two-dimensional exponential functions. In [1], the authors obtained the asymptotic expansion of the operator $W_\alpha(f;r,t)$ as $t\to 0^+$, for functions $f\in E$ being sufficiently smooth at a point $r>0$. If $f$ belonging to $E$ is a real analytic function, then

$$W_\alpha(f;r,t)\sim\sum_{n=0}^{\infty}c_n(\alpha,f,r)t^n\quad(t\to 0^+),\tag{7}$$

where the coefficients are given by

$$c_n(\alpha,f,r)=\frac{4^n}{n!r^{2n}}\left(\frac{\partial}{\partial w}\right)^n\left[w^{n+\alpha}\left(\frac{\partial}{\partial w}\right)^n f\left(r\sqrt{w}\right)\right]\Big|_{w=1}\tag{8}$$

( [1, *Theorem 3*]). In this paper, we study simultaneous approximation by the operators $W_\alpha(f;r,t)$. The main result of this paper (Theorem 6) states that the expansion (7) can be differentiated term by term, i.e.,

$$\left(\frac{\partial}{\partial r}\right)^m W_\alpha(f;r,t)\sim\sum_{n=0}^{\infty}\left[\left(\frac{\partial}{\partial r}\right)^m c_n(\alpha,f,r)\right]t^n\quad(t\to 0^+).$$

To obtain an autonomous exposure, in this preliminary section, we recall some notions which will be used in establishing our results.

The factorial powers (falling and rising factorial, respectively) are marked as follows:

$$u^{\underline{j}}=\prod_{l=0}^{j-1}(u-l),\quad u^{\overline{j}}=\prod_{l=0}^{j-1}(u+l),\quad j\in\mathbb{N}.$$

An empty product ($j=0$) is taken to be 1.

For $|z|<1$ and generic parameters $a,b$, and $c$, the Gauss hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(a,b;c;z)=\sum_{j\geq 0}\frac{a^{\overline{j}}b^{\overline{j}}}{c^{\overline{j}}}\frac{z^j}{j!},\tag{9}$$

with this series being convergent; see [2, *Chapter 15*]. Outside the disk with unit radius, the function is defined as the analytic continuation with respect to $z$ of this sum, with the parameters $a,b,c$, held fixed. For a particular case $z=1$, the identity

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \mathrm{Re}(c-a-b)>0,\tag{10}$$

$c\neq 0,-1,-2,\ldots$, takes place.

## 3. Results

In the first stage, we establish some technical formulas gathered in a few lemmas. Set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

The first result is an explicit representation of $\left(\frac{\partial}{\partial r}\right)^m W_\alpha(f;r,t)$ in terms of $W_{\alpha+j}(f;r,t)$, $j \in \mathbb{N}_0$. We mention the identity that we will state contains a finite sum.

**Lemma 1.** *Let $f$ belong to $E_K$ and let $W_\alpha f$ be defined by (4). For any $m \in \mathbb{N}_0$,*

$$\left(\frac{\partial}{\partial r}\right)^m W_\alpha(f;r,t) = \sum_{j\geq 0} \frac{(2j)!}{2^j j!}\binom{m}{2j} r^{m-2j}\left(\frac{1}{2t}\Delta\right)^{m-j} W_\alpha(f;r,t) \quad (11)$$

*holds, where $\Delta$ denotes the forward difference of step one with respect to $\alpha$.*

*Proof.* If $f \in E_K$, for $t > 0$, we have

$$\frac{\partial}{\partial r} W_\alpha(f;r,t) = \frac{1}{2t}\int_0^\infty s^{\alpha+1}\frac{\partial}{\partial r}\left(\exp\left(-\frac{r^2+s^2}{4t}\right) r^{-\alpha} I_\alpha\left(\frac{rs}{2t}\right)\right) f(s)ds.$$

Formula (3) yields the relation:

$$\frac{\partial}{\partial r} W_\alpha(f;r,t) = \frac{r}{2t}(W_{\alpha+1}(f;r,t) - W_\alpha(f;r,t)) = \frac{r}{2t}\Delta W_\alpha(f;r,t). \quad (12)$$

We will establish the proof of identity (12) by mathematical induction. Obviously, the assertion is valid for $m = 0$. Assuming that it is true for an arbitrary $m$, we show that it takes place for $m+1$. Relations (11) and (12) imply

$$\begin{aligned}
\left(\frac{\partial}{\partial r}\right)^{m+1} W_\alpha(f;r,t) &= \sum_{j\geq 0} \frac{(2j)!}{2^j j!}\binom{m}{2j} \\
&\quad \times \left[(m-2j) r^{m-2j-1}\left(\frac{1}{2t}\Delta\right)^{m-j} + r^{m-2j+1}\left(\frac{1}{2t}\Delta\right)^{m-j+1}\right] W_\alpha(f;r,t) \\
&= r^{m+1}\left(\frac{1}{2t}\Delta\right)^{m+1} W_\alpha(f;r,t) \\
&\quad + \sum_{j\geq 1}\left[\frac{(2j-2)!}{2^{j-1}(j-1)!}\binom{m}{2j-2}(m-2j+2) + \frac{(2j)!}{2^j j!}\binom{m}{2j}\right] \\
&\quad \times r^{m-2j+1}\left(\frac{1}{2t}\Delta\right)^{m+1-j} W_\alpha(f;r,t) \\
&= r^{m+1}\left(\frac{1}{2t}\Delta\right)^{m+1} W_\alpha(f;r,t) \\
&\quad + \sum_{j\geq 1}\frac{(2j)!}{2^j j!}\left[\binom{m}{2j-1} + \binom{m}{2j}\right] r^{m+1-2j}\left(\frac{1}{2t}\Delta\right)^{m+1-j} W_\alpha(f;r,t).
\end{aligned}$$

Considering the elementary identity

$$\binom{m}{2j-1}+\binom{m}{2j}=\binom{m+1}{2j},$$

we obtain that (11) is valid for $m+1$, and the induction is completed. $\square$

Recall that in the following, $\Delta$ stands for the forward difference of the step $h=1$ with respect to $\alpha$.

**Lemma 2.** *Let $f\in E_K$ be a real analytic function and let $W_\alpha f$ be defined by (4). For any $m\in\mathbb{N}_0$,*

$$\Delta^m W_\alpha(f;r,t)\sim\sum_{n=m}^{\infty}\Delta^m c_n(\alpha,f,r)t^n \quad (t\to 0^+),$$

*where the quantities $c_n(\alpha,f,r)$ are described by (8). For $m\le n$,*

$$\Delta^m c_n(\alpha,f,r)=\frac{4^n}{(n-m)!r^{2n}}\left(\frac{\partial}{\partial w}\right)^{n-m}\left[w^{n+\alpha}\left(\frac{\partial}{\partial w}\right)^n f\left(r\sqrt{w}\right)\right]\Big|_{w=1} \tag{13}$$

*and for $m<n$, $\Delta^n c_n(\alpha,f,r)=0$.*

*Proof.* Based on the $m$-th order forward difference with the step $h=1$, we get

$$\Delta^m w^{m+\alpha}=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}w^{n+\alpha+k}=w^{n+\alpha}(w-1)^m.$$

Consequently, taking into account (8), we can write

$$\Delta^m c_n(\alpha,f,r)=\frac{4^n}{n!r^{2n}}\left(\frac{\partial}{\partial w}\right)^n\left[w^{n+\alpha}(w-1)^m\left(\frac{\partial}{\partial w}\right)^n f\left(f\sqrt{w}\right)\right]\Big|_{w=1}.$$

Since

$$\left(\frac{\partial}{\partial w}\right)^k(w-1)^m\Big|_{w=1}=0$$

for $k\neq m$, we deduce that $\Delta^m c_n(\alpha,f,r)$ is null for $m>n$. Otherwise, if $m\le n$, application of the Leibniz rule for differentiation yields

$$\Delta^m c_n(\alpha,f,r)=\frac{4^n}{n!r^{2n}}\binom{n}{m}m!\left(\frac{\partial}{\partial w}\right)^{n-m}\left[w^{n+\alpha}\left(\frac{\partial}{\partial w}\right)^n f\left(r\sqrt{w}\right)\right]\Big|_{w=1},$$

which leads us to (13), and the proof of our lemma is completed. $\square$

Our first main result can be read as follows.

**Theorem 3.** *Let $f\in E_K$ be a real analytic function and let $W_\alpha f$ be defined by (4). For any $m\in\mathbb{N}_0$, the relation*

$$\left(\frac{\partial}{\partial r}\right)^m W_\alpha(f;r,t)\sim\sum_{n=0}^{\infty}c_n^{[m]}(\alpha,f,r)t^n \quad (t\to 0^+) \tag{14}$$

*holds, where the coefficients are given by*

$$c_n^{[m]}(\alpha,f,r)=\frac{1}{n!}\left(\frac{2}{r}\right)^{2n+m}\sum_{j\geq 0}\frac{(2j)!}{4^j j!}\binom{m}{2j}\left(\frac{\partial}{\partial w}\right)^n \times\left[w^{n+m-j+\alpha}\left(\frac{\partial}{\partial w}\right)^{n+m-j}f\left(r\sqrt{w}\right)\right]\Big|_{w=1}. \tag{15}$$

*Proof.* Let $m\in\mathbb{N}_0$ be fixed. Concatenating the conclusions of Lemmas 1 and 2, we can write

$$\begin{aligned}&\left(\frac{\partial}{\partial r}\right)^m W_\alpha(f;r,t)\\ &\sim\frac{1}{2^m}\sum_{j\geq 0}\frac{(2j)!}{j!}\binom{m}{2j}r^{m-2j}\sum_{n=m-j}^{\infty}\Delta^{m-j}c_n(\alpha,f,r)t^{n-m+j}\\ &=\frac{1}{2^m}\sum_{j\geq 0}\frac{(2j)!}{j!}\binom{m}{2j}r^{m-2j}\sum_{n=0}^{\infty}\Delta^{m-j}c_{n+m-j}(\alpha,f,r)t^n\quad(t\to 0^+)\end{aligned}\tag{16}$$

Further, with the help of the relation (13) applied for $\Delta^{m-j}c_{n+m-j}(\alpha,f,r)$, for each $n\in\mathbb{N}_0$, the coefficient of $t^n$ can be found to be

$$\begin{aligned}&\frac{1}{2^m}\sum_{j\geq 0}\frac{(2j)!}{j!}\binom{m}{2j}r^{m-2j}\Delta^{m-j}c_{n+m-j}(\alpha,f,r)\\ &=\frac{1}{2^m}\sum_{j\geq 0}\frac{(2j)!}{j!}\binom{m}{2j}r^{m-2j}\frac{4^{n+m-j}}{n!r^{2(n+m-j)}}\\ &\times\left(\frac{\partial}{\partial w}\right)^n\left[w^{n+m-j+\alpha}\left(\frac{\partial}{\partial w}\right)^{n+m-j}f\left(r\sqrt{w}\right)\right]\Big|_{w=1}:=c_n^{[m]}(\alpha,f,r);\end{aligned}$$

see (15). Returning to (16), the statement (14) is proven. □

*Remark.* Choosing in (15) $m=0$, the sum is reduced to a single term ($j=0$) and $c_n^{[0]}(\alpha,f,r)$ coincides with $c_n(\alpha,f,r)$ defined at (8).

Now, we show that

$$c_n^{[m]}\left(\alpha,f,r\right)=\left(\frac{\partial}{\partial r}\right)^m c_n\left(\alpha,f,r\right), \tag{17}$$

where $c_n^{[m]}\left(\alpha,f,r\right)$ and $c_n\left(\alpha,f,r\right)$ are as defined by (15) and (8).

To present our result (see Proposition 5), we first need to establish some identities that involve hypergeometric functions defined by (9) and (10).

**Lemma 4.** *Let $m\in\mathbb{N}_0$ and $x>m-1$, $x\notin\mathbb{Z}$, be arbitrarily chosen.*

*(i) For any $z$, $|z|<1/4$, the identity*

$$\sum_{j\geq 0}\binom{2j}{j}\binom{x}{m-j}\left(\binom{m}{2j}\Big/\binom{m}{j}\right)z^j=\binom{x}{m}\,{}_2F_1\left(\frac{-m}{2},\frac{-m+1}{2};x-m+1;4z\right), \tag{18}$$

*holds, where ${}_2F_1$ is given by (9).*

*(ii) For $z = 1/4$, the identity*

$$\sum_{j\geq 0} \frac{(2j)!}{4^j j!}\binom{m}{2j} x^{\underline{m-j}} = 2^{-m}(2x)^{\underline{m}} \tag{19}$$

*holds.*

*Proof.* (i) The sum of the left-hand side of (18) is finite and has terms only for integer values of $j$ satisfying $0 \leq j \leq \lfloor m/2 \rfloor$, where $\lfloor \cdot \rfloor$ stands for the floor function. Using the obvious formulas

$$\frac{x^{\underline{m}}}{m!} = \binom{x}{m}, \quad w^{\overline{p}} = (-1)^p(-w)^{\underline{p}} = (w+p-1)^{\underline{p}},$$

valid for any $p \in \mathbb{N}_0$ and real or complex $w$, we can write the next set of identities:

$$\begin{aligned}
&\binom{2j}{j}\binom{x}{m-j}\left(\binom{m}{2j}\Big/\binom{m}{j}\right) \\
&= \frac{m^{\underline{2j}}}{j!m!}x^{\underline{m-j}} = \frac{(-m)^{\overline{2j}}}{j!m!}\,\frac{x^{\underline{m}}}{(x-m+j)^{\underline{j}}} \\
&= \frac{x^{\underline{m}}}{m!}\,\frac{2^j\left(-\frac{m}{2}\right)^{\overline{j}} 2^j\left(-\frac{m-1}{2}\right)^{\overline{j}}}{(x-m+1)^{\overline{j}}j!}.
\end{aligned}$$

From the above relations, we obtain

$$\sum_{j\geq 0}\binom{2j}{j}\binom{x}{m-j}\left(\binom{m}{2j}\Big/\binom{m}{j}\right)z^j = \binom{x}{m}\sum_{j\geq 0}\frac{\left(-\frac{m}{2}\right)^{\overline{j}}\left(-\frac{m-1}{2}\right)^{\overline{j}}}{(x-m+1)^{\overline{j}}}\,\frac{(4z)^j}{j!}.$$

Considering (9), the relation (18) is proved.

(ii) Choosing $z = 1/4$ in (18) and using (10), this identity can thus be rewritten as

$$\sum_{j\geq 0}\frac{(2j)!}{4^j j!}\binom{m}{2j}x^{\underline{m-j}} = x^{\underline{m}}\,\frac{\Gamma(x-m+1)\Gamma\left(x+\frac{1}{2}\right)}{\Gamma\left(x-\frac{m}{2}+1\right)\Gamma\left(x-\frac{m}{2}+\frac{1}{2}\right)}. \tag{20}$$

The Legendre duplication formula for the Gamma function

$$\Gamma(2w) = \frac{1}{\sqrt{\pi}}\,2^{2w-1}\Gamma(w)\Gamma\left(w+\frac{1}{2}\right),\ 2w \neq 0, -1, -2, \ldots,$$

see, e.g., [7, *Eq. 5.5.5*], allows us to write

$$\begin{aligned}
\frac{\Gamma(x-m+1)\Gamma\left(x+\frac{1}{2}\right)}{\Gamma\left(x-\frac{m}{2}+1\right)\Gamma\left(x-\frac{m}{2}+\frac{1}{2}\right)} &= \frac{\Gamma(x-m+1)\Gamma(2x+1)2^{-2x}}{\Gamma(x+1)\Gamma(2x-m+1)2^{-2x+m}} \\
&= 2^{-m}\,\frac{(2x)^{\underline{m}}}{x^{\underline{m}}}.
\end{aligned}$$

Returning to (20), identity (19) is obtained, and our lemma is completely motivated. $\square$

Since $c_n(\alpha, f, r)$ is a finite linear combination of derivatives of $f$, it is sufficient to prove Identity (17) for monomials $e_i$, $e_i(r) = r^i$ $(i \in \mathbb{N}_0)$. We confirm this in the next proposition.

**Proposition 5.** *For any $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$, the identity*

$$c_n^{[m]}(\alpha, e_i, r) = \left(\frac{\partial}{\partial r}\right)^m c_n(\alpha, e_i, r) \tag{21}$$

*holds for all monomials $e_i$ $(i \in \mathbb{N}_0)$, where $c_n^{[m]}(\alpha, \cdot, r)$ and $c_n(\alpha, \cdot, r)$ are respectively defined by (15) and (8).*

*Proof.* For $m = 0$, we have highlighted this identity; see Remark. Further, we consider $m \in \mathbb{N}$.

$$\begin{aligned}
&c_n^{[m]}(\alpha, e_i, r) \\
&= \frac{1}{n!}\left(\frac{2}{r}\right)^{2n+m} \sum_{j \ge 0} \frac{(2j)!}{4^j j!}\binom{m}{2j}\left(\frac{\partial}{\partial w}\right)^n \\
&\quad \left[w^{n+m-j+\alpha}\left(\frac{\partial}{\partial w}\right)^{n+m-j} r^i w^{i/2}\right]\Big|_{w=1} \\
&= \frac{r^i}{n!}\left(\frac{2}{r}\right)^{2n+m} \sum_{j \ge 0} \frac{(2j)!}{4^j j!}\binom{m}{2j}\left(\frac{\partial}{\partial w}\right)^n \left[\left(\frac{i}{2}\right)^{\underline{n+m-j}} w^{i/2+\alpha}\right]\Big|_{w=1} \\
&= \frac{r^i}{n!}\left(\frac{2}{r}\right)^{2n+m} \sum_{j \ge 0} \frac{(2j)!}{4^j j!}\binom{m}{2j}\left(\frac{i}{2}\right)^{\underline{n+m-j}}\left(\frac{i}{2}+\alpha\right)^{\underline{n}}.
\end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned}
&\left(\frac{\partial}{\partial r}\right)^m c_n(\alpha, e_i, r) \\
&\quad = \frac{4^n}{n!}\left[\left(\frac{\partial}{\partial r}\right)^m \frac{r^i}{r^{2n}}\right]\left(\frac{\partial}{\partial w}\right)^n \left[w^{n+\alpha}\left(\frac{\partial}{\partial w}\right)^n w^{i/2}\right]\Big|_{w=1} \\
&\quad = \frac{4^n}{n!}(i-2n)^{\underline{m}} r^{i-2n-m}\left(\frac{\partial}{\partial w}\right)^n \left[w^{n+\alpha}\left(\frac{i}{2}\right)^{\underline{n}} w^{i/2-n}\right]\Big|_{w=1} \\
&\quad = \frac{r^i}{n!}\,\frac{2^{2n}}{i^{2n+m}}(i-2n)^{\underline{m}}\left(\frac{i}{2}\right)^{\underline{n}}\left(\frac{i}{2}+\alpha\right)^{\underline{n}}.
\end{aligned} \tag{23}$$

To match the expressions (22) and (23), it remains to be shown that

$$2^m \sum_{j \ge 0} \frac{(2j)!}{4^j j!}\binom{m}{2j}\left(\frac{i}{2}-n\right)^{\underline{m-j}} = (i-2n)^{\underline{m}}. \tag{24}$$

We have taken into account that

$$\left(\frac{i}{2}\right)^{\underline{n+m-j}} \Big/ \left(\frac{i}{2}\right)^{\underline{n}} = \left(\frac{i}{2}-n\right)^{\underline{m-j}}.$$

Now, we turn to Lemma 4. By choosing $x := i/2 - n$ in the identity (19), we obtain exactly (24). Thus, relation (21) is proven. □

Now, we can state our main result.

**Theorem 6.** *Let $f \in E_K$ be a real analytic function and let $W_\alpha f$ be defined by (4). For any $m \in \mathbb{N}_0$, the relation*

$$\left(\frac{\partial}{\partial r}\right)^m W_\alpha(f;r,t) \sim \sum_{n=0}^{\infty} \left[\left(\frac{\partial}{\partial r}\right)^m c_n(\alpha,f,r)\right] t^n \quad (t \to 0^+)$$

*holds, where the coefficients $c_n(\alpha,f,r)$ are given by (8).*

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