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ASYMPTOTIC PROPERTIES OF KANTOROVICH-TYPE SZÁSZ–MIRAKJAN OPERATORS OF HIGHER ORDER

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Dedicated to Professor Vijay Gupta on the occasion of his 60th birthday anniversary

Abstract. In this paper we study local approximation properties of a higher order Kantorovich-type Szász–Mirakjan operator recently introduced by Sabancigil, Kara, and Mahmudov. We derive the complete asymptotic expansion for these operators. They generalize the Szász–Mirakjan operators of Kantorovich-type and approximate locally integrable functions satisfying a certain growth condition on the infinite interval $(0,\infty)$.

1. Introduction.

As the Bernstein operators represent the most investigated linear positive approximation process of functions defined on a compact interval, the Szász operators and their generalizations are the most studied approximation operators for functions defined on an unbounded interval. For the complete information of the readers, we will present the first expressions of these operators keeping the same notations used in the cited works.

In 1950, for the infinite interval $(0,\infty)$ Szász [32, Eq. (2)] defined the transform

$$P(u;f)=e^{-ux}\sum_{\nu=0}^{\infty}\frac{1}{\nu!}(ux)^{\nu}f\left(\frac{\nu}{u}\right),\ u>0,$$

proving the following main result: suppose that $f(x)$ is bounded in every finite interval; if $f(x)=\mathcal{O}(x^k)$ for some $k>0$ as $x\to\infty$ and if $f(x)$ is continuous at a point ξ, then $P(u;f)$ converges uniformly to $f(x)$ at $x=\xi$ ([32, Theorem 1]).

As mentioned in [12, p. 547], the Polish mathematician Mark Kac pointed out to Butzer that he considered the above transform several years ago and made use of it in his lectures, but never published his result. In fact, the definition of this approximation process has been carried out over time by several mathematicians. Thus, in 1941 Mirakyan [27] considered a finite sum. The above series was also studied in 1944 by Favard [17].

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The domain of definition of operators was considered in papers that appeared in different forms. For example, Sikkema [31, Application II] denoted $H^{(s)}(\xi)$ the set of all real functions $f(x)$ which are defined on the whole of the real x-axis and possess the following three properties: $f(x)$ is s times ($s \geq 1$) differentiable at $x = \xi$, $f(x)$ is bounded on every finite interval of the x-axis and $f(x) = \mathcal{O}(|x|^s)$ if $|x| \to \infty$. Further, Sikkema considered the operators $S_{n,p} : H^{(q)}(\xi) \to C([0, b])$, $\xi \in [0, b]$, where b denotes an arbitrary positive number and p is non-negative

$$S_{n,p}(f(t); x) = e^{-(n+p)x} \sum_{k=0}^{\infty} \frac{(n+p)^k x^k}{k!} f\left(\frac{k}{n}\right) \quad (n = 1, 2, \ldots).$$

In case $p = 0$ we obtain the Mirakjan operators [27] which also have been considered by Szász in 1950. Case $p > 0$ occurs in a paper by Schurer [30].

Altomare and Campiti in their monograph [6, Eqs. (5.3.54)-(5.3.57)] considered the Banach lattice

$$E_2 := \{f \in C(\mathbb{R}_+) \mid \frac{f(x)}{1 + x^2} \text{ is convergent as } x \to \infty\}$$

which becomes the domain for the n-th Szász operator, $S_n f \equiv P(n; f)$, $n \geq 1$, proving that for any $f \in E_2$ the series is absolutely convergent. Moreover, every S_n maps $C_B(\mathbb{R}_+)$, respectively $C_0(\mathbb{R}_+)$, into itself. Here $C_B(\mathbb{R}_+) = C(\mathbb{R}_+) \cap B(\mathbb{R}_+)$ and $C_0(\mathbb{R}_+)$ is the space of all functions $f \in C(\mathbb{R}_+)$ which vanish at infinity. $B(\mathbb{R}_+)$ stands for the space of all real valued bounded functions defined on $\mathbb{R}_+$.

In our opinion, the most appropriate domain for operators is the space E representing the class of all locally integrable functions of exponential type on $\mathbb{R}_+$ with the property $|f(t)| \leq Me^{At}$ ($t \geq 0$) for some constants $M, A > 0$.

The Szász–Mirakjan operators associate to each function $f \in E$ the series

$$(S_n f)(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right) \qquad (x \geq 0),$$

where

$$p_{n,\nu}(x) = e^{-nx} \frac{(nx)^\nu}{\nu!} \qquad (\nu = 0, 1, 2, \ldots).$$

Results for approximating L_p functions may have potential applications in machine learning generalization analysis. Moreover, problems for approximating functions in higher dimensions are more and more important in various applications of big data.

2. **Preliminaries.** Butzer [12, Section 3] published, for the first time, an integral generalization in the Kantorovich sense of the genuine Szász operators. Using the P_u^f notation for Szász operators, Butzer considered a function f Lebesgue integrable over the interval $0 \leq x \leq R$ for every $R > 0$ and $F(x) = \mathcal{O}(x^k)$ as $x \to \infty$, for some $k > 0$, where

$$F(x) = \int_0^x f(s) \mathrm{d}s.$$

The new operators have the following form

$$W_u^f(x) = ue^{-ux} \sum_{\nu=0}^{\infty} \left(\int_{\nu/u}^{(\nu+1)/u} f(s) \mathrm{d}s \right) \frac{(ux)^\nu}{\nu!}, \ u > 0,$$

which can be written as

$$W_u^f(x) = \int_0^\infty K_u(x; s) f(s) \mathrm{d}s,$$

where

$$K_u(x;s) = ue^{-ux}\frac{(ux)^\nu}{\nu!}$$

for $\nu/u < s \le (\nu+1)/u$, $\nu = 0,1,2,\ldots$; $K_u(x;0) = 0$, $x \in \mathbb{R}_+$.

The main approximation property established by Butzer [12, Theorem 2] is read as follows: if $f(x) \in L(0,\infty)$, then

$$\lim_{u\to\infty} W_u^f(x) = f(x)$$

at every point where $f(x) = F'(x)$, i.e. almost everywhere in $(0,\infty)$.

These integral operators were studied in different functional spaces, for example in $L^p(0,\infty)$, $1 < p < \infty$, spaces the investigation has been achieved by Totik [33].

It is worth emphasizing that direct and converse results for exponential-type operators (including Szász operators) and their Kantorovich analogue are analyzed in the monograph [16, Section 9.3].

The Kantorovich-type variant W_u^f of Szász operators can be rewritten in the form

$$(K_n f)(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}(x) \int_0^1 f\left(\frac{\nu+t}{n}\right) dt \quad (x \ge 0),$$

where $p_{n,\nu}(x) = e^{-nx}(nx)^\nu/\nu!$ $(\nu = 0,1,\ldots)$.

Recently (2022), Sabancigil, Kara, and Mahmudov [29] introduced, for $\ell \in \mathbb{N}$, an ℓ-th order Kantorovich-type Szász–Mirakjan operator

$$\left(K_n^\ell f\right)(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}(x) \int_0^1 \cdots \int_0^1 f\left(\frac{\nu+t_1+\cdots+t_\ell}{n+\ell}\right) \mathrm{d}t_1\cdots\mathrm{d}t_\ell \qquad (x \ge 0).$$

Since $K_n^{\ell=1}$ does not coincide with K_n we define, for $\alpha \ge 0$,

$$\left(K_n^{\ell,\alpha} f\right)(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}(x) \int_0^1 \cdots \int_0^1 f\left(\frac{\nu+t_1+\cdots+t_\ell}{n+\alpha}\right) \mathrm{d}t_1\cdots\mathrm{d}t_\ell \qquad (x \ge 0),$$

such that $K_n^\ell = K_n^{\ell,\ell}$. A natural generalization of the operators K_n are $K_n^{\ell,0}$ and $K_n^{\ell,\ell-1}$. Note that in the special case $\ell = 1$, these operators both reduce to the Kantorovich-type variant $K_n^{1,0} = K_n$.

It is easy to see that $K_n^{\ell,\alpha}$ preserves constant functions, i.e., $K_n^{\ell,\alpha} e_0 = e_0$, where e_r denote the monomials given by $e_r(x) = x^r$ $(r = 0,1,2,\ldots)$. Since

$$\int_0^1 \cdots \int_0^1 \frac{\nu+t_1+\cdots+t_\ell}{n+\alpha}\mathrm{d}t_1\cdots\mathrm{d}t_\ell = \frac{\nu+\ell/2}{n+\alpha},$$

we obtain

$$K_n^{\ell,\alpha} e_1 = \frac{n}{n+\alpha}\left(S_n e_1 + \frac{\ell}{2} S_n e_0\right) = \frac{n}{n+\alpha}\left(e_1 + \frac{\ell}{2n} e_0\right) \to e_1 \qquad (n\to\infty).$$

A direct computation yields

$$K_n^{\ell,\alpha} e_2 = \left(\frac{n}{n+\alpha}\right)^2 e_2 + \frac{n}{(n+\alpha)^2}(\ell+1)\,nx + \frac{3\ell^2+\ell}{12\,(n+\alpha)^2} \to e_2 \qquad (n\to\infty).$$

Thus, the Popoviciu–Bohman–Korovkin (see [28, 9, 25]) theorem implies

$$\lim_{n\to\infty}\left(K_n^{\ell,\alpha} f\right)(x) = f(x),$$

for all bounded continuous functions $f \in E$. Later on, we shall see that the approximation property is valid for all $f \in E$.

The power and, at the same time, the simpleness of the Popoviciu–Bohman–Korovkin criterion turned it into the main tool for studying linear positive approximation processes. Unfortunately, for operators acting on spaces of functions defined on unbounded intervals, the uniform convergence is not guaranteed by this theorem. This is true even if we restrict at a compact interval the functions obtained as images. An example in this direction was given by Ditzian [15]. To obtain the extension of this theorem in the case of non-compact intervals, additional conditions are required. We highlight that these conditions are not uniquely determined, they are varied depending on the considered function spaces. We propose a very short foray into this domain of research in order to point out some achievements. A pioneering activity in this direction was carried out by Boyanov and Veselinov [10]. In their approach they took into account functions that have a finite limit at infinity. We also note the significant results of Gadžhiev [19] who studied this subject in weighted spaces.

The approximation property of Szász transform $S_n f$ was settled in polynomial weight spaces by Becker [7] and in exponential weight spaces by Becker, Kucharski and Nessel [8]. To obtain results at the uniform convergence of the $S_n f$, $n \geq 1$, operators, Totik worked in the $C_B(\mathbb{R}_+)$ space involving modulus of continuity, see [34, Theorems 1-3]. By using the representation of the operators in terms of appropriate stochastic processes, J. de la Cal and Cárcamo [14] obtained new results on uniform approximation for families of operators of probabilistic type over non-compact interval. The same concerns to establish sufficient conditions of uniform convergence appeared, for example, in [11] and [4]. Among the particular cases presented in these two papers are the Szász operators.

We also mention that in more abstract context of metric spaces, Altomare [5] presented general results of Korovkin-type establishing conditions which ensure the uniform convergence.

In this paper, we derive the complete asymptotic expansion for the sequence of operators $K_n^{\ell,\alpha}$ in the form

$$\left(K_n^{\ell,\alpha} f\right)(x) \sim f(x) + \sum_{k=1}^{\infty} a_k\left(f, \ell, \alpha; x\right)\ n^{-k} \qquad (n \to \infty), \tag{1}$$

provided that f admits derivatives of sufficiently high order at $x > 0$. Formula (1) means that, for all $q = 1, 2, \ldots$, there holds

$$\left(K_n^{\ell,\alpha} f\right)(x) = f(x) + \sum_{k=1}^{q} a_k\left(f, \ell, \alpha; x\right)\ n^{-k} + o(n^{-q}) \qquad (n \to \infty).$$

The coefficients $a_k\left(f, \ell, \alpha; x\right)$, independent of n, will be given in an explicit form. It turns out that Stirling numbers of the second kind play an important role. As a special case we obtain the complete asymptotic expansion for the sequence of Szász–Mirakjan–Kantorovich operators K_n.

Asymptotic expansions for the Bernstein-Kantorovich operators are derived in [1]. Asymptotic expansions for the Szász–Mirakjan operators and their generalizations can be found in [2].

3. **Asymptotic expansion.** For $q \in \mathbb{N}$ and $x \in (0, \infty)$, let $K[q; x]$ be the class of all functions $f \in E$ which are q times differentiable at x. The following theorem presents as our main result the complete asymptotic expansion for the operators $K_n^{\ell,\alpha}$.

Theorem 3.1. *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in K[2q;x]$, the operators $K_n^{\ell,\alpha}$ possess the asymptotic expansion*

$$\left(K_n^{\ell,\alpha} f\right)(x) = f(x) + \sum_{k=1}^{q} a_k(f,\ell,\alpha;x)\ n^{-k} + \mathrm{o}(n^{-q}) \qquad (n\to\infty) \tag{2}$$

with the coefficients

$$a_k(f,\ell,\alpha;x) = \sum_{s=1}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{m=0}^{k} x^{s-m}(-\alpha)^{k-m}\, T_\ell(s,k,m) \qquad (k=1,2,\ldots), \tag{3}$$

where the numbers $T_\ell(s,k,m)$ are defined by

$$\begin{aligned} & T_\ell(s,k,m) \\ = \ & \sum_{j=0}^{k} \binom{s}{j}\binom{j+\ell}{\ell}^{-1} \left\{ {\ell+j \atop \ell} \right\} \sum_{r=j}^{s} (-1)^{s-r} \binom{s-j}{r-j} \left\{ {r-j \atop r-m} \right\} \binom{k-m+r-1}{k-m}. \end{aligned} \tag{4}$$

Here and in the following, the quantities $\left\{ {\ell \atop j} \right\}$ denote the Stirling numbers of the second kind defined through

$$z^{\ell} = \sum_{j=0}^{\ell} \left\{ {\ell \atop j} \right\} z^{\underline{j}} \qquad (\ell = 0,1,2,\ldots), \tag{5}$$

where $z^{\underline{0}} = 1$, $z^{\underline{j}} = z(z-1)\cdots(z-j+1)$, $j\in\mathbb{N}$, are the falling factorials (we follow the convention to define $\left\{ {\ell \atop j} \right\} = 0$, for all negative integers j).

Remark 3.2. If $f \in \bigcap_{q=1}^{\infty} K[q;x]$, the operators $K_n^{\ell,\alpha}$ possess the complete asymptotic expansion

$$\left(K_n^{\ell,\alpha} f\right)(x) = f(x) + \sum_{k=1}^{\infty} a_k(f,\ell,\alpha;x)\ n^{-k} \qquad (n\to\infty),$$

where the coefficients $a_k(f,\ell,\alpha;x)$ are as defined in (3).

Remark 3.3. For the convenience of the reader, we list the explicit expressions for the initial coefficients $a_k(f,\ell,\alpha;x)$:

$$\begin{aligned} a_1(f,\ell,\alpha;x) &= \frac{1}{2}(\ell-2\alpha x) f'(x) + \frac{x}{2} f''(x), \\ a_2(f,\ell,\alpha;x) &= -\frac{1}{2}\alpha(\ell-2\alpha x) f'(x) + \frac{\left(\ell+3\ell^2\right) - 12(\ell+2)\alpha x + 12\alpha^2 x^2}{24} f''(x) \\ &\quad + \frac{x(2+3\ell-6\alpha x)}{12} f^{(3)}(x) + \frac{x^2}{8} f^{(4)}(x), \end{aligned}$$

$$\begin{aligned} & a_3(f,\ell,\alpha;x) \\ = \ & \frac{1}{2}\alpha^2(\ell-2\alpha x) f'(x) \\ & -\frac{\alpha}{12}\left(\ell+3\ell^2-(12\ell+18)\alpha x+12\alpha^2x^2\right) f''(x) \\ & +\frac{1}{48}\left(\ell^2+\ell^3-\left(24+38\ell+6\ell^2\right)\alpha x+(72+12\ell)\alpha^2x^2-8\alpha^3x^3\right) f^{(3)}(x) \\ & +\frac{x}{48}\left(2+5\ell+3\ell^2-(32+12\ell)\alpha x+12\alpha^2x^2\right) f^{(4)}(x) \end{aligned}$$

$$+\frac{x^2}{48}\left(4+3\ell-6\alpha x\right)f^{(5)}\left(x\right)+\frac{x^3}{48}f^{(6)}\left(x\right).$$

In the case $q=1$, an immediate consequence of Theorem 3.1 is the following Voronovskaja-type formula.

Corollary 3.4. *Let $x\in(0,\infty)$. For each function $f\in K[2;x]$, the operators $K_n^{\ell,\alpha}$ satisfy the asymptotic relation*

$$\lim_{n\to\infty}n\left(\left(K_n^{\ell,\alpha}f\right)(x)-f(x)\right)=\frac{1}{2}\left(\ell-2\alpha x\right)f'(x)+\frac{x}{2}f''(x).$$

In the case $\alpha=\ell$, we have, for $f\in K[2;x]$, the Voronovskaja-type formula

$$\lim_{n\to\infty}n\left(\left(K_n^{\ell,\ell}f\right)(x)-f(x)\right)=\frac{\ell}{2}\left(1-2x\right)f'(x)+\frac{x}{2}f''(x),$$

which was derived in [29, Theorem 10], for bounded functions having bounded first and second derivatives on $[0,\infty)$.

Corollary 3.5. *For each $f\in K[2;x]$, the sequence of operators $K_n^{\ell,0}$ satisfies the asymptotic relation*

$$\lim_{n\to\infty}n\left(\left(K_n^{\ell,0}f\right)(x)-f(x)\right)=\frac{\ell}{2}f'(x)+\frac{x}{2}f''(x).$$

In the special case $\alpha=0$ we obtain the following result for the Szász–Mirakjan-Kantorovich operators $K_n^{\ell,0}$.

Corollary 3.6. *Let $q\in\mathbb{N}$ and $x\in(0,\infty)$. For each function $f\in K[2q;x]$, the Szász–Mirakjan–Kantorovich operators possess the asymptotic expansion*

$$\left(K_n^{\ell,0}f\right)(x)=f(x)+\sum_{k=1}^{q}a_k\left(f,\ell,\alpha=0;x\right)n^{-k}+o\left(n^{-q}\right)\qquad(n\to\infty)$$

with the coefficients

$$a_k\left(f,\ell,\alpha=0;x\right)=\sum_{s=k}^{2k}\frac{f^{(s)}(x)}{s!}x^{s-k}T_\ell\left(s,k,k\right)\qquad(k=1,2,\ldots),$$

where

$$T_\ell\left(s,k,m=k\right)=\sum_{j=0}^{k}\binom{s}{j}\binom{j+\ell}{\ell}^{-1}\left\{\begin{matrix}\ell+j\\ \ell\end{matrix}\right\}\sum_{r=j}^{s}\left(-1\right)^{s-r}\binom{s-j}{r-j}\left\{\begin{matrix}r-j\\ r-k\end{matrix}\right\}.$$

We emphasize the fact that $a_k\left(f,\ell,\alpha=0;x\right)$ contains only derivatives $f^{(s)}(x)$ of orders $s=k,\ldots,2k$.

We give the series explicitly, for $q=3$:

$$\begin{aligned}&\left(K_n^{\ell,0}f\right)(x)\\ =\;&f(x)+\frac{\ell f'(x)+xf''(x)}{2n}+\frac{\left(\ell+3\ell^2\right)f''(x)+2x\left(2+3\ell\right)f^{(3)}(x)+3x^2f^{(4)}(x)}{24n^2}\\ &+\frac{1}{48n^3}\Big(\left(\ell^2+\ell^3\right)f^{(3)}(x)+x\left(2+5\ell+3\ell^2\right)f^{(4)}(x)\\ &+x^2\left(4+3\ell\right)f^{(5)}(x)+x^3f^{(6)}(x)\Big)+o\left(n^{-3}\right)\qquad\text{as } n\to\infty.\end{aligned}$$

4. **Auxiliary results and proofs.** In order to prove our main result, we shall need some auxiliary results. Recall that throughout the paper e_r denote the monomials $e_r(t) = t^r$ $(r = 0, 1, 2, \ldots)$ and, for each real x, we put $\psi_x = e_1 - xe_0$. The proof of Theorem 3.1 is based on several lemmas, which are gathered in this section.

We recall some known facts about Stirling numbers which will be useful in the sequel. The Stirling numbers of the second kind possess the representation

$$\left\{ {r \atop r-m} \right\} = \sum_{i=m}^{2m} \sigma_2(i, i-m) \binom{r}{i} = \sum_{i=0}^{m} \sigma_2(i+m, i) \binom{r}{i+m} \qquad (0 \le m \le r) \tag{6}$$

(see [13, page 226, Ex. 16]). The coefficients $\sigma_2(i, i-m)$, called associated Stirling numbers of the second kind, are independent of r. Furthermore, we make use of the following formula for iterated integrals (see [23, page 202, Eq. (4)]).

Lemma 4.1. *For $j \in \mathbb{N}$, and let $G \subseteq \mathbb{C}$ be a region such that $\{z \in \mathbb{C} \mid |z| \le \ell\} \subset G$. For functions g, analytic in G, it holds*

$$\int_0^1 \cdots \int_0^1 g(t_1 + \cdots + t_\ell) \,\mathrm{d}t_1 \cdots \mathrm{d}t_\ell = \sum_{j=0}^{\infty} \frac{\ell!}{(\ell+j)!} \left\{ {\ell+j \atop \ell} \right\} g^{(j)}(0).$$

Lemma 4.2. *For $r = 0, 1, 2, \ldots,$ the moments of the operators $K_n^{\ell,\alpha}$ have the representation*

$$K_n^{\ell,\alpha} e_r = \sum_{j=0}^{r} \binom{r}{j} \binom{j+\ell}{\ell}^{-1} \left\{ {\ell+j \atop \ell} \right\} \frac{n^{r-j}}{(n+\alpha)^r} S_n e_{r-j}.$$

More explicitly, it holds

$$\left(K_n^{\ell,\alpha} e_r\right)(x) = \left(\frac{n}{n+\alpha}\right)^r \sum_{k=0}^{r} \frac{x^{r-k}}{n^k} \sum_{j=0}^{k} \binom{r}{j} \binom{j+\ell}{\ell}^{-1} \left\{ {\ell+j \atop \ell} \right\} \left\{ {r-j \atop r-k} \right\}. \tag{7}$$

Proof. Applying Lemma 4.1 to the function $g(t) = f\left(\frac{\nu+t}{n+\alpha}\right)$ we obtain the formula

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 f\left(\frac{\nu + t_1 + \cdots + t_\ell}{n+\alpha}\right) \mathrm{d}t_1 \cdots \mathrm{d}t_\ell \\ &= \sum_{j=0}^{\infty} \frac{\ell!}{(\ell+j)!} \left\{ {\ell+j \atop \ell} \right\} \frac{1}{(n+\alpha)^j} f^{(j)}\left(\frac{\nu}{n+\alpha}\right). \end{aligned}$$

In particular, for $f = e_r$, we have

$$\begin{aligned} &\left(K_n^{\ell,\alpha} e_r\right)(x) \\ &= \frac{1}{(n+\alpha)^r} \sum_{\nu=0}^{\infty} p_{n,\nu}(x) \sum_{j=0}^{r} \frac{\ell!}{(\ell+j)!} \left\{ {\ell+j \atop \ell} \right\} r^{\underline{j}} \nu^{r-j} \\ &= \frac{1}{(n+\alpha)^r} \sum_{j=0}^{r} \binom{r}{j} \binom{j+\ell}{\ell}^{-1} \left\{ {\ell+j \atop \ell} \right\} \sum_{\nu=0}^{\infty} p_{n,\nu}(x) \nu^{r-j}. \end{aligned}$$

Taking advantage of Eq. (5), we obtain

$$\sum_{\nu=0}^{\infty} p_{n,\nu}(x) \nu^{r-j} = \sum_{i=0}^{r-j} \left\{ {r-j \atop i} \right\} \sum_{\nu=i}^{\infty} e^{-nx} \frac{(nx)^\nu}{\nu!} \nu^{\underline{i}} = \sum_{i=0}^{r-j} \left\{ {r-j \atop i} \right\} (nx)^i.$$

Hence,

$$\left(K_n^{\ell,\alpha}e_r\right)(x)=\frac{1}{(n+\alpha)^r}\sum_{j=0}^{r}\binom{r}{j}\binom{j+\ell}{\ell}^{-1}\left\{\begin{matrix}\ell+j\\ \ell\end{matrix}\right\}\sum_{i=0}^{r-j}\left\{\begin{matrix}r-j\\ r-j-i\end{matrix}\right\}(nx)^{r-j-i}.$$

Collecting all terms with $k=i+j$ we obtain Eq. (7). □

Lemma 4.3. *For $s=0,1,2,\ldots$, the central moments of the operators $K_n^{\ell,\alpha}$ have, for $n>\alpha$, the representation*

$$\left(K_n^{\ell,\alpha}\psi_x^s\right)(x)=\sum_{k=0}^{\infty}\frac{1}{n^k}\sum_{m=0}^{k}x^{s-m}(-\alpha)^{k-m}T_\ell(s,k,m). \tag{8}$$

Proof. Let $n>\alpha$. Using the expansion

$$\left(\frac{n}{n+\alpha}\right)^r=\frac{1}{(1+\alpha/n)^r}=\sum_{\rho=0}^{\infty}\binom{\rho+r-1}{\rho}\left(\frac{-\alpha}{n}\right)^\rho,$$

Lemma 4.2 leads to

$$\begin{aligned}\left(K_n^{\ell,\alpha}e_r\right)(x)&=\left(\frac{n}{n+\alpha}\right)^r\sum_{m=0}^{r}\frac{x^{r-m}}{n^m}A_\ell(m,r)\\&=\sum_{k=0}^{\infty}\frac{1}{n^k}\sum_{m=0}^{k}x^{r-m}(-\alpha)^{k-m}A_\ell(m,r)\binom{k-m+r-1}{k-m},\end{aligned}$$

where

$$A_\ell(m,r)=\sum_{j=0}^{k}\binom{r}{j}\binom{j+\ell}{\ell}^{-1}\left\{\begin{matrix}\ell+j\\ \ell\end{matrix}\right\}\left\{\begin{matrix}r-j\\ r-m\end{matrix}\right\}.$$

Application of the binomial formula yields for the central moments

$$\begin{aligned}\left(K_n^{\ell,\alpha}\psi_x^s\right)(x)&=\sum_{r=0}^{s}(-x)^{s-r}\binom{s}{r}\left(K_n^{\ell,\alpha}e_r\right)(x)\\&=\sum_{k=0}^{\infty}\frac{1}{n^k}\sum_{m=0}^{k}x^{s-m}(-\alpha)^{k-m}T_\ell(s,k,m),\end{aligned}$$

where

$$T_\ell(s,k,m)=\sum_{r=0}^{s}(-1)^{s-r}\binom{s}{r}A_\ell(m,r)\binom{k-m+r-1}{k-m}.$$

Using the identity $\binom{s}{r}\binom{r}{j}=\binom{s}{j}\binom{s-j}{r-j}$ and reversing the order of summations completes the proof of Lemma 4.3. □

Remark 4.4. The second central moment of the operators $K_n^{\ell,\alpha}$ is given by

$$\left(K_n^{\ell,\alpha}\psi_x^2\right)(x)=\frac{12nx+\ell+3\ell^2-12\ell\alpha x+12\alpha^2x^2}{12(n+\alpha)^2}.$$

Hence, the second central moment satisfies $\left(K_n^{\ell,\alpha}\psi_x^2\right)(x)=O(1/n)$ as $n\to\infty$. More generally, we show

Lemma 4.5. *For $s=0,1,2,\ldots$, the central moments of the operators $K_n^{\ell,\alpha}$ satisfy the asymptotic relation*

$$\left(K_n^{\ell,\alpha}\psi_x^s\right)(x)=O\left(n^{-\lfloor(s+1)/2\rfloor}\right)\qquad(n\to\infty).$$

A direct consequence is the representation

$$\left(K_n^{\ell,\alpha}\psi_x^s\right)(x)=\sum_{k=\lfloor (s+1)/2\rfloor}^{\infty}\frac{1}{n^k}\sum_{m=0}^{k}x^{s-m}\left(-\alpha\right)^{k-m}T_\ell\left(s,k,m\right),$$

provided that $n>\alpha$.

Proof. We have to prove that $\left(K_n^{\ell,\alpha}\psi_x^s\right)(x)=O\left(n^{-\lfloor (s+1)/2\rfloor}\right)$ as $n\to\infty$. This is a consequence of $K_n^{\ell,\alpha}e_0=e_0$ in the case $s=0$. Now suppose that $s>0$. By Lemma 4.3, we have to show that $T_\ell\left(s,k,m\right)=0$, for $s>2k$ and $0\le m\le k$. Since by Eq. (4),

$$T_\ell\left(s,k,m\right)=\sum_{j=0}^{k}\binom{s}{j}\binom{j+\ell}{\ell}^{-1}\left\{{\ell+j \atop \ell}\right\}V\left(s,k,j,m\right)$$

with

$$V\left(s,k,j,m\right):=\sum_{r=0}^{s-j}\left(-1\right)^{s-j-r}\binom{s-j}{r}\left\{{r \atop r+j-m}\right\}\binom{k-m+r+j-1}{k-m}$$

it is sufficient to prove that $V\left(s,k,j,m\right)=0$, for any integers j,m with $0\le j\le m\le k$. By Eq. (6), we have

$$\left\{{r \atop r-(m-j)}\right\}=\sum_{i=m-j}^{2(m-j)}\sigma_2\left(i,i-(m-j)\right)\binom{r}{i}.$$

Since $\binom{s-j}{r}\binom{r}{i}=\binom{s-j}{i}\binom{s-j-i}{r-i}$ we obtain

$$\begin{aligned}V\left(s,k,j,m\right)&=\sum_{i=m-j}^{2(m-j)}\sigma_2\left(i,i-(m-j)\right)\binom{s-j}{i}\\&\quad\times\sum_{r=i}^{s-j}\left(-1\right)^{s-j-r}\binom{s-j-i}{r-i}\binom{k-m+r+j-1}{k-m}.\end{aligned}$$

The inner sum

$$\sum_{r=0}^{s-j-i}\left(-1\right)^{s-j-i-r}\binom{s-j-i}{r}\binom{k-m+r+i+j-1}{k-m}$$

is equal to zero, if $\binom{k-m+r+i+j-1}{k-m}$ is a polynomial in r of degree less than $s-j-i$, i.e., if $k-m<s-j-i$. Since $i\le 2\left(m-j\right)$, this inequality is fulfilled if $k<s+m-j-2\left(m-j\right)=s-\left(m-j\right)$. Since $m-j\le k$, we infer that $V\left(s,k,j,m\right)=0$ if $k<s-k$, i.e., $2k<s$. This implies that $T_\ell\left(s,k,m\right)=0$, for $s>2k$. □

In order to extend our main result from bounded functions to functions of exponential growth, we need a localization result.

For the proof of Theorem 3.1, we apply the following localization theorem.

Proposition 4.6. *Let $x>0$. If $f\in E$ vanishes in a neighborhood $\left(x-\delta,x+\delta\right)\cap[0,+\infty)$ of x, then there exists a positive constant c such that*

$$\left(K_n^{\ell,\alpha}f\right)(x)=O\left(\exp\left(-cn\right)\right)\qquad\left(n\to\infty\right).$$

In order to derive Theorem 3.1, a general approximation theorem due to Sikkema [31, Theorem 3] will be applied. To this end, we need some notation. Let I be a real interval and $x \in I$.

An inspection of the proof of Sikkema's result reveals that it can be stated in the following form which is more appropriate for our purposes.

Lemma 4.7. *Let $q \in \mathbb{N}$ and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators, $L_n : H^{(2q)}(x) \to C[c,d]$, $x \in [c,d]$. Suppose that the operators L_n apply to ψ_x^{2q+1} and to ψ_x^{2q+2}. Then the condition*

$$(L_n \psi_x^s)(x) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n \to \infty), \qquad \text{for } s = 0, 1, \ldots, 2q+2,$$

implies, for each function $f \in H^{(2q)}(x)$, the asymptotic relation

$$(L_n f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (L_n \psi_x^s)(x) + o\left(n^{-q}\right) \qquad (n \to \infty).$$

In the application used in the proof of Theorem 3.1, we restrict $H^{(s)}(x)$ to consist only of locally integrable functions. We proceed with the proof of the localization result.

Proof of Proposition 4.6. For $f \in E$, set

$$\Phi_{n,\nu}(f) = \int_0^1 \cdots \int_0^1 f\left(\frac{\nu + t_1 + \cdots + t_\ell}{n+\alpha}\right) \mathrm{d}t_1 \cdots \mathrm{d}t_\ell.$$

Observe that

$$\frac{n}{n+\alpha}\frac{\nu}{n} \le \frac{\nu + t_1 + \cdots + t_\ell}{n+\alpha} \le \frac{n}{n+\alpha}\frac{\nu}{n} + \frac{\ell}{n+\alpha}.$$

From $|f(t)| \le M e^{At}$ $(t \ge 0)$ we obtain the estimate

$$|\Phi_{n,\nu}(f)| \le M \exp\left(A\frac{\ell}{n+\alpha}\right)\exp\left(A\frac{\nu}{n}\right) \qquad (\nu \in \mathbb{N}_0).$$

Hence, $K_n^{\ell,\alpha} f$ is well-defined. Put $U_\delta(x) = (x-\delta, x+\delta) \cap [0,+\infty)$. The condition $f(t) = 0$, for $t \in U_\delta(x)$, implies that $\Phi_{n,\nu}(f) = 0$ if $\frac{\nu}{n} \in U_{\delta/2}(x)$, for sufficiently large values of n. Hence, the result follows from the localization theorem for the classical Szász–Mirakjan S_n operators. $\square$

Proof of Theorem 3.1. Let $x > 0$ and put $U_r(x) = (x-r, x+r) \cap [0,+\infty)$, for $r > 0$. Let $\delta > 0$ be given. Suppose that $f^{(2q)}(x)$ exists. Choose a function $\varphi \in C^\infty([0,+\infty))$ with $\varphi(x) = 1$ on $U_\delta(x)$ and $\varphi(x) = 0$ on $[0,+\infty) \setminus U_{2\delta}(x)$. Put $\widetilde{f} = \varphi f$. Then we have $\widetilde{f} \equiv f$ on $U_\delta(x)$ which implies $\widetilde{f}^{(s)}(x) = f^{(s)}(x)$, for $s = 0, \ldots, 2q$, and $\widetilde{f} \equiv 0$ on $[0,+\infty) \setminus U_{2\delta}(x)$. By the localization theorem (Proposition 4.6), $\left(K_n^{\ell,\alpha}\left(f - \widetilde{f}\right)\right)(x)$ decays exponentially fast as $n \to \infty$. Consequently, $\widetilde{f}$ and f possess the same asymptotic expansion of the form (1). Therefore, without loss of generality, we can assume that $f \equiv 0$ on $[0,+\infty) \setminus U_{2\delta}(x)$. By Lemma 4.3, we have $\left(K_n^{\ell,\alpha}\psi_x^{2s}\right)(x) = O(n^{-s})$ as $n \to \infty$. Under these conditions, Lemma 4.7 implies that

$$\left(K_n^{\ell,\alpha} f\right)(x) = f(x) + \sum_{s=1}^{2q} \frac{f^{(s)}(x)}{s!}\left(K_n^{\ell,\alpha}\psi_x^s\right)(x) + o\left(n^{-q}\right) \qquad (n \to \infty).$$

By Lemma 4.3 and Lemma 4.5, we obtain

$$\begin{aligned}&\sum_{s=1}^{2q}\frac{f^{(s)}(x)}{s!}\left(K_n^{\ell,\alpha}\psi_x^s\right)(x)\\ =\ &\sum_{s=1}^{2q}\frac{f^{(s)}(x)}{s!}\sum_{k=\lfloor (s+1)/2\rfloor}^{\infty}\frac{1}{n^k}\sum_{m=0}^{k}x^{s-m}(-\alpha)^{k-m}T_\ell(s,k,m).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}&\left(K_n^{\ell,\alpha}f\right)(x)\\ &=f(x)+\sum_{k=1}^{q}\frac{1}{n^k}\sum_{s=1}^{2k}\frac{f^{(s)}(x)}{s!}\sum_{m=0}^{k}x^{s-m}(-\alpha)^{k-m}T_\ell(s,k,m)+o\left(n^{-q}\right)\end{aligned}$$

as $n\to\infty$. This is the desired expansion (2) with coefficients $T_\ell(s,k,m)$ as defined in Eq. (4). □

Proof of Corollary 3.6. The proof runs along the lines of the proof of Theorem 3.1. The only difference is the fact that, in the special case $\alpha=0$, the central moments (8) can be represented by a finite sum, viz.,

$$\left(K_n^{\ell,0}\psi_x^s\right)(x)=\sum_{k=0}^{s}\frac{1}{n^k}x^{s-k}T_\ell(s,k,k),$$

such that $s\geq k$. □

Final Remark. One reason why we chose the study of a Szász-type sequence is that Vijay Gupta approached this transform in numerous papers that include statistical convergence, q-calculus, and different ingenious generalizations involving, for example, functions belonging to spaces with polynomial or exponential weights, functions with bounded variations, smooth functions. His track record includes dozens of papers with Szász-type operators. Instead of citing a part of these papers, we prefer to refer to his books that include a complete and unified presentation of the operators in question. The references in the monographs [20] and [21] prove our statement.

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