CONFERENCE ON ANALYSIS, FUNCTIONAL EQUATIONS,
APPROXIMATION AND CONVEXITY
IN HONOUR OF PROFESSOR ELENA POPOVICIU
CLUJ-NAPOCA, OCTOBER 15-16, 1999; pp. 257–263

Behavior Properties and Ordinary Differential Equations

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(CLUJ-NAPOCA)

ABSTRACT: The goal of this paper is to discuss the implications of the behavior properties from classical analysis (positivity, monotonicity, convexity, convexity of high order) [PE72], [PT44], [Pr85], into the qualitative theory of ordinary differential equations. We survey our own results concerning this subject in connection with other contributions in literature.

1 Positivity and Linear Equations

Let us consider the differential operator

$$L_0 u = -(pu')' \quad (p \in C^1[0,1],\ p \geq c > 0 \text{ on } [0,1])$$

and the two-point boundary value problem

$$\begin{cases} Lu \equiv L_0 u + q(x)\, u = f(x), & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

$(q,\ f \in C[0,1])$.

The deep implication of the positivity into the qualitative analysis of problem (1) is shown by the following well known theorem.

Theorem 1 *The following conditions are equivalent:*

(i) For each $f \in C[0,1]$, *(1) has a unique solution* $u = L^{-1} f \in C^2[0,1]$ *and*

$$L^{-1} f \geq 0 \text{ if } f \geq 0$$

(positivity of L^{-1}*), or equivalently*

$$Lu \geq 0 \text{ implies } u \geq 0$$

(weak maximum principle).

(ii) There exists the Green's function $G(x,y)$ *for (1) and*

$$G(x,y) > 0 \text{ for all } x,\ y \in (0,1)$$

(positivity of Green's function).

(iii) The first eigenvalue

$$\lambda_1(L) = \inf\left\{\int_0^1 \left(pu'^2 + qu^2\right)dx;\ u \in C_0^1,\ \|u\|_{L^2} = 1\right\}$$

(here $C_0^1 = \{u \in C^1[0,1];\ u(0) = u(1) = 0\}$) is positive, equivalently, there exists $\mu > 0$ (take $\mu = \lambda_1(L)$) such that for each $u \in C_0^1$, one has

$$\int_0^1 \left(pu'^2 + qu^2\right)dx \geq \mu \int_0^1 u^2 dx$$

(coercivity of L).

For proof and several generalizations see [Pro67]. A similar result [Prap2] holds for the operator

$$L_0 u = (-1)^m u^{(2m)} \quad (m \in \mathbf{N},\ m \geq 1)$$

and the focal boundary value

$$\begin{cases} Lu \equiv L_0 u + q(x)u = f(x), & x \in [0,1] \\ u^{(k)}(0) = 0, & 0 \leq k \leq m-1 \\ u^{(k)}(1) = 0, & m \leq k \leq 2m-1 \end{cases} \tag{2}$$

$(q, f \in C[0,1])$. Let $C_B^{2m-1} = \{u \in C^{2m-1}[0,1];\ u^{(k)}(0) = 0,\ 0 \leq k \leq m-1,\ u^{(k)}(1) = 0,\ m \leq k \leq 2m-1\}$. In this respect, in [RePr], we proved the following result:

Theorem 2 *1) The number*

$$\inf\left\{\int_0^1 \left(u^{(m)2} + qu^2\right)dx;\ u \in C_B^{2m-1},\ \|u\|_{L^2} = 1\right\}$$

is the smallest (first) eigenvalue $\lambda_1(L)$ of the problem

$$\begin{cases} (-1)^m u^{(2m)} + q(x)u = \lambda u, & x \in [0,1] \\ u^{(k)}(0) = 0, & 0 \leq k \leq m-1 \\ u^{(k)}(1) = 0, & m \leq k \leq 2m-1. \end{cases}$$

2) If $\lambda_1(L) > 0$, then for each $f \in C[0,1]$, (2) has a unique solution which is nonnegative if $f \geq 0$.

2 Monotonicity and Nonlinear Equations

Monotonicity is for nonlinear equations what is positivity for the linear ones. To be more explicit, let us first deal with an abstract example.

Let $(X, |.|)$ be a Banach space and $K \subset X$ a *cone* (i.e., a closed set with $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \in \mathbf{R}_+$ and $K \cap (-K) = \emptyset$). Let $\leq$ be the order relation on X given by $x \leq y$ iff $y - x \in K$. For every two elements $x, y \in X$ with $x \leq y$, we define the interval $[x, y]$ by $[x, y] = \{z \in X;\ x \leq z \leq y\}$.

The cone K is said to be *normal* if whenever

$$0 \leq x_n \leq y_n,\ n = 1, 2, \ldots \text{ and } y_n \to 0 \text{ as } n \to \infty,$$

we have $x_n \to 0$ as $n \to \infty$ too. The cone K is said to be *regular* if every increasing sequence which is bounded from above is already convergent. Notice any regular cone is normal. For example, in $C[0,1]$, the positive cone (of all nonnegative functions on $[0,1]$) is normal but not regular. In $L^p(0,1)$, $1 \leq p < \infty$, the positive cone is regular by Beppo-Levi's theorem. The following theorem is known as the *monotone iterations principle* (see [De85], [Kr64], or [La85]).

Theorem 3 *Let X be a Banach space, $K \subset X$ a cone, $\underline{x}_0, \bar{x}_0 \in X$ with $\underline{x}_0 \leq \bar{x}_0$ and let $A : [\underline{x}_0, \bar{x}_0] \to [\underline{x}_0, \bar{x}_0]$ nondecreasing in the sense that $\underline{x}_0 \leq x \leq y \leq \bar{x}_0$ implies $A(x) \leq A(y)$. Let the sequences $(\underline{x}_n)$ and $(\bar{x}_n)$ be given by*

$$\underline{x}_n = A(\underline{x}_{n-1}),\quad \bar{x}_n = A(\bar{x}_{n-1}),\quad n = 1, 2, \ldots.$$

Suppose that one of the following two conditions is satisfied:

(i) K is regular and A is continuous.

(ii) K is normal, A is continuous with $\overline{A([\underline{x}_0, \bar{x}_0])}$ compact in X.

Then $(\underline{x}_n)$, $(\bar{x}_n)$ converge to some $\underline{x}$ and $\bar{x}$, respectively, $A(\underline{x}) = \underline{x}$, $A(\bar{x}) = \bar{x}$ and

$$\underline{x}_0 \leq \underline{x}_1 \leq \ldots \leq \underline{x}_n \leq \ldots \leq \underline{x} \leq$$
$$\leq \bar{x} \leq \ldots \leq \bar{x}_n \leq \ldots \leq \bar{x}_1 \leq \bar{x}_0.$$

Moreover, $\underline{x}$ (respectively, $\bar{x}$) is the minimal (respectively, maximal) fixed point of A.

Notice that if there exists a $\gamma > 0$ with $0 < \underline{x}_0 \leq \gamma\bar{x} \leq \underline{x}$ and A satisfies

$$\gamma^\alpha A(x) \leq A(\gamma x)$$

for all $\gamma \in (0,1)$, $x \in [\underline{x}_0, \bar{x}_0]$ with $\gamma x \in [\underline{x}_0, \bar{x}_0]$, and some $\alpha < 1$, then $\underline{x} = \bar{x}$. Indeed, let γ_0 is the biggest $\gamma > 0$ satisfying $\underline{x}_0 \leq \gamma\bar{x} \leq \underline{x}$. It is clear that $0 < \gamma_0 \leq 1$. If we suppose that $\gamma_0 < 1$, then $\gamma_0 < \gamma_0^\alpha$, and since A is nondecreasing we find

$$\underline{x}_0 \leq \gamma_0\bar{x} \leq \gamma_0^\alpha\bar{x} = \gamma_0^\alpha A(\bar{x}) \leq A(\gamma_0\bar{x}) \leq A(\underline{x}) = \underline{x},$$

which contradicts the maximality of γ_0. Thus $\gamma_0 = 1$, so $\underline{x} = \bar{x}$.

In [Pr96], we have given an analogue result for nonincreasing operators with respect to a normal cone. Here is a version of that result for both normal and regular cases.

Theorem 4 *Suppose all the assumptions in Theorem 3 hold where this time, A is nonincreasing. Then the sequences $(\underline{x}_{2n})$, $(\bar{x}_{2n+1})$ converge to some $\underline{x}$ and $(\underline{x}_{2n+1})$, $(\bar{x}_{2n})$ converge to some $\bar{x}$, $A(\underline{x}) = \bar{x}$, $A(\bar{x}) = \underline{x}$ and*

$$\underline{x}_0 \leq \bar{x}_1 \leq \underline{x}_2 \leq \bar{x}_3 \leq \underline{x}_4 \leq \ldots \leq \underline{x}_{2n} \leq \bar{x}_{2n+1} \leq \ldots \leq \underline{x} \leq$$
$$\leq \bar{x} \leq \ldots \leq \underline{x}_{2n+1} \leq \bar{x}_{2n} \leq \ldots \leq \bar{x}_2 \leq \underline{x}_1 \leq \bar{x}_0.$$

If in addition, there exists a $\gamma > 0$ with $0 < \underline{x}_0 \leq \gamma\bar{x} \leq \underline{x}$ and A satisfies

$$\gamma^\alpha A(x) \geq A(\gamma x)$$

for all $\gamma \in (0,1)$, $x \in [\underline{x}_0, \bar{x}_0]$ with $\gamma x \in [\underline{x}_0, \bar{x}_0]$, and some $\alpha > -1$, then $\underline{x} = \bar{x}$.

For an example, consider the nonlinear integral equation

$$u(t) = \int_0^1 f(t, s, u(s))\, ds, \quad 0 \le t \le 1. \tag{3}$$

Theorem 5 *Let $f : [0,1]^2 \times [a,b] \to \mathbf{R}$ be continuous. Suppose that $0 < a < b$, $f(t,s,.)$ is nondecreasing on $[a,b]$ for each $(t,s) \in [0,1]^2$,*

$$a \le \int_0^1 f(t,s,a)\, ds \quad \text{and} \quad \int_0^1 f(t,s,b)\, ds \le b \quad \text{on } [0,1].$$

Then, (3) has a minimal solution $\underline{u}$ and a maximal solution $\bar{u}$ in $[a,b]$, $a \le \underline{u}(t) \le \bar{u}(t) \le b$ and

$$\underline{u}_n(t) = \int_0^1 f(t, s, \underline{u}_{n-1}(s))\, ds, \quad n = 1, 2, \ldots$$

is nondecreasing and uniformly convergent to $\underline{u}(t)$; and

$$\bar{u}_n(t) = \int_0^1 f(t, s, \bar{u}_{n-1}(s))\, ds, \quad n = 1, 2, \ldots$$

is nonincreasing and uniformly convergent to $\bar{u}(t)$. If in addition, there exists an $\alpha < 1$ such that

$$f(t, s, \gamma u) \ge \gamma^\alpha f(t, s, u)$$

for all $t, s \in [0,1]$, $u \in [a,b]$ and $\gamma \in (0,1)$ with $\gamma u \in [a,b]$, then $\underline{u} = \bar{u}$ is the unique solution of (3).

Theorem 6 *Let $f : [0,1]^2 \times [a,b] \to \mathbf{R}$ be continuous. Suppose that $0 < a < b$, $f(t,s,.)$ is nonincreasing on $[a,b]$ for each $(t,s) \in [0,1]^2$,*

$$a \le \int_0^1 f(t,s,b)\, ds \quad \text{and} \quad \int_0^1 f(t,s,a)\, ds \le b \quad \text{on } [0,1].$$

In addition suppose that there exists an $\alpha > -1$ such that

$$f(t, s, \gamma u) \le \gamma^\alpha f(t, s, u)$$

for all $t, s \in [0,1]$, $u \in [a,b]$ and $\gamma \in (0,1)$ with $\gamma u \in [a,b]$. Then (3) has a unique solution $u \in C([0,1]; [a,b])$ which can be bilaterally approximated by monotone iterations.

For other applications see [Pr94] and [Pr95].

3 Convexity and Quadratic Approximation of Solutions

The approximate solutions given by the monotone iterative methods described in the previous section converge to the solution only linearly. A quadratic convergence is guaranteed by the so called *quasiliniarization* method provided that the nonlinear term of the equation is convex or difference of two convex functions [Be65], [La95].

In [Prap1], we applied the quasiliniarization method to the following class of problems of interest in epidemics:

$$\begin{cases} x'(t) = f(t, x(t)) + g(t-\tau, \widetilde{x}(t-\tau)), & 0 < t \le T \\ x(0) = \phi_0, \end{cases} \tag{4}$$

where $\widetilde{x}(t) = x(t)$ for $t \in (0,T]$, $\widetilde{x}(t) = \phi(t)$ if $t \in [-\tau, 0]$, τ being a positive number and ϕ a given function on $[-\tau, 0]$.

The main result in [Prap1] is the following theorem:

Theorem 7 *Suppose $\phi \in C[-\tau,0]$, $u_0, v_0 \in C^1[0,T]$, $u_0(0) = v_0(0) = \phi(0)$ and $u_0(t) < v_0(t)$ on $[0,T]$. Let $\Omega = \{(t,x);\ t \in (0,T],\ u_0(t) < x < v_0(t)$ on $(0,T]\}$ and let $\widetilde{\Omega} = \bar{\Omega} \cup \{(t, \phi(t));\ -\tau \le t < 0\}$. In addition assume that*

(h1) for $0 < t \le T$, we have

$$u_0'(t) \le f(t, x(t)) + g(t-\tau, \widetilde{u}_0(t-\tau)),$$
$$v_0'(t) \ge f(t, v_0(t)) + g(t-\tau, \widetilde{v}_0(t-\tau));$$

(h2) $f \in C(\bar{\Omega})$, $g \in C(\widetilde{\Omega})$, the derivatives f_x, f_{xx}, g_x, g_{xx} exist and are continuous on $\bar{\Omega}$, and satisfy

$$f_{xx} \ge 0, \ g_x \ge 0 \ \text{and} \ g_{xx} \le 0 \ \text{on } \bar{\Omega}.$$

Then there exist the sequences (u_n) nondecreasing and (v_n) nonincreasing which converge uniformly on $[0,T]$ to the unique solution $x \in C^1[0,T]$ of (4) satisfying $u_0(t) \le x(t) \le v_0(t)$ on $[0,T]$, and the convergence is quadratic.

Notice that a bilateral monotone approximation of the solution with an order of convergence higher than quadratic is possible in case that the nonlinear term of the equation is a smooth function, nonconcave of an order higher than one (see [De98]). An open problem is to provide monotone sequences of iterations with the k-th order of convergence, without assuming the smoothness of the nonlinear term.

4 Convexity in Nonlinear Eigenvalue Problems

Let X be a Banach space, $K \subset X$ a subset and $F : K \times \mathbf{R}_+ \to X$ a continuous map. Consider the equation

$$u = F(u, \lambda)$$

where $\lambda \in \mathbf{R}_+$ and $u \in K$. A *solution* of this equation is a pair $(u, \lambda) \in K \times \mathbf{R}_+$. Equations of this form are called *nonlinear eigenvalue problems* [Ra73], [Sch95]. The interest is in studying the structure of the set of solutions and its behavior with respect to λ.

In [RePr], we have discussed the following nonlinear eigenvalue problem:

$$\begin{cases} L_0 u \equiv (-1)^m u^{(2m)} = \lambda f(u), & x \in [0,1] \\ u^{(k)}(0) = 0, & 0 \le k \le m-1 \\ u^{(k)}(1) = 0, & m \le k \le 2m-1 \end{cases} \tag{5}$$

and we proved the following theorem:

Theorem 8 *Suppose $f \in C^1(\mathbf{R}_+; (0,\infty))$ is increasing and nonconcave. Then there exists a maximal $\lambda^* \in (0,\infty)$ and a C^1 function $\lambda \longmapsto u_\lambda$ from $[0,\lambda^*)$ into C_B^{2m} such that $u_0 = 0$ and for each $\lambda \in (0,\lambda^*)$, $u_\lambda(t) > 0$ on $(0,1)$, solves (5) and is stable in the sense that*

$$\lambda_1(L) > 0 \quad (Lu := L_0 u - \lambda f'(u_\lambda)\, u).$$

Moreover,

$$\begin{cases} \left(\frac{du_\lambda}{d\lambda}\right)^{(k)} > 0, & 0 \le k \le m \\ (-1)^{k-m}\left(\frac{du_\lambda}{d\lambda}\right)^{(k)} > 0, & m+1 \le k \le 2m. \end{cases} \tag{6}$$

We point out the role of the nonconcavity of f to guarantee that the maximal λ^* is finite, and also the "good" behavior (6) of the continuum of solutions $\{u_\lambda;\ \lambda \in [0,\lambda^*)\}$ with respect to λ.

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