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Nontrivial solvability of Hammerstein integral equations in Hilbert spaces

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ABSTRACT. In this note some well-known existence and multiplicity results of nontrivial solutions for scalar Hammerstein equations [1], [3] are extended to equations in Hilbert spaces. The tools are a mountain pass theorem on closed convex subsets of a Hilbert space due to Guo-Sun-Qi [1] and a new technique of checking the Palais-Smale compactness condition which was first presented in [4]. The results complement those established in [4].

KEY WORDS: Hammerstein integral equation, compactness, critical point theory.

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1 Preliminaries

1.1 A mountain pass theorem

Let X be a real Hilbert space, $D \subset X$ closed convex and $E \in C^1(X;\mathbf{R})$. Let U be an open subset in D, $x_0 \in U$ and $x_1 \in D^{ri} \setminus \overline{U}$. Here D^{ri} is the set of all relatively algebraic interior points of D. Denote

$$\Phi = \{\phi \in C([0,1];D) : \phi(0) = x_0,\ \phi(1) = x_1\},$$

$$c = \inf_{\phi \in \Phi} \max_{t \in [0,1]} E(\phi(t)),$$

$$\mathcal{K}_c = \{x \in D : E(x) = c, \; E'(x) = 0\}.$$

We say that E satisfies the *Palais-Smale (P-S) condition on* D, if

$$\begin{cases} \{x_n\} \subset D, \; E(x_n) \to \mu \in \mathbf{R}, \; E'(x_n) \to 0 \\ \text{imply that } \{x_n\} \text{ has a convergent subsequence.} \end{cases}$$

The following result is due to Guo-Sun-Qi [1] and represents a variant on closed convex sets of the mountain pass theorem of Ambrosetti-Rabinowitz [5].

Theorem 1.1 (Guo-Sun-Qi) *Assume that* E *satisfies the (P-S) condition on* D *and*

$$\max\{E(x_0), E(x_1)\} \leq \inf_{x \in \partial U} E(x) \tag{1.1}$$

where ∂U *is the boundary of* U *with respect to* D. *Also suppose that*

$$(I - E')(D) \subset D. \tag{1.2}$$

Then $\mathcal{K}_c \setminus \{x_0, x_1\} \neq \emptyset$.

1.2 Abstract Hammerstein equations

Here we discuss the abstract Hammerstein equation

$$y = KF(y), \quad y \in Y, \tag{1.3}$$

where Y is a real Banach space, $F : Y \to Y^*$ and $K : Y^* \to Y$ is linear.

Suppose that K splits into

$$\begin{cases} K = AA^* \text{ with } A : X \to Y \text{ and } A^* : Y^* \to X, \\ \text{where } X \text{ is a real Hilbert space.} \end{cases} \tag{1.4}$$

Then (1.3) can be converted to an equation on X, namely

$$x = A^* F A(x), \quad x \in X. \tag{1.5}$$

Indeed, if y solves (1.3), then $x = A^* F(y)$ is a solution of (1.5) and, conversely, if x solves (1.5), then $y = A(x)$ is a solution of (1.3).

If in addition, we suppose

$$F = G' \text{ for some } G \in C^1(Y; \mathbf{R}) \tag{1.6}$$

and that

(1.7)
A is linear bounded, one-to-one and A^* is the adjoint of A,

then (1.5) is equivalent with the critical point problem

$$E'(x) = 0, \quad x \in X$$

for the energy functional

$$E : X \to \mathbf{R}, \quad E(x) = \frac{1}{2} |x|_X^2 - GA(x).$$

It is easy to prove that $E \in C^1(X; \mathbf{R})$ and

$$E'(x) = x - A^* F A(x), \quad x \in X.$$

Sufficient conditions for (1.4) and (1.7) can be found in Krasnoselskii [2] (see also [3]).

2 Main results

Our first result is concerning with the nontrivial solvability of abstract Hammerstein equations in wedges. Recall that by a wedge of Y, one means a nonempty closed convex subset $P \subset Y$ such that $\lambda y \in P$ for all $y \in P$ and $\lambda \in \mathbf{R}_+$.

Theorem 2.1 *Assume (1.4), (1.6) and (1.7). In addition suppose that* $P \subset Y$ *is a wedge and there exists* $y_1 \in P \setminus \{0\}$ *and* $r \in \left(0, \left|A^{-1}(y_1)\right|_X\right)$ *such that the following conditions are satisfied:*

$$G(0) = 0 \text{ and } G(y_1) \geq \left|A^{-1}(y_1)\right|_X^2 /2; \tag{2.1}$$

$$G(y) \leq r^2/2 \text{ for } y \in P \text{ with } \left|A^{-1}(y)\right|_X = r; \tag{2.2}$$

$$KF(P) \subset P; \tag{2.3}$$

(2.4)
$$\begin{cases} \text{if } y_n \in P,\ \frac{1}{2}\left|A^{-1}(y_n)\right|_X^2 - G(y_n) \to \mu \in \mathbf{R},\ y_n - KF(y_n) \to 0 \\ \text{in } Y, \text{ then } \{y_n\} \text{ has a subsequence convergent in } Y. \end{cases}$$

Then (1.3) has at least one solution $y \in P \setminus \{0, y_1\}$.

Proof. We shall apply Theorem 1.1. Here $D = A^{-1}(P)$, $x_1 = A^{-1}(y_1)$ and

$$U = \{x \in D : |x|_X < r\}.$$

It is clear that $x_0 = 0 \in U$, $x_1 \in D^{ri} \setminus U$ and (1.1) holds by (2.1) and (2.2). To check (1.2), let $x \in A^{-1}(P)$. Then $A(x) \in P$. Also,

$$\left(I - E'\right)(x) = A^* FA(x).$$

Let $x' = A^* FA(x)$. Then, using (2.3), we obtain

$$A\left(x'\right) = KFA(x) \in KF(P) \subset P.$$

This shows that $A(x') \in P$, that is $x' \in D$. Thus (1.2) holds.

Next we show that E satisfies the (P-S) condition on D. For this, let $\{x_n\} \subset D$ be any sequence such that

$$E(x_n) \to \mu \in \mathbf{R},\ E'(x_n) \to 0. \tag{2.5}$$

Let $y_n = A(x_n)$. From (2.5), it follows that

$$y_n \in P,\ \frac{1}{2}\left|A^{-1}(y_n)\right|_X^2 - G(y_n) \to \mu,\ y_n - KF(y_n) \to 0.$$

Now (2.4) guarantees that $\{y_n\}$ has a subsequence convergent in Y. Then, from

$$E'(x_n) = x_n - A^*FA(x_n) = x_n - A^*F(y_n) \to 0$$

(in Y) and the continuity of A^* and F, it follows that the corresponding subsequence of $\{x_n\}$ converges in X. Thus all the assumptions of Theorem 1.1 are satisfied.

For an example of application of Theorem 2.1, let us consider the Hammerstein integral equation

$$(2.6)\qquad y(t) = \int_J k(t,s)\, f(s, y(s))\, ds, \quad \text{a.e. } t \in J.$$

Here J is a compact real interval, k is a real function, while y and f take values in a real Hilbert space H with the inner product $(.,.)$.

We use the following notion. A function $\psi : J \times D \to Y$, where $D \subset X$ and X, Y are two Banach spaces, is said to be (q,p) - *Carathéodory* ($1 \le q \le \infty$, $1 \le p < \infty$) if $\psi(.,x)$ is strongly measurable for each $x \in D$, $\psi(t,.)$ is continuous for a.e. $t \in J$ and

$$|\psi(t,x)|_Y \le \psi_0(t) + \alpha |x|_X^p$$

for a.e. $t \in J$ and all $x \in D$, where $\psi_0 \in L^q(J;\mathbf{R}_+)$, $\alpha \in \mathbf{R}_+$.

Our assumptions are as follows:

(a1) $k \in L^p(J \times J; \mathbf{R}_+)$ $(2 < p < \infty)$ and the map $K : L^q(J;H) \to L^p(J;H)$ given by

$$K(z)(t) = \int_J k(t,s)\, z(s)\, ds$$

$(1/p + 1/q = 1)$ is well defined and satisfies (1.4), (1.7) with $X = L^2(J;H)$ and $Y = L^p(J;H)$.

(a2) $f : J \times H \to H$ is $(q, p-1)$-Carathéodory and there exists $g : J \times H \to \mathbf{R}$ (∞, p)-Carathéodory such that $g(t, 0) = 0$ and

$$g(t, x+y) - g(t, x) = (f(t, x), y) + \omega(t, x, y) \tag{2.7}$$

for a.e. $t \in J$ and all $x, y \in H$, where

$$\omega(t, x, y) / |y| \to 0 \quad \text{as } y \to 0 \tag{2.8}$$

uniformly for $x \in H$ and a.e. $t \in J$.

(a3) $C \subset H$ is a wedge, $f(t, C) \subset C$ for a.e. $t \in J$, and there exists $x_0 \in H$ with $A(x_0)(t) \in C \setminus \{0\}$ for a.e. $t \in J$.

(a4) the following inequality is satisfied

$$\limsup_{x \to 0,\, x \in C} g(t, x) / |x|^2 \leq 0 \tag{2.9}$$

uniformly for a.e. $t \in J$.

(a5) there exist $\alpha > 2$, $\rho > 0$, $m \geq 0$ and $\gamma \in L^1(J; (0, \infty))$ such that

$$g(t, x) \geq \gamma(t) |x|^\alpha \tag{2.10}$$

$$(f(t, x), x) - \alpha g(t, x) \geq -m \tag{2.11}$$

for a.e. $t \in J$ and all $x \in C$ with $|x| \geq \rho$.

(a6) there exists $w : J \times \mathbf{R}_+ \to \mathbf{R}_+$ $(q, p-1)$-Carathéodory with

$$\beta(f(t, M)) \leq w(t, \beta(M)) \tag{2.12}$$

for every bounded set $M \subset C$, a.e. $t \in J$, such that $\varphi \equiv 0$ is the unique solution in $L^p(J; \mathbf{R}_+)$ of the inequality

$$\varphi(t) \leq \int_J k(t, s)\, w(s, \varphi(s))\, ds, \quad \text{a.e. } t \in J. \tag{2.13}$$

The main result is the following theorem.

Theorem 2.2 *If (a1)-(a6) hold, then (2.6) has at least one solution* $y \in L^p(J;H) \setminus \{0\}$ *with* $y(t) \in C$ *for a.e.* $t \in J$.

Proof. Let $X = L^2(J;H)$, $Y = L^p(J;H)$. We shall denote by $(.,.)_2$ the usual inner-product of $L^2(J;H)$ and by $|\cdot|_p$ the norm of $L^p(J;H)$.

Let $F : L^p(J;H) \to L^q(J;H)$ be given by

$$F(y)(t) = f(t, y(t)),$$

and let $G : L^p(J;H) \to \mathbf{R}$,

$$G(y) = \int_J g(t, y(t))\, dt.$$

Since f is $(q, p-1)$-Carathéodory and g is (∞, p)-Carathéodory, we have

$$|f(t,x)| \le f_0(t) + a|x|^{p-1}, \quad x \in H, \text{ a.e. } t \in J, \tag{2.14}$$

for some $f_0 \in L^q(J;\mathbf{R}_+)$, $a \in \mathbf{R}_+$, and

$$|g(t,x)| \le b(1 + |x|^p), \quad x \in H, \text{ a.e. } t \in J, \tag{2.15}$$

where $b \in \mathbf{R}_+$. Consequently, the maps F and G are well defined. Also, from (2.7) and (2.8), we have (1.6), while $g(t,0) = 0$ implies $G(0) = 0$.

Define $P = \{y \in L^p(J;H) : y(t) \in C \text{ a.e. } t \in J\}$. From (a3) and $k \ge 0$, we easily see that (2.3) holds.

From (2.9), we have that for each $\varepsilon > 0$ there exists $\delta > 0$ with $g(t,x) \le \varepsilon |x|^2$ for a.e. $t \in J$ and all $x \in C$ satisfying $|x| < \delta$. Furthermore, from (2.15), it follows that

$$g(t,x) \le b\left(|x|^{-p} + 1\right)|x|^p \le b\left(\delta^{-p} + 1\right)|x|^p$$

for $|x| \ge \delta$. Hence, if we let $c(\varepsilon) = b(\delta^{-p} + 1)$, then

$$g(t,x) \le \varepsilon |x|^2 + c(\varepsilon)|x|^p \tag{2.16}$$

for a.e. $t \in J$ and all $x \in C$. Consequently, for every $x \in D$, we have

$$E(x) = \frac{1}{2}|x|_2^2 - G(A(x)) = \frac{1}{2}|x|_2^2 - \int_J g(t, A(x)(t))\,dt$$

$$\geq \frac{1}{2}|x|_2^2 - \varepsilon|A(x)|_2^2 - c(\varepsilon)|A(x)|_p^p$$

$$\geq \frac{1}{2}|x|_2^2 - \varepsilon|A||A^*||x|_2^2 - \bar{c}(\varepsilon)|x|_2^p$$

$$= \left(1/2 - \varepsilon|A||A^*| - \bar{c}(\varepsilon)|x|_2^{p-2}\right)|x|_2^2. \tag{2.17}$$

Here we have used

$$|A(x)|_p \leq c_0|x|_2 \tag{2.18}$$

(since A is linear bounded) and

$$|A(x)|_2^2 = (A^*A(x), x) \leq |A^*||A||x|_2^2.$$

Now, if we choose $\varepsilon > 0$ such that $1/2 - \varepsilon|A||A^*| > 0$, and any $r > 0$ small enough that

$$1/2 - \varepsilon|A||A^*| - \bar{c}(\varepsilon)r^{p-2} \geq 0$$

(recall $p > 2$), then from (2.17) we have $E(x) \geq 0$ for all $x \in D$ with $|x|_2 = r$. Hence (2.2) holds.

Now we look for a function y_1 satisfying (2.1) in the form $y_1(t) = \lambda A(x_0)(t)$ for some $\lambda > 0$. Looking x_0 as an element of $L^2(J; D)$, from (a5), we obtain

$$E(\lambda x_0) = (\lambda^2/2)|x_0|_2^2 - \int_J g(t, \lambda A(x_0)(t))\,dt$$

$$\leq (\lambda^2/2)|x_0|_2^2 - \lambda^\alpha \int_{\lambda|A(x_0)(t)|\geq\rho} \gamma(t)|A(x_0)(t)|^\alpha\,dt$$

$$(2.19) \qquad - \int_{\lambda |A(x_0)(t)| < \rho} g(t, \lambda A(x_0)(t))\, dt.$$

Let $\lambda_0 > 0$ be such that the measure of the set $\{t : \lambda_0 |A(x_0)(t)| \geq \rho\}$ is positive (such an λ_0 exists since $A(x_0)(t) \neq 0$ for a.e. $t \in J$). Then

$$\eta := \int_{\lambda_0 |A(x_0)(t)| \geq \rho} \gamma(t) |A(x_0)(t)|^{\alpha}\, dt > 0.$$

Since for $\lambda \geq \lambda_0 > 0$,

$$\int_{\lambda |A(x_0)(t)| \geq \rho} \gamma(t) |A(x_0)(t)|^{\alpha}\, dt \geq \eta$$

and the last integral in (2.19) is bounded by (2.15), we have that

$$E(\lambda x_0) \leq (\lambda^2/2) |x_0|_2^2 - \eta \lambda^{\alpha} + M$$

for all $\lambda \geq \lambda_0$. Since $\alpha > 2$, $E(\lambda x_0) \to -\infty$ as $\lambda \to \infty$. Thus we may chose $\lambda > \max\{\lambda_0, r/|x_0|_2\}$ such that $E(\lambda x_0) < 0$. Therefore (2.1) is fulfilled.

Finally we prove that the (P-S) condition on D is satisfied. For this, let $\{x_n\}$ be any sequence of D with:

$$(2.20) \qquad E(x_n) = |x_n|_2^2 /2 - G(y_n) \quad \text{bounded}$$

and

$$(2.21) \qquad E'(x_n) = x_n - A^* F(y_n) =: z_n \to 0 \text{ as } n \to \infty$$

where $y_n = A(x_n)$. We have

$$(x_n, z_n)_2 = |x_n|_2^2 - \int_J (f(t, y_n(t)), y_n(t))\, dt$$

$$(2.22) \qquad = |x_n|_2^2 - \alpha \int_J g(t, y_n(t))\, dt$$

$$-\int_J [(f(t, y_n(t)), y_n(t)) - \alpha g(t, y_n(t))] \, dt.$$

From (2.14), (2.15) and (2.11), it follows that there exists $c_1 \geq 0$ such that

$$\int_J [(f(t, y_n(t)), y_n(t)) - \alpha g(t, y_n(t))] \, dt \geq -c_1 \tag{2.23}$$

for all $n \in \mathbf{N}$. Now from (2.20)-(2.23) we obtain that

$$(\alpha/2 - 1) |x_n|_2^2 \leq \alpha E(x_n) - (x_n, z_n)_2 + c_1 \leq |x_n|_2 + c$$

for some constant c. Since $\alpha > 2$, this implies that $\{x_n\}$ is bounded in $L^2(J; H)$. Next the existence of a convergent subsequence of $\{x_n\}$ can be proved as in [4, Theorem 4.1] using (a6) and a compactness criterium in $L^p(J; H)$ (see [4, Theorem 2.5]). Now the conclusion follows from Theorem 2.1. □

Remark 2.1 *Theorems 2.1 and 2.2 yield multiplicity results for (1.3) and (2.6) in case that their assumptions hold for different wedges P such that each two of them have in common only the origin. The most usual case is that where P is a cone, i.e. $P \cap (-P) = \{0\}$ and the hypotheses of Theorems 2.1 and 2.2 are fulfilled both by P and $-P$.*

Remark 2.2 *In the scalar case when $H = \mathbf{R}$, we have*

$$g(t, x) = \int_0^x f(t, s) \, ds$$

and (a6) trivially holds with $w \equiv 0$. Also, a sufficient condition for (a5) (see [3, Theorem 9.2]) is

(a5') *there exist $\alpha > 2$ and $\rho > 0$ such that*

$$f(t, x) x \geq \alpha g(t, x) > 0$$

for a.e. $t \in J$ and all $x \in C$ with $|x| \geq \rho$.

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