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Continuation Method for Contractive Maps on Spaces Endowed with Vector-valued Metrics

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Abstract. A continuation result for contractive maps on spaces endowed with vector-valued metrics is established.

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1 Introduction

The Banach contraction principle was generalized by Perov (see [3] and [5]) for contractive maps on spaces endowed with vector-valued metrics. Also, Granas [1] proved that the property of having a fixed point is invariant by homotopy for contractions on complete metric spaces. This result was completed in [4] by an iterative procedure of discrete continuation along the fixed points curve. The goal of this paper is to extend this result to contractive maps on spaces endowed with vector-valued metrics.

Let X be a nonempty set. By a *vector-valued metric* on X we mean a map $d : X \times X \to \mathbf{R}^m$ with the following properties:

(i) $d(x,y) \geq 0\ \forall x,y \in X$; $d(x,y) = 0 \iff x = y$
(ii) $d(x,y) = d(y,x)\ \forall x,y \in X$
(iii) $d(x,y) \leq d(x,z) + d(z,y)\ \forall x,y,z \in X$.

Here, if $u, v \in \mathbf{R}^m$, $u = (u_1, u_2, ..., u_m)$ and $v = (v_1, v_2, ..., v_m)$, by $u \leq v$ we mean that $u_i \leq v_i$ for $i = 1, 2, ..., m$.

A set X endowed with a vector-valued metric d is said to be a *generalized metric space*. For the generalized metric spaces, the notions of a convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Definition 1.1 *Let (X,d) be a generalized metric space. A map $T : X \to X$ is said to be contractive if there exists a matrix $A \in M_{m\times m}(\mathbf{R}_+)$ such that*

$$A^k \to 0 \text{ as } k \to \infty \tag{1.1}$$

and

$$d(T(x), T(y)) \leq A\, d(x,y)\ \forall x,y \in X.$$

The matrix A is said to be a Lipschitz matrix for T.

Recall that for a matrix $A \in M_{m\times m}(\mathbf{R}_+)$ the property (1.1) is equivalent to the fact that $I - A$ is nonsingular and

$$(I-A)^{-1} = I + A + A^2 + ... \tag{1.2}$$

(see [5], Theorem 4.1.1). From (1.2) we see that $\rho \leq (I-A)^{-1}\rho$ for every $\rho \in [0,\infty)^m$.

Theorem 1.1 (Perov) *Let (X,d) be a complete generalized metric space and $T : X \to X$ be contractive with the Lipschitz matrix A. Then T has a unique fixed point x^* and for each $x_0 \in X$ one has*

$$d\left(T^k(x_0), x^*\right) \leq A^k (I-A)^{-1} d(x_0, T(x_0))$$

for every $k \in \mathbf{N}$.

2 Main result

Theorem 2.1 *Let (X,d) be a complete generalized metric space with $d : X \times X \to \mathbf{R}^m$ and U be an open set of X. Let $H : \bar{U} \times [0,1] \to X$ and assume that the following conditions are satisfied:*

(a1) there is $A \in M_{m\times m}(\mathbf{R}_+)$ such that $A^k \to 0$ as $k \to \infty$,

$$(I-A)\rho \in (0,\infty)^m \text{ for every } \rho \in (0,\infty)^m \tag{2.1}$$

and

$$d(H(x,\lambda), H(y,\lambda)) \leq A\, d(x,y) \tag{2.2}$$

for all $x, y \in \bar{U}$ and $\lambda \in [0,1]$;

(a2) $H(x,\lambda) \neq x$ for all $x \in \partial U$ and $\lambda \in [0,1]$;

(a3) H is continuous in λ, uniformly for $x \in \bar{U}$, i.e. for each $\varepsilon \in (0,\infty)^m$ and $\lambda \in [0,1]$, there is $\rho \in (0,\infty)$ such that $d(H(x,\lambda), H(x,\mu)) < \varepsilon$ whenever $x \in \bar{U}$ and $|\lambda - \mu| < \rho$.

In addition suppose that $H_0 := H(.,0)$ has a fixed point. Then, for each $\lambda \in [0,1]$, there exists a unique fixed point $x(\lambda)$ of $H_\lambda := H(.,\lambda)$. Moreover, $x(\lambda)$ depends continuously on λ and there exists $r \in (0,\infty]^m$, integers $m, n_1, n_2, ..., n_{m-1}$ and numbers $0 < \lambda_1 < \lambda_2 < ... < \lambda_{m-1} < \lambda_m = 1$ such that for any $x_0 \in X$ satisfying $d(x_0, x(0)) \leq r$, the sequences $(x_{j,k})_{k\geq 0}$, $j = 1, 2, ..., m$,

$$\begin{aligned} x_{1,0} &= x_0 \\ x_{j,k+1} &= H_{\lambda_j}(x_{j,k}), \quad k = 0, 1, ... \\ x_{j+1,0} &= x_{j,n_j}, \quad j = 1, 2, ..., m-1 \end{aligned}$$

are well defined and satisfy

$$d(x_{j,k}, x(\lambda_j)) \leq A^k (I-A)^{-1} d(x_{j,0}, H_{\lambda_j}(x_{j,0})) \quad (k \in \mathbf{N}). \tag{2.3}$$

Proof.

1) First we prove that for each $\lambda \in [0,1]$, H_λ has a fixed point. Let

$$\Lambda = \{\lambda \in [0,1];\ H(x,\lambda) = x \text{ for some } x \in U\}.$$

We have $0 \in \Lambda$ by the assumption that H_0 has a fixed point. Hence Λ is nonempty. We will show that Λ is both closed and open in $[0,1]$ and so, by the connectedness of $[0,1]$, $\Lambda = [0,1]$.

To prove that Λ is closed, let $\lambda_k \in \Lambda$ with $\lambda_k \to \lambda$ as $k \to \infty$. Since $\lambda_k \in \Lambda$, there is $x_k \in U$ so that $H(x_k, \lambda_k) = x_k$. Then, by

), we obtain

$$d(x_k,x_j) = d(H(x_k,\lambda_k),H(x_j,\lambda_j)) \le d(H(x_k,\lambda_k),H(x_k,\lambda))$$
$$+d(H(x_k,\lambda),H(x_j,\lambda)) + d(H(x_j,\lambda),H(x_j,\lambda_j))$$
$$\le d(H(x_k,\lambda_k),H(x_k,\lambda)) + A\,d(x_k,x_j) + d(H(x_j,\lambda),H(x_j,\lambda_j)).$$

It follows that

$$d(x_k,x_j)$$
$$\le (I-A)^{-1}[d(H(x_k,\lambda_k),H(x_k,\lambda)) + d(H(x_j,\lambda),H(x_j,\lambda_j))].$$

This, by (a3), shows that (x_k) is a Cauchy sequence. Then, since X is complete, there is $x \in X$ with $d(x_k,x) \to 0$ as $k \to \infty$. Clearly, $x \in \overline{U}$. Then

$$d(x_k,H(x,\lambda)) \to d(x,H(x,\lambda))$$

and, by (2.2) and (a3),

$$d(x_k,H(x,\lambda)) = d(H(x_k,\lambda_k),H(x,\lambda)) \to 0.$$

Hence $d(x,H(x,\lambda)) = 0$, that is $H(x,\lambda) = x$. By (a2), $x \in U$ and so $\lambda \in \Lambda$. To prove that Λ is open in $[0,1]$, let $\mu \in \Lambda$ and $z \in U$ such that $H(z,\mu) = z$. Since U is open, there exists $\rho \in (0,\infty)^m$ such that

$$d(x,z) \le \rho \quad \text{implies} \quad x \in U.$$

Also, by (a3), there is $\eta = \eta(\rho) \in (0,\infty)$ such that

$$d(z,H(z,\lambda)) = d(H(z,\mu),H(z,\lambda)) \le (I-A)\rho \tag{2.4}$$

for $|\lambda-\mu| \le \eta$. Notice $(I-A)\rho \in (0,\infty)^m$ according to (2.1). Consequently,

$$\begin{aligned} d(z,H(x,\lambda)) &\le d(z,H(z,\lambda)) + d(H(z,\lambda),H(x,\lambda)) \\ &\le (I-A)\rho + Ad(z,x) \le \rho \end{aligned}$$

whenever $d(z,x) \le \rho$ and $|\lambda-\mu| \le \eta$. This shows that for $|\lambda-\mu| \le \eta$, H_λ sends B into itself, where $B = \{x \in X;\ d(z,x) \le \rho\}$. Now we may apply Theorem 1.1 to $T = H_\lambda$. Consequently, there is $x(\lambda) \in \bar{B} \subset \overline{U}$ a fixed point of H_λ for $|\lambda-\mu| \le \eta$.

This shows that μ is an interior point of Λ and hence Λ is open in $[0,1]$. Notice that for every $x \in B$ and $|\lambda-\mu| \le \eta$, we also have by Theorem 1.1, that the sequence $(H_\lambda^k(x))_{k\ge0}$ is well defined and

$$d(H_\lambda^k(x),x(\lambda)) \le A^k(I-A)^{-1}d(x,H_\lambda(x)) \quad (k \in \mathbf{N}).$$

2) The uniqueness of $x(\lambda)$ is a simple consequence of (2.2).

3) $x(\lambda)$ is continuous on $[0,1]$.

Indeed,

$$d(x(\lambda),x(\mu)) = d(H(x(\lambda),\lambda),H(x(\mu),\mu))$$
$$\le d(H(x(\lambda),\lambda),H(x(\mu),\lambda)) + d(H(x(\mu),\lambda),H(x(\mu),\mu))$$
$$\le Ad(x(\lambda),x(\mu)) + d(H(x(\mu),\lambda),H(x(\mu),\mu)).$$

This, by (a3), implies

$$d(x(\lambda),x(\mu)) \le (I-A)^{-1}d(H(x(\mu),\lambda),H(x(\mu),\mu)) \to 0$$

as $\lambda \to \mu$.

4) Obtention of r. For any $\mu \in [0,1]$ and each $i \in \{1,2,...,m\}$ denote

$$r_i(\mu) = \inf\{d_i(x,x(\mu));\ x \in X \setminus U\}.$$

Here $d = (d_1,d_2,...,d_m)$. Since $x(\mu) \in U$ and U is open, $r_i(\mu) > 0$. We claim that

$$\inf\{r_i(\mu);\ \mu \in [0,1]\} > 0. \tag{2.5}$$

To prove this, assume the contrary. Then, there are $\mu_k \in [0,1]$ such that $r_i(\mu_k) \to 0$ as $k \to \infty$. Clearly, we may assume that $\mu_k \to \mu$ for some $\mu \in [0,1]$. Then, from the continuity of $x(\lambda)$, we have

$$d_i(x(\mu_k),x(\mu)) < r_i(\mu)/2 \quad \text{for } k \ge k_1. \tag{2.6}$$

On the other hand, since $r_i(\mu_k) \to 0$,

$$r_i(\mu_k) < r_i(\mu)/2 \quad \text{for } k \ge k_2. \tag{2.7}$$

Let $k_0 = \max\{k_1,k_2\}$. By (2.7) and the definition of $r_i(\mu_{k_0})$ as infimum, there is $x \in X \setminus U$ with

$$d_i(x, x(\mu_{k_0})) < r_i(\mu)/2. \tag{2.8}$$

Then, by (2.6) and (2.8), we obtain

$$d_i(x, x(\mu)) \le d_i(x, x(\mu_{k_0})) + d_i(x(\mu_{k_0}), x(\mu)) < 2r_i(\mu)/2 = r_i(\mu)$$

a contradiction. Thus (2.5) holds as claimed. Now we choose any $r_i > 0$ less than the infimum in (2.5), with the convention that $r_i = \infty$ if the infimum equals infinity. Then take $r = (r_1, r_2, ..., r_m)$.

5) Obtention of m and $0 < \lambda_1 < \lambda_2 < ..., \lambda_{m-1} < 1$. Let $h = \eta(r)$, where r was fixed at the anterior step and $\eta(r)$ is chosen as in (2.4). Then, by what was shown at the end of step 1), for each $\mu \in [0, 1]$,

$$\begin{aligned} d(x, x(\mu)) &\le r \text{ and } |\lambda - \mu| \le h \quad \text{imply} \\ &(H_\lambda^k(x))_{k\ge 0} \text{ is well defined and} \\ d(H_\lambda^k(x), x(\lambda)) &\le A^k (I - A)^{-1} d(x, H_\lambda(x)) \quad (k \in \mathbf{N}) \end{aligned} \tag{2.9}$$

Now we choose any partition $0 = \lambda_0 < \lambda_1 < ... < \lambda_{m-1} < \lambda_m = 1$ of $[0, 1]$ such that $\lambda_{j+1} - \lambda_j \le h$, $j = 0, 1, ..., m-1$.

6) Finding of integers $n_1, n_2, ..., n_{m-1}$. From $d(x_{1,0}, x(0)) = d(x_0, x(0)) \le r$ and $\lambda_1 - \lambda_0 \le h$, by (2.9), we have that $(x_{1,k})_{k\ge 0}$ is well defined and satisfies (2.3). By (2.3), we may choose $n_1 \in \mathbf{N}$ such that $d(x_{1,n_1}, x(\lambda_1)) \le r$. Now

$$d(x_{2,0}, x(\lambda_1)) = d(x_{1,n_1}, x(\lambda_1)) \le r \text{ and } \lambda_2 - \lambda_1 \le h$$

and we repeat the above argument in order to show that $(x_{2,k})_{k\ge 0}$ is well defined and satisfies (2.3). In general, at step j ($1 \le j \le m-1$) we choose $n_j \in \mathbf{N}$ such that $d(x_{j,n_j}, x(\lambda_j)) \le r$. Then

$$d(x_{j+1,0}, x(\lambda_j)) = d(x_{j,n_j}, x(\lambda_j)) \le r \text{ and } \lambda_{j+1} - \lambda_j \le h,$$

by (2.9), imply that sequence $(x_{j+1,k})_{k\ge 0}$ is well defined and satisfies (2.3). ■

The above proof yields the following algorithm for the approximation of $x(1)$ under the assumptions of Theorem 2.1:

Suppose we know r and h and we wish to obtain an approximation $\bar{x}_1$ of $x(1)$ with $d(\bar{x}_1, x(1)) \le \varepsilon$ for some $\varepsilon \in (0, \infty)^m$. Then we choose any partition $0 = \lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_{m-1} < \lambda_m = 1$ of $[0, 1]$ with $\lambda_{j+1} - \lambda_j \le h$, $j = 0, 1, ..., m-1$, any element x_0 with $d(x_0, x(0)) \le r$ and we follow the next

Iterative procedure:

Set $n_0 := 0$ and $x_{0,n_0} := x_0$;
For $j := 1$ to $m - 1$ do
  $x_{j,0} := x_{j-1,n_{j-1}}$
  $k := 0$
  While $A^k (I - A)^{-1} d(x_{j,0}, H_{\lambda_j}(x_{j,0})) \not\le r$
    $x_{j,k+1} := H_{\lambda_j}(x_{j,k})$
    $k := k + 1$
  $n_j = k$
Set $k := 0$
While $A^k (I - A)^{-1} d(x_{m,0}, H_1(x_{m,0})) \not\le \varepsilon$
  $x_{m,k+1} = H_1(x_{m,k})$
  $k := k + 1$
Finally take $\bar{x}_1 = x_{m,k}$.

Notice for $m = 1$, Theorem 2.1 reduces to Corollary 2.5 in [4].

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