Séminaire de la Théorie de la Meilleure Approximation, Convexité et Optimisation
Cluj–Napoca, le 12 décembre 2002, pp. 141–149

On the Method of Upper and Lower Solutions

Radu Precup

(Cluj-Napoca)

Abstract. Some results concerning the method of upper and lower solutions for nonlinear integral equations of Hammerstein type are presented.

Key Words: Nonlinear integral equation, Hammerstein equation, Upper and lower solutions.

2001 AMS Subject Classification Code: 47H30, 47J25

1. One of the most useful methods for solving nonlinear equations is the method of upper and lower solutions. It consists in localizing solutions in an order interval $[u_0, v_0]$, where u_0 is a lower solution, v_0 is an upper solution, and $u_0 \leq v_0$. Thus a basic problem is to find comparable lower and upper solutions. In this paper we present such type of results for the abstract Hammerstein equation in $\mathbf{R}^n$

$$u(x) = AN_f(u)(x) \quad \text{a.e. on } \Omega. \tag{0.1}$$

Here N_f is Nemytskii's superposition operator associated to a given function $f : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ ($\Omega \subset \mathbf{R}^N$ bounded open), and A is a bounded linear operator from $L^q(\Omega; \mathbf{R}^n)$ to $L^p(\Omega; \mathbf{R})$.

2. To obtain lower and upper solutions we need information about f and A, in particular, about the spectrum of A.

Theorem 1. *Let $p, q \in [1,\infty)$, $A : L^q(\Omega;\mathbf{R}^n) \to L^p(\Omega;\mathbf{R}^n)$ an increasing linear operator and $f : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ a (p,q)-Carathéodory function. Assume that there are $c \in \mathbf{R}_+$ and $g \in L^q(\Omega;\mathbf{R}^n_+)$ such that*

$$f(x,z) \le cz + g(x) \tag{0.2}$$

for a.e. $x \in \Omega$ and all $z \ge 0$. Then any solution $v_0 \ge 0$ of the equation

$$(I - cA)(v) = A(g) \tag{0.3}$$

(if there is one) is an upper solution of the equation $u = AN_f(u)$. If in addition

$$-f(x,-z) \le cz + g(x) \tag{0.4}$$

for a.e. $x \in \Omega$ and all $z \ge 0$, then $u_0 = -v_0$ is a lower solution.

Proof. Assume $v_0 \ge 0$ solves (0.3). Then, from (0.2) we have

$$f(x, v_0(x)) \le c\, v_0(x) + g(x)$$

and since A is increasing,

$$AN_f(v_0) \le cA(v_0) + A(g) = v_0.$$

Hence v_0 is a upper solution. We leave to the reader to check that $-v_0$ is a lower solution.

In what follows we present two much applicable results involving spectral properties of A.

First we establish an *abstract Poincaré inequality.*

Lemma 2. *Let X be a Hilbert space and $A : X \to X$ be a positive self-adjoint operator. Then*

$$|A(u)|^2 \le |A|\,(A(u), u), \quad u \in X. \tag{0.5}$$

Proof. Since A is positive, for all $u, v \in X$ and $t \in \mathbf{R}$, we have

$$(A(u+tv), u+tv) \ge 0,$$

that is

$$(A(v), v)\, t^2 + 2(A(u), v)\, t + (A(u), u) \ge 0.$$

Consequently

$$(A(u), v)^2 \le (A(v), v)\,(A(u), u).$$

For $v = A(u)$ this inequality becomes

$$|A(u)|^4 \le \left(A^2(u), A(u)\right)(A(u), u). \tag{0.6}$$

On the other hand,

$$\left(A^2(u), A(u)\right) \le |A|\,|A(u)|^2. \tag{0.7}$$

Now (0.6) and (0.7) yield (0.5).

Our next result is an abstract weak maximum principle in $L^2(\Omega;\mathbf{R}^n)$. For a function $u : \Omega \to \mathbf{R}^n$, we let u^+, u^- be the functions defined by

$$u_i^+(x) = \max\{0, u_i(x)\}, \quad u_i^-(x) = \max\{0, -u_i(x)\},$$

$i = 1, 2, ..., n$. Clearly, $u = u^+ - u^-$, $u^+ \ge 0$ and $u^- \ge 0$. Also, for a function u one has $u \ge 0$, if and only if $u^- = 0$.

Lemma 3. *Let $A : L^2(\Omega;\mathbf{R}^n) \to L^2(\Omega;\mathbf{R}^n)$ be a positive self-adjoint operator. Assume the following conditions are satisfied:*

$$A(u) \ge 0 \ \text{for}\ u \ge 0; \quad A(u) \ne 0 \ \text{for}\ u \ne 0 \tag{0.8}$$

and

$$\left(A(u^+), A(u^-)\right)_2 = \left(A(u^+), u^-\right)_2 = 0, \quad u \in L^2(\Omega;\mathbf{R}^n). \tag{0.9}$$

Then for any constant $c < |A|^{-1}$,

$$(I - cA)^{-1}(u) \ge 0 \ \text{for all}\ u \ge 0. \tag{0.10}$$

Proof. Let $\sigma(A)$ be the spectrum of A, that is

$$\sigma(A) = \mathbf{R} \setminus \{\lambda \in \mathbf{R} : A - \lambda I \text{ is bijective}\}.$$

It is known that

$$\sigma(A) \subset [-|A|, |A|]$$

(see Brezis [1], p 94). Since $c < |A|^{-1}$ the operator $I - cA$ is invertible. Let $u \geq 0$ and let $v = (I - cA)^{-1}(u)$. Clearly

$$v - cA(v) = u. \tag{0.11}$$

We have to show that $v \geq 0$, equivalently $v^- = 0$. Assume the contrary, i.e., $v^- \neq 0$. Then (0.8) guarantees that $A(v^-) \geq 0$ and $A(v^-) \neq 0$. If we multiply (0.11) by $A(v^-)$, and we use (0.9), we obtain

$$-\left(A(v^-), v^-\right)_2 + c\left(A(v^-), A(v^-)\right)_2 = \left(A(v^-), u\right)_2.$$

Since both u and $A(v^-)$ are positive $(A(v^-), u)_2 \geq 0$. Therefore

$$c \geq \frac{(A(v^-), v^-)_2}{|A(v^-)|_2^2}.$$

This together with (0.5) implies $c \geq |A|^{-1}$, a contradiction. Thus $v^- = 0$.

Theorem 4. *Let $A : L^2(\Omega; \mathbf{R}^n) \to L^2(\Omega; \mathbf{R}^n)$ be a positive self-adjoint operator such that* (0.8) *and* (0.9) *hold. Let $f : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ be a map satisfying the Carathéodory conditions, such that for each $m \in (0, \infty)$ there is a constant $a_m \in \mathbf{R}_+$ with*

$$f(x, z) + a_m z \quad \text{increasing in } z \text{ on } [-m, m]$$

for a.e. $x \in \Omega$. Assume that

$$f(x, z) \leq cz + c', \quad f(x, -z) \geq -cz - c' \tag{0.12}$$

for a.e. $x \in \Omega$, all $z \in \mathbf{R}^n$ with $z \geq 0$, and some $c \in \mathbf{R}_+$ with $c < |A|^{-1}$ and $c' \in \mathbf{R}^n_+$. In addition assume that the solution of the equation

$$u - cA(u) = A(c')$$

belongs to $L^\infty(\Omega; \mathbf{R}^n)$. Then the equation $u = AN_f(u)$ has at least one solution in $L^2(\Omega; \mathbf{R}^n)$. Moreover, if the set $\mathcal{S}_+$ ($\mathcal{S}_-$) of all solutions $u \geq 0$ (respectively, $u \leq 0$) is nonempty, then it has a maximal (respectively, minimal) element.

Proof. Since $c < |A|^{-1}$, the operator $I - cA$ is bijective and so the equation (0.3) has a unique solution v_0 for each g. Here $g = c'$. By Lemma 3, $v_0 \geq 0$. Now, (0.12) guarantees both (0.2), (0.4). Thus, by Theorem 1, v_0 is an upper solution and $u_0 = -v_0$ is a lower solution. Since v_0 belongs to $L^\infty(\Omega; \mathbf{R}^n)$, there is $m \in (0, \infty)$ with $v_0 \leq m$. Then the function f_m given by

$$f_m(x, z) = f(x, z) + a_m z$$

is increasing in z on $[-m, m]$. Also the equation $u = AN_f(u)$ is equivalent to

$$u = (I + a_m A)^{-1} AN_{f_m}(u).$$

Let

$$T_m = (I + a_m A)^{-1} AN_{f_m}.$$

Clearly,

$$u_0 \leq T_m(u_0), \quad T_m(v_0) \leq v_0$$

and T_m is continuous and increasing on $[u_0, v_0]$. Let u^*, v^* be the minimal, respectively maximal solution in $[u_0, v_0]$. We have

$$-v_0 \leq u^* \leq v^* \leq v_0.$$

We now show that if $w \in L^2(\Omega; \mathbf{R}^n)$, $w \geq 0$, solves $w = AN_f(w)$ then $w \leq v_0$. Indeed, from

$$w = AN_f(w) \leq A(cw + c') = cA(w) + A(c')$$

and

$$v_0 = cA(v_0) + A(c'), \tag{0.13}$$

by subtraction we obtain

$$v_0 - w \geq cA(v_0 - w).$$

Then by the maximum principle, Lemma 3, $v_0 - w \geq 0$. Hence v^* is maximal in $\mathcal{S}_+$. Similarly, if $w \in L^2(\Omega)$, $w \leq 0$ and $w = AN_f(w)$, then $-v_0 \leq w$. Hence u^* is minimal in $\mathcal{S}_-$.

The last theorem is an existence and localization result of a nonnegative non-zero solution.

Theorem 5. *Let $A : L^2(\Omega; \mathbf{R}^n) \to L^2(\Omega; \mathbf{R}^n)$ be a completely continuous positive self-adjoint operator such that (0.8) and (0.9) hold. Let $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuous map such that $f(0) = 0$ and for each $m \in (0, \infty)$, there is a constant $a_m \in \mathbf{R}_+$ with*

$$f_m(z) := f(z) + a_m z \text{ increasing on } [0, m],$$

Assume that

$$f(z) \leq cz + c'$$

for all $z \in \mathbf{R}^n_+$, and some $c \in \mathbf{R}_+$ with $c < |A|^{-1}$, $c' \in (0, \infty)^n$, and

$$f(z) \geq |A|^{-1} z \tag{0.14}$$

for all $z \in \mathbf{R}^n_+$ with $|z| \leq \varepsilon_0$, where $\varepsilon_0 > 0$. In addition assume that the solutions of the equations

$$u - cA(u) = A(c') \text{ and } u - |A|^{-1} A(u) = 0$$

belong to $L^\infty(\Omega; \mathbf{R}^n)$. Then the equation $u = AN_f(u)$ has a maximal solution u in $L^2(\Omega; \mathbf{R}^n_+)$ and $u \neq 0$.

Proof. As above, the unique solution v_0 of the equation $u - cA(u) = A(c')$ belongs to $L^\infty(\Omega; \mathbf{R}^n_+)$ and is an upper solution of the equation $u = AN_f(u)$. Since $f(0) = 0$ the null function is a solution, and so a lower solution. Now we apply the Monotone Iteration Principle (see Deimling [2] and Precup [3]) to deduce the existence of a maximal fixed point v^* in $[0, v_0]$ of the operator

$$T_m = (I + a_m A)^{-1} AN_{f_m},$$

where $m \in (0, \infty)$ satisfies $v_0(x) \leq m$, a. e. $x \in \Omega$. As in the proof of Theorem 4 we can show that v^* is maximal in the set of all nonnegative solutions. To show that $v^* \neq 0$, we prove that v^* is the maximal fixed point of T_m in an order subinterval $[u_0, v_0] \subset [0, v_0]$ with $u_0 \neq 0$.

Since A is completely continuous and positive, there exists a u_1 with $|u_1|_2 = 1$ such that

$$|A| = (A(u_1), u_1)_2.$$

Then, according to (0.9), we have

$$\begin{aligned} |A| &= (A(u_1^+ - u_1^-), u_1^+ - u_1^-)_2 \\ &= (A(u_1^+), u_1^+)_2 + (A(u_1^-), u_1^-)_2 \\ &= (A(u_1^+ + u_1^-), u_1^+ + u_1^-)_2. \end{aligned}$$

Hence we may assume that $u_1 \geq 0$. For any fixed $v \in L^2(\Omega; \mathbf{R}^n)$ we consider the function

$$g(t) = \frac{(A(u_1 + tv), u_1 + tv)_2}{|u_1 + tv|_2^2},$$

which can be defined on a neighborhood of $t = 0$. This function attains its maximum $|A|$ at $t = 0$, so $g'(0) = 0$. Notice

$$g'(0) = 2[(A(u_1), v)_2 - |A|(u_1, v)_2].$$

Hence

$$u_1 = |A|^{-1} A(u_1)$$

(i.e., $|A|$ is the largest eigenvalue of A and u_1 is an eigenfunction). Also, by hypothesis u_1 belongs to $L^\infty(\Omega; \mathbf{R}^n)$. Let $u_0 = \varepsilon |u_1|_\infty^{-1} u_1$, where $0 < \varepsilon \leq \varepsilon_0$. Clearly

$$u_0 \geq 0, \quad u_0 \neq 0, \quad |u_0(x)| \leq \varepsilon \text{ a.e. on } \Omega, \quad u_0 = |A|^{-1} A(u_0).$$

Using (0.14), we deduce

$$\begin{aligned} u_0 &= |A|^{-1} A(u_0) = A\left(|A|^{-1} u_0\right) \\ &\leq AN_f(u_0). \end{aligned}$$

Thus u_0 is a lower solution of $u = AN_f(u)$. Also, from

$$v_0 = cA(v_0) + A(c'), \quad u_0 = |A|^{-1} A(u_0),$$

we have

$$v_0 - u_0 = cA(v_0 - u_0) + \left(c - |A|^{-1}\right) A(u_0) + A(c').$$

Now we choose $\varepsilon > 0$ small enough so that

$$\left(c - |A|^{-1}\right) u_0(x) + c' \geq 0 \quad \text{a.e. on } \Omega.$$

Then

$$v_0 - u_0 - cA(v_0 - u_0) \geq 0,$$

and by the maximum principle, $v_0 - u_0 \geq 0$. Next we apply the Monotone Iteration Principle to deduce the existence of a maximal fixed point in $[u_0, v_0]$ of T_m. Clearly it is equal to v^*.

Example 6. Let $n = 1$. The operator $A = (-\Delta)^{-1}$ has all the properties required by Theorems 4-5. Moreover, in this case u^*, v^* are, respectively, the minimal and maximal solutions in the set of all solutions in $L^2(\Omega)$. Indeed, if $w \in L^2(\Omega)$ is any solution and we let f_w be defined by

$$f_w(x, z) = f(x, z) \text{ if } w(x) > 0, \quad f_w(x, z) = 0 \text{ if } w(x) \leq 0,$$

then

$$-\Delta w^+ = f_w(x, w^+) \leq c\,w^+(x) + c' \quad \text{a.e. on } \Omega.$$

Hence

$$w^+ \leq cA(w^+) + A(c').$$

This together with (0.13) implies

$$v_0 - w^+ \geq cA(v_0 - w^+).$$

Thus $w^+ \leq v_0$. Similarly $-v_0 \leq -w^-$. Therefore $-v_0 \leq w \leq v_0$.

References

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Faculty of Mathematics and Computer Science
Babeş-Bolyai University
Str. M. Kogălniceanu, nr. 1
3400 Cluj-Napoca
Romania
E-mail: r.precup@math.ubbcluj.ro