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SOME EXISTENCE RESULTS FOR DIFFERENTIAL EQUATIONS WITH BOTH RETARDED AND ADVANCED ARGUMENTS

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Abstract. Existence, uniqueness and monotone approximation of solutions to the Cauchy problem for differential equations with both advanced and retarded arguments are obtained.

AMS Subject Classification: 34K15.

1. Introduction

In this paper we study the existence of solutions to the Cauchy problem

$$(1)\qquad \begin{cases} x'(t) = f(t, x(t), x(\phi(t)), x(\psi(t))), & 0 \leqslant t \leqslant T \\ x(t) = a(t), & -A \leqslant t \leqslant 0, \end{cases}$$

where $0 < T < \infty$ and

$$\phi \;:\; [0,T] \to [-A,T] \;\; (0 \leqslant A < \infty), \;\; \phi(t) \leqslant t \text{ on } [0,T];$$

$$\psi \;:\; [0,T] \to [-A, T+B] \;\; (0 \leqslant B < \infty),$$

$$a \;:\; [-A,0] \to \mathbb{R},$$

$$f \;:\; [0,T] \times \mathbb{R}^3 \to \mathbb{R}$$

are continuous functions.

By a solution of (1) we mean a function $x \in C^1[0,T] \cap C[-A, T+B]$.

In (1), the argument $\phi(t)$ is retarded if $\phi(t) < t$, while $\psi(t)$ is advanced if $\psi(t) > t$.

In the study of (1), two cases are possible:

Case I: $B = 0$;

Case II: $B > 0$. This case can be reduced to *Case I* if one considers continuous extensions of the functions f, ϕ and ψ as follows

$$\widetilde{f} \;:\; [0,\widetilde{T}] \times \mathbb{R}^3 \to \mathbb{R}, \text{ where } \widetilde{T} = T + B,$$

$$\widetilde{\phi} \;:\; [0,\widetilde{T}] \to [-A,\widetilde{T}] \text{ with } \widetilde{\phi}(t) \leqslant t \text{ on } [0,\widetilde{T}],$$

$$\widetilde{\psi} \;:\; [0,\widetilde{T}] \to [-A,\widetilde{T}].$$

Then we may extend (1) to the following problem on $[0,\widetilde{T}]$

$$(2)\qquad \begin{cases} x'(t)=\widetilde{f}\left(t,x(t),x\left(\widetilde{\phi}(t)\right),x\left(\widetilde{\psi}(t)\right)\right), & 0\leqslant t\leqslant \widetilde{T} \\ x(t)=a(t), & -A\leqslant t\leqslant 0, \end{cases}$$

which is in Case I. Clearly the solutions of (2) depend on the choosen extensions of f, ϕ, and ψ.

Our tools will be Banach contraction principle, Schauder fixed point theorem, Leray-Schauder continuation principle, the technique of a priori bounds and the monotone iterative (lower and upper solutions) method.

The literature in differential equations with modified arguments, especially of retarded type, is now very extensive. We refer the reader to the following monographs: D. Bainov-D.P. Mishev [1], L.E. Elsgolts-S.B. Norkin [2], K. Gopalsamy [3], J. Hale [4], V. Kolmanovskii-A. Myshkis [5], Y. Kuang [6], V. Lakshmikantham-L. Wen-B. Zhang [7], V. Mureşan [8], and to our papers [9], [10]. The case of equations with advanced argument and with both retarded and advanced arguments has been less studied. So our results complement in this respect the existing literature.

2. Results

Let $0<T<\infty$ and $0\leqslant A<\infty$. Our basic assumptions are as follows.

(h1) $\phi\in C([0,T]\ ;\ [-A,T])$ and $\phi(t)\leqslant t$ on $[0,T]$;
(h2) $\psi\in C([0,T]\ ;\ [-A,T])$;
(h3) $a\in C[-A,0]$;
(h4) $f\in C([0,T]\times\mathbb{R}^3)$.

The first existence result is of the same type like Theorem 15.2.2 in I.A. Rus [11].

THEOREM 1. *Assume* **(h1)–(h4)** *hold. In addition suppose*

(h5) *there exist* $\alpha,\beta,\gamma\in\mathbb{R}_+$ *such that*

$$(3)\qquad |f(t,x,y,z)-f(t,\bar{x},\bar{y},\bar{z})|\leqslant\alpha\,|x-\bar{x}|+\beta\,|y-\bar{y}|+\gamma\,|z-\bar{z}|$$

for all $(t,x,y,z)\in[0,T]\times\mathbb{R}^3$, *and*

$$(4)\qquad \gamma<\frac{1}{M}\mathrm{e}^{-1-M(\alpha+\beta)},$$

where

$$(5)\qquad M=\max\left(0,\ \max_{t\in[0,T]}\,(\psi(t)-t)\right).$$

Then we may extend (1) to the following problem on $[0,\widetilde{T}]$

$$(2)\qquad \begin{cases} x'(t)=\widetilde{f}\left(t,x(t),x\left(\widetilde{\phi}(t)\right),x\left(\widetilde{\psi}(t)\right)\right), & 0\leqslant t\leqslant \widetilde{T} \\ x(t)=a(t), & -A\leqslant t\leqslant 0, \end{cases}$$

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where

$$(5)\qquad M=\max\left(0,\ \max_{t\in[0,T]}\left(\psi(t)-t\right)\right).$$

Then (1) *has a unique solution* $x \in C^1[0,T] \cap C[-A,T]$.

Proof. Clearly (1) is equivalent with the integral equation

$$x(t) = a(0) + \int_0^t f(s, x(s), \widetilde{x}(\phi(s)), \widetilde{x}(\psi(s)))\mathrm{d}\, s \quad (0 \leqslant t \leqslant T),$$

where $\widetilde{x}(t) = x(t)$ for $t \in [0,T]$ and $\widetilde{x}(t) = a(t)$ if $t \in [-A,0)$. Let

$$K = \{x \in C[0,T] : \ x(0) = a(0)\}$$

and $N : K \to K$ be given by

$$N(x)(t) = a(0) + \int_0^t f(s, x(s), \widetilde{x}(\phi(s)), \widetilde{x}(\psi(s)))\mathrm{d}\, s \quad (0 \leqslant t \leqslant T). \tag{6}$$

We show that N is a contraction on K with respect to the norm $\|x\|_\eta = \max_{t\in[0,T]} |x(t)|\, \mathrm{e}^{-\eta t}$ if $\eta > 0$ is sufficiently large. Indeed, one has

$$|N(x)(t) - N(y)(t)| \leqslant$$

$$\leqslant \alpha \int_0^t |x(s)-y(s)|\, \mathrm{d}\, s + \beta \int_0^t |\widetilde{x}(\phi(s)) - \widetilde{y}(\phi(s))|\, \mathrm{d}\, s + \gamma \int_0^t |\widetilde{x}(\psi(s)) - \widetilde{y}(\psi(s))|\, \mathrm{d}\, s \leqslant$$

$$\leqslant \|x-y\|_\eta \int_0^t \left(\alpha \mathrm{e}^{\eta s} + \beta \mathrm{e}^{\eta\phi(s)} + \gamma \mathrm{e}^{\eta\psi(s)}\right) \mathrm{d}\, s \leqslant$$

$$\leqslant \|x-y\|_\eta \int_0^t \left[\alpha + \beta + \gamma \mathrm{e}^{\eta(\psi(s)-s)}\right] \mathrm{e}^{\eta s} \mathrm{d}\, s \leqslant$$

$$\leqslant \left(\alpha + \beta + \gamma \mathrm{e}^{\eta M}\right) \eta^{-1} \|x-y\|_\eta\, \mathrm{e}^{\eta t}.$$

It follows that

$$\|N(x) - N(y)\|_\eta \leqslant \left(\alpha + \beta + \gamma \mathrm{e}^{\eta M}\right) \eta^{-1} \|x-y\|_\eta\, .$$

Hence, N is a contraction if we choose an $\eta > 0$ with

$$\left(\alpha + \beta + \gamma \mathrm{e}^{\eta M}\right) \eta^{-1} < 1$$

or equivalently

$$\gamma < (\eta - \alpha - \beta)\mathrm{e}^{-\eta M}.$$

Such an η exists provided that

$$\gamma < \sup_{u>0} (u - \alpha - \beta)\mathrm{e}^{-uM}.$$

An elementary calculus shows that the supremum in the above formula equals $M^{-1}e^{-1-M(\alpha+\beta)}$ and is attained for $u=\alpha+\beta+1/M$.

Now the conclusion follows from Banach contraction theorem. □

Remark 1. If $\psi(t)\leqslant t$ on $[0,T]$ (no advanced arguments), then $M=0$ and (4) becomes $\gamma<\infty$. Thus, in this case no restriction on γ is required. The inequality (4) also shows that γ is supposed to be very small if M is very large.

In the next result, more generally, f is assumed to satisfy a growth condition, instead of the Lipschitz condition (3).

THEOREM 2. *Assume* **(h1)–(h4)** *hold. In addition, suppose*

(h6) *there exist* $\alpha,\ \beta,\ \gamma,\ \delta\in\mathbb{R}_+$ *such that*

$$|f(t,x,y,z)|\leqslant\alpha\,|x|+\beta\,|y|+\gamma\,|z|+\delta$$

for all $(t,x,y,z)\in[0,T]\times\mathbb{R}^3$, *and (4) be satisfied.*

Then (1) has at least one solution $x\in C^1[0,T]\cap C[-A,T]$.

Proof. Similar estimates like those in the proof of Theorem 1 yield

$$\|N(x)\|_\eta\leqslant|a(0)|+\delta T+\left(\alpha+\beta+\gamma e^{\eta M}\right)\eta^{-1}\max\left(\|x\|_\eta\,,\theta\right),$$

where $\theta=\max\limits_{t\in[-A,0]}|a(t)|\,e^{-\eta t}$. Since by (4) there exists an $\eta>0$ with $\left(\alpha+\beta+\gamma e^{\eta M}\right)\eta^{-1}<1$, we may find an $R>\theta$ such that

$$|a(0)|+\delta T+\left(\alpha+\beta+\gamma e^{\eta M}\right)\eta^{-1}R\leqslant R.$$

Then

$$N(K_R)\subset K_R,\ \text{where } K_R=\{x\in K:\|x\|_\eta\leqslant R\}.$$

On the other hand, by Arzela-Ascoli theorem, N is completely continuous. Now the conclusion follows from Schauder fixed point theorem. □

The next result is a generalization of Theorem 2.

THEOREM 3. *Assume* **(h1)–(h4)** *hold. In addition suppose*

(h7) *there exists a function* $\rho:\mathbb{R}_+^3\to\mathbb{R}_+$ *nondecreasing in each variable with*

$$\rho(\lambda x,\lambda y,\lambda z)\leqslant\lambda\rho(x,y,z)\tag{7}$$

for all $x,y,z\in\mathbb{R}_+$, $\lambda\geqslant1$, *such that*

$$|f(t,x,y,z)|\leqslant\rho(|x|,\ |y|,\ |z|)$$

on $[0,T]\times\mathbb{R}^3$ *and there exist* $\eta>0$, $R>0$ *with*

(8) $$u\leqslant |a(0)|+\eta^{-1}\rho\left(u,u,\mathrm{e}^{\eta M}u\right)\Rightarrow u\leqslant R,$$

where M *is given by* (5).
Then (1) *has a solution.*

Proof. We shall use Leray-Schauder fixed point theorem. Let x be any solution to

(9) $$x(t)=a(0)+\lambda\int_0^t f(s,x(s),\widetilde{x}(\phi(s)),\widetilde{x}(\psi(s)))\mathrm{d}\,s$$

for $\lambda\in(0,1)$. With the above notations, we have

$$|x(t)|\leqslant |a(0)|+\int_0^t\rho\left(|x(s)|,\ |\widetilde{x}(\phi(s))|,\ |\widetilde{x}(\psi(s))|\right)\mathrm{d}\,s=|a(0)|+$$

$$+\int_0^t\rho\left(|x(s)|\,\mathrm{e}^{-\eta s},\mathrm{e}^{\eta s},|\widetilde{x}(\phi(s))|\,\mathrm{e}^{-\eta\phi(s)}\mathrm{e}^{\eta\phi(s)},|\widetilde{x}(\psi(s))|\,\mathrm{e}^{-\eta\psi(s)}\mathrm{e}^{\eta\psi(s)}\right)\mathrm{d}\,s\leqslant$$

$$\leqslant|a(0)|+\int_0^t\rho\left(\|x\|_\eta\,\mathrm{e}^{\eta s},\max\left(\|x\|_\eta,\theta\right)\mathrm{e}^{\eta s},\max\left(\|x\|_\eta,\theta\right)\mathrm{e}^{\eta M}\mathrm{e}^{\eta s}\right)\mathrm{d}\,s\leqslant$$

$$\leqslant|a(0)|+\eta^{-1}\mathrm{e}^{\eta t}\rho\left(\|x\|_\eta,\max\left(\|x\|_\eta,\theta\right),\max\left(\|x\|_\eta,\theta\right)\mathrm{e}^{\eta M}\right).$$

It follows that

$$\|x\|_\eta\leqslant|a(0)|+\eta^{-1}\rho\left(\|x\|_\eta,\max\left(\|x\|_\eta,\theta\right),\max\left(\|x\|_\eta,\theta\right)\mathrm{e}^{\eta M}\right).$$

Now suppose that η and R are like in **(h7)**. Then

$$\|x\|_\eta\leqslant\max,(R,\theta).$$

Thus, the solutions are bounded independently on λ and Leray-Schauder fixed point theorem applies. □

Remark 2. Notice **(h6)** implies **(h7)**. Take $\rho(x,y,z)=\alpha x+\beta y+\gamma z+\delta$ and observe that (4) guarantees (8).

In the next theorem no subhomogenity condition of type (7) is required. It can be considered as a typical result for functional differential equations with both retarded and advanced arguments.

THEOREM 4. *Let* $\phi\in C([0,T];[-A,T])$ *and* $\psi\in C([0,T];[0,T])$ *with*

(10) $$\phi(t)\leqslant t\leqslant\psi(t)\ \ on\ \ [0,T].$$

Assume $a\in C([-A,0];\mathbb{R}_+)$ *is nondecreasing and* $f\in C\left([0,T]\times\mathbb{R}^3_+;\mathbb{R}_+\right)$.

In addition suppose that the following condition is satisfied:

(h8) *there exists a continuous function* $\beta : \mathbb{R}_+^3 \to (0, \infty)$ *and* $\alpha \in L^1[0,T]$ *such that* $\beta(x,y,z)$ *is nondecreasing in* y *and nonincreasing in* z,

$$f(t,x,y,z) \leqslant \alpha(t)\beta(x,y,z)$$

on $[0,T] \times \mathbb{R}_+^3$ *and*

$$\int_0^T \alpha(t)\,\mathrm{d}\,t < \int_{a(0)}^{\infty} \frac{\mathrm{d}\,u}{\beta(u,u,u)}.$$

Then (1) *has a nonnegative and nondecreasing solution* $x \in C^1[0,T] \cap C[-A,T]$.

Proof. We use the Leray-Schauder continuation principle. Let x be any solution to (10). Then

$$0 \leqslant x'(t) = \lambda f(t, x(t), \widetilde{x}(\phi(t)), \widetilde{x}(\psi(t))), \quad 0 \leqslant t \leqslant T.$$

Using (10) and the monotonicity properties of f, we obtain

$$x'(t) \leqslant \alpha(t)\beta(x(t), x(t), x(t)), 0 \leqslant t \leqslant T.$$

It follows that

$$\int_{a(0)}^{x(T)} \frac{\mathrm{d}\,u}{\beta(u,u,u)} = \int_0^T \frac{x'(t)}{\beta(x(t), x(t), x(t))}\mathrm{d}\,t \leqslant \int_0^T \alpha(t)\,\mathrm{d}\,t.$$

Hence $x(T) \leqslant R$, where R is so that

$$\int_0^T \alpha(t)\,\mathrm{d}\,t = \int_{a(0)}^{R} \frac{\mathrm{d}\,u}{\beta(u,u,u)}.$$

Thus the solutions of (10) are bounded independently on λ and the Leray-Schauder continuation principle applies. □

The next result is based on the monotone iterative method.

THEOREM 5. *Let* $\phi \in C([0,T];[-A,T]), \psi \in C([0,T];[-A,T]), a \in C([-A,0];\mathbb{R}_+)$ *and* $f \in C([0,T] \times \mathbb{R}_+^3; \mathbb{R}_+)$. *Suppose that* $f(t,x,y,z)$ *is nondecreasing in* x, y, z *and that there exists a function* $w \in C[0,T]$ *with*

$$w(t) \geqslant a(0) + \int_0^t f(s, w(s), \widetilde{w}(\phi(s)), \widetilde{w}\,(\psi(s)))\mathrm{d}\,s, \quad 0 \leqslant t \leqslant T.$$

Let

$$U_0(t) \equiv a(0), V_0(t) = w(t), U_{n+1} = N(U_n) \text{ and } V_{n+1} = N(V_n),$$

$(t \in [0,T])$, $n = 0, 1, \ldots$ *Then*

$$a(0) = U_0 \leqslant U_1 \leqslant \ldots \leqslant U_n \leqslant \ldots \leqslant V_n \leqslant \ldots \leqslant V_1 \leqslant V_0 = w, \tag{11}$$

$$0 \leqslant U_1' \leqslant \ldots \leqslant U_n' \leqslant \cdots \leqslant V_n' \leqslant \cdots \leqslant V_1' \tag{12}$$

on $[0,T]$. *Also, the following limits exist*

$$\underline{x}(t) = \lim_{n\to\infty} U_n(t), \; \overline{x}(t) = \lim_{n\to\infty} V_n(t)$$

uniformly on $[0,T]$. *Moreover,* $\underline{x}, \overline{x}$ *are the minimal and maximal solutions of* (1) *in* K *satisfying* $x \leqslant w$ *on* $[0,T]$.

Proof. From $a(0) \leqslant w$ it follows that $a(0) \leqslant N(a(0)) \leqslant N(w) \leqslant w$ and $0 \leqslant N(a(0))' \leqslant N(w)'$, i.e. $U_0 \leqslant U_1 \leqslant V_1 \leqslant V_0$ and $0 \leqslant U_1' \leqslant V_1'$. Further, (11) and (12) follow successively. Now, since

$$\{U_n : n \geqslant 1\} = N\left(\{U_n : n \geqslant 0\}\right)$$

and N is completely continuous, the sequence $(U_n)_{n\geqslant 0}$ contains a convergent subsequence. By the monotonicity, the entire sequence $(U_n)_{n\geqslant 0}$ converges. Similarly, $(V_n)_{n\geqslant 0}$ is convergent. □

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