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POSITIVE SOLUTIONS OF NONLINEAR SINGULAR INTEGRAL EQUATIONS IN ORDERED BANACH SPACES

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Abstract. By using a specially constructed cone and the fixed point theory in cone for strict set contraction operators, this paper investigates the existence of multiple positive solutions for a class of nonlinear singular integral equations in ordered Banach spaces. Two examples are included to illustrate the main result.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and $P \subset E$ be a cone of E. The purpose of this paper is to investigate the existence of multiple positive solutions of the following nonlinear singular integral equation

$$x(t) = \int_0^1 K(t,s) f(s, x(s)) ds, \tag{1.1}$$

where $K : [0,1] \times [0,1] \to R^+$ and $f : (0,1) \times (P \setminus \{\theta\}) \to P$ are both continuous; θ is the zero element of E. The main tool used here is the fixed point theory in cone for strict set contraction operators. This technique (including Krasnoselskii's compression-expansion fixed point theorem) has been extensively applied in the literature to scalar equations, when $E = R$, see [1], [4], [5], [16] and [17] and references therein. Recently, in paper [11], the authors used the Krasnoselskii's compression-expansion fixed point theorem to discuss nonlinear integral equations in Banach spaces and obtained the existence of nonnegative solutions. On the other hand, the theory of singular boundary

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value problems has become an important area of investigation (see [2], [10], [13] and [15] and references therein) in last twenty years. Most of such problems can be converted into nonlinear integral equations. Without doubt, it is interesting to investigate directly the theory of singular integral equations, especially, in abstract spaces.

The main features of this paper are as follows. First, the existence result obtained is about positive solution, not nonnegative solution as in [11]. Secondly, comparing with [11], the number of solutions is multiple. Finally, the nonlinear term $f(t,x)$ may be *singular* at $t=0$, 1, and $x=\theta$, that is, $\lim\limits_{t\to 0^+}\|f(t,\cdot)\|=\infty$, $\lim\limits_{t\to 1^-}\|f(t,\cdot)\|=\infty$, and $\lim\limits_{x\to\theta, x\in P}\|f(\cdot,x)\|=+\infty$ (for details, see our example). To our knowledge, no paper considered singular integral equations in abstract spaces.

The organization of this paper is as follows. We shall introduce some lemmas and notations in the rest of this section. The main result will be stated and proved in Section 2. In Section 3, two examples are included to illustrate the main result.

Basic facts about ordered Banach spaces can be found in [3], [9] and [12]. Here we just recall a few of them. The cone P in E induces a *partial order* on E, i.e., $x\le y$ if and only if $y-x\in P$. P is said to be *normal* if there exists a positive constant N such that $\theta\le x\le y$ implies $\|x\|\le N\|y\|$. Without loss of generality, we suppose in the present paper that $N=1$. We consider the integral equation (1.1) in $C[J,E]$, with $J=[0,1]$. Evidently, $(C[J,E],\|\cdot\|_C)$ is a Banach space with norm $\|x\|_C=\max\limits_{t\in J}\|x(t)\|$ for $x\in C[J,E]$. In what follows, $x\in C[J,E]$ is called a *solution* of the integral equation (1.1) if it satisfies (1.1). The function x is a *positive solution* of (1.1) if, in addition, $x(t)>\theta$ for $t\in(0,1)$. Let $x:(0,1]\to E$ be continuous. The abstract generalized integral $\int_0^1 x(t)dt$ is called *convergent* if the limit $\lim\limits_{\epsilon\to 0^+}\int_\epsilon^1 x(t)dt$ exits. The convergency or divergency of other kinds of generalized integrals can be defined similarly.

For a bounded set V in a Banach space E, we denote by $\alpha(V)$ the Kuratowski *measure of noncompactness* (see [3], [9] and [12]). The operator $A:\ D\to E$ $(D\subset E)$ is said to be a k- *set contraction* if it is continuous, bounded and there is a constant $k\ge 0$ such that $\alpha(A(S))\le k\alpha(S)$ for any bounded set $S\subset D$; a k- set contraction with $k<1$ is called a *strict set contraction.*

In the paper, we shall denote by $\alpha(\cdot)$ and $\alpha_C(\cdot)$ the Kuratowski measure of noncompactness of a bounded subset in E and in $C[J,E]$, respectively.

For the application in the sequel, we first state the following lemmas which can be found in [9], pp13 and pp22.

Lemma 1.1. *Let* $V \subset C[J, E]$ *be bounded and equicontinuous on* J. *Then* $\alpha(V(t))$ *is continuous on* J *and* $\alpha_C(V) = \max_{t \in J} \alpha(V(t))$, *where* $V(t) = \{x(t) | x \in V\}$.

Lemma 1.2. *Let* P *be a cone of the Banach space* E *and* $P_{r,s} = \{x \in P : r \leq \|x\| \leq s\}$ *with* $s > r > 0$. *Suppose that* $A : P_{r,s} \to P$ *is a strict set contraction such that one of the following two conditions is satisfied:*

(i) $Ax \not\leq x$ *for* $x \in P$, $\|x\| = r$ *and* $Ax \not\geq x$ *for* $x \in P$, $\|x\| = s$.
(ii) $Ax \not\geq x$ *for* $x \in P$, $\|x\| = r$ *and* $Ax \not\leq x$ *for* $x \in P$, $\|x\| = s$.

Then, A *has a fixed point* $x \in P$ *such that* $r < \|x\| < s$.

For some applications of Lemma 1.2 and related topics see [6], [8], [14] and [18].

2. Main results

To establish the existence of multiple positive solutions in $C[J, P]$ of the integral equation (1.1), let us list the following assumptions:

(H1) There exists $\mu \in C[J, R^+]$ with $\mu(t) > 0$ for $t \in (0, 1)$ such that

$$K(t, s) \geq \mu(t) K(\tau, s),$$

for all $t, s, \tau \in J$.

(H2) For any three positive numbers R, r and δ with $R > r$ and $\delta < \frac{1}{4}$, $f(t, x)$ is uniformly continuous with respect to t on $[\delta, 1 - \delta] \times P_{r,R}$, and

$$\int_0^1 f_{r,R}(s) ds < +\infty$$

where $P_{r,R} = \{x \in P : r \leq \|x\| \leq R\}$ and $f_{r,R}(s) = \sup\{\|f(s, x)\| : x \in P \text{ and } \|x\| \in [r\mu(s), R]\}$ for all $s \in (0, 1)$. In addition, there exists a positive number l such that $\max_{t \in J} \int_0^1 K(t, s) f_{l,l}(s) ds < l$.

(H3) There exists a nonnegative number L with $2k_0 L < 1$ such that

$$\alpha(f(t, D)) \leq L \alpha(D),$$

for all $t \in (0, 1)$ and $D \subset P_{r,R}$, where $k_0 = \max_{t,s \in J \times J} K(t, s) > 0$.

(H4) There exist $\varphi^* \in P^*$ with $\|\varphi^*\| = 1$ and $\psi_R \in C[J, R^+]$ ($\psi_R(t) \not\equiv 0$) for all $R > 0$ such that

$$\varphi^*(f(t,x)) \geq \psi_R(t),$$

for $t \in (0,1), x \in P_R \setminus \{\theta\}$, where P^* is the dual cone of P and $P_R =: \{x \in P \big| \ \|x\| < R\}$.

(H5) There exist $\psi^* \in P^*$ with $\|\psi^*\| = 1$ and $[a,b] \subset (0,1)$ such that

$$\liminf_{\|x\| \to +\infty,\, x \in P} \frac{\psi^*(f(t,x))}{\|x\|} = +\infty$$

uniformly with respect to $t \in [a,b]$.

Remark 2.1. Assumption (H4) indicates that $f(t,x)$ may be singular at $t = 0, 1$ and $x = \theta$. Assumption (H5) shows that f is superlinear at ∞.

The following theorems are the main results of this paper.

Theorem 2.1. *Let assumptions* (H1)-(H5) *be satisfied. Then* (1.1) *has at least two positive solutions.*

Corollary 2.1. *Let assumptions* (H1)-(H3) *be satisfied. In addition, assume either* (H4) *or* (H5) *holds. Then* (1.1) *has at least one positive solution.*

Before proving the main result, we first give some preliminaries and lemmas. Let

$$Q =: \{x \in C[J,P] \big| \ x(t) \geq \mu(t)x(\tau) \ \text{ for all } t, \tau \in J\}.$$

From condition (H1), we have $\mu(t) \leq 1$ for $t \in J$, which means that Q is not an empty set. On the other hand, it is easy to see that Q is a closed and convex subset of $C[J,P]$. Moreover, Q is a cone of the Banach space $C[J,E]$. Since the cone P is normal with normality constant $N = 1$, we have for $x \in Q$ that

$$\|x\|_C \geq \|x(t)\| \geq \mu(t)\|x\|_C, \quad \text{for all } t \in J. \tag{2.1}$$

Obviously, x is a positive solution of (1.1) if $x \in Q \setminus \{\theta\}$ is a solution of (1.1).

For the sake of applying the fixed point theory in cone, define an operator A on $Q \setminus \{\theta\}$ by

$$(Ax)(t) =: \int_0^1 K(t,s) f(s, x(s)) ds. \tag{2.2}$$

From (2.1), (2.2) and (H2), it is easy to see that A is well defined on $Q \setminus \{\theta\}$. Thus, we need to investigate only the existence of a fixed point of A in $Q \setminus \{\theta\}$.

Lemma 2.1. *Assume that conditions* (H1) *and* (H2) *hold. Then for any* $R > r > 0$, $A : Q_{r,R} \to Q$ *is continuous and bounded, where* $Q_{r,R} = \{x \in Q : r \leq \|x\| \leq R\}$.

The proof of Lemma 2.1 is similar to that of Lemma 2.1.1 [9].

Lemma 2.2. *Suppose hypotheses* (H1)-(H4) *hold. Then for any* $R > r > 0$, $A : Q_{r,R} \to Q$ *is a strict set contraction.*

Proof. For any $R > r > 0$, assume $S \subset Q_{r,R}$. It is easy to see that AS is bounded and equicontinuous on J. Therefore, Lemma 1.1 guarantees that

$$\alpha_C(AS) = \sup_{t \in J} \alpha((AS)(t)), \tag{2.6}$$

where $(AS)(t) = \{(Ax)(t) : x \in S\}$ for $t \in J$. For $\delta \in (0, \frac{1}{4})$, let

$$D(t) =: \Big\{ \int_\delta^{1-\delta} K(t,s) f(s, x(s)) ds : \ x \in S \Big\}.$$

By (H2) we have for all $x \in S$ and $t \in J$ that

$$\begin{aligned} &\| \int_\delta^{1-\delta} K(t,s) f(s,x(s)) ds - \int_0^1 K(t,s) f(s,x(s)) ds \| \\ &\leq k_0 \Big(\int_0^\delta f_{r,R}(s) ds + \int_{1-\delta}^1 f_{r,R}(s) ds \Big), \end{aligned}$$

which means

$$d_H(D(t), \ (AS)(t)) \to 0 \quad \text{as } \delta \to 0 \quad \text{ for each } t \in J.$$

Here $d_H(\cdot, \ \cdot)$ denotes the Pompeiu–Hausdorff metric. Thus, by the property of the Kuratowski measure of noncompactness, one can see

$$\alpha\big((AS)(t)\big) = \lim_{\delta \to 0^+} \alpha(D(t)), \qquad \text{for all } t \in J. \tag{2.7}$$

Now we estimate $\alpha(D(t))$. Note that

$$\int_\delta^{1-\delta} K(t,s) f(s,x(s)) ds \in (1-2\delta) \overline{co} \big\{ K(t,s) f(s,x(s)) : \ s \in [\delta, 1-\delta] \big\},$$

where $\overline{co}\{\cdot\}$ denotes the convex closure. By (H2), (H3), and Lemma 1.1, we deduce

$$\begin{aligned}\alpha(D(t)) &= \alpha\Big(\Big\{\int_{\delta}^{1-\delta} K(t,s)f(s,x(s))ds:\ x\in S\Big\}\Big)\\
&\le (1-2\delta)\alpha\big(\overline{co}\big\{K(t,s)f(s,x(s)):\ s\in[\delta,1-\delta],\ x\in S\big\}\big)\\
&\le k_0 \max_{s\in[\delta,1-\delta]} \alpha\big(f(s,S(I))\big)\\
&\le k_0 L\alpha(S(I)) \le 2k_0 L\alpha_C(S),\end{aligned} \tag{2.8}$$

where $I=[\delta,1-\delta]$, $S(I)=\{x(s):\ x\in S,\ s\in I\}$. Therefore, from (2.6)-(2.8), we have

$$\alpha_C(AS)\le 2k_0L\alpha_C(S).$$

This together with $2k_0L<1$ implies that $A:\ Q_{r,R}\to Q$ is a strict set contraction. The proof of Lemma 2.2 is thus completed. □

Proof of Theorem 2.1. First we choose

$$r_1=\min\Big\{\frac{l}{2},\ \frac{1}{2}\max_{t\in J}\int_0^1 K(t,s)\psi_l(s)\Big\},$$

where $\psi_l(s)$ is the same as in hypothesis (H4). Then $0<r_1<l$. We claim that

$$Ax\not\le x \quad \text{for } x\in\partial Q_{r_1}. \tag{2.9}$$

Assume the contrary, i.e., there exists a $x\in\partial Q_{r_1}$ with $Ax\le x$. Then

$$x(t)\ge (Ax)(t)=\int_0^1 K(t,s)f(s,x(s))ds, \quad \text{for all } t\in J. \tag{2.10}$$

Notice that $\|x\|_C=r_1<l$. Using condition (H4) and (2.10), we obtain

$$\begin{aligned}r_1 &\ge \|x(t)\|\ge \varphi^*(x(t))\\
&\ge \varphi^*\Big(\int_0^1 K(t,s)f(s,x(s))ds\Big)\\
&\ge \int_0^1 K(t,s)\varphi^*\big(f(s,x(s))\big)ds\\
&\ge \int_0^1 K(t,s)\psi_l(s)ds,\end{aligned}$$

which is a contradiction with the choice of r_1. This means that (2.9) holds. Secondly, from hypothesis (H2) and the normality of cone, it follows that

$$Ax \ngeq x \quad \text{for} \;\; x \in \partial Q_l. \tag{2.11}$$

Assume the contrary, i.e., $Ax \geq x$ for some $x \in \partial Q_l$. Then, from $x \in Q$, we have $x(s) \geq \mu(s) x(\tau)$ for all $s, \tau \in J$. Consequently, $\mu(s) \|x(\tau)\| \leq \|x(s)\| \leq l$ for all $s, \tau \in J$. It follows that $\mu(s) l \leq \|x(s)\| \leq l$ for every $s \in J$. Now, if $t_0 \in J$ is such that $l = \|x(t_0)\|$, then from $x \leq Ax$ we obtain

$$\begin{aligned} l &\leq \int_0^1 K(t_0, s) \|f(s, x(s))\| \, ds \\ &\leq \int_0^1 K(t_0, s) f_{l,l}(s) \, ds < l, \end{aligned}$$

a contradiction. Thirdly, choose $\bar{R} = \Big(\max\limits_{t \in J} \int_a^b K(t,s)\mu(s) ds \Big)^{-1} + 1$. Then, by condition (H5), there exists $M > 0$ such that $\psi^*\big(f(t,x)\big) \geq \bar{R}\|x\|$ for $t \in [a,b]$ and $\|x\| \geq M$. Again let

$$R_1 = \max\Big\{ l+1, \; \frac{M}{\min\limits_{t \in [a,b]} \mu(t)} \Big\}. \tag{2.12}$$

We claim that

$$Ax \nleq x \quad \text{for} \; x \in \partial Q_{R_1}. \tag{2.13}$$

If not, then there exists $x \in \partial Q_{R_1}$ such that $Ax \leq x$. Using condition (H5), (2.1), and (2.12), we obtain

$$\begin{aligned} R_1 &\geq \psi^*(x(t)) \geq \psi^*\big((Ax)(t)\big) \\ &\geq \int_0^1 K(t,s)\psi^*\big(f(s,x(s))\big) ds \\ &\geq \int_a^b K(t,s)\psi^*\big(f(s,x(s))\big) ds \\ &\geq \bar{R} \int_a^b K(t,s)\|x(s)\| ds \\ &\geq \bar{R} \int_a^b K(t,s)\mu(s)\|x\|_c ds \\ &= \bar{R}R_1 \int_a^b K(t,s)\mu(s) ds > R_1, \end{aligned}$$

which is a contradiction. Thus (2.13) holds. Finally, using (2.9), (2.11), (2.13), and Lemma 1.2 twice, we obtain that the integral equation (1.1) has two positive solutions $x(t)$ and $y(t)$ satisfying

$$r_1 < \|x\|_C < l < \|y\|_C < R_1.$$

This completes the proof of Theorem 2.1. □

3. Examples

Example 1. Consider the following finite system for scalar integral equations:

$$\begin{aligned} x_n(t) = \int_0^1 K(t,s)\frac{\sqrt{1-s}}{5}\Big(\frac{1}{m\sum\limits_{k=1}^{m}|x_k|} + x_{n+1} + x_{n+2}^2\Big)ds,& \\ t \in (0,1),\ n = 1,2,\cdots,m.& \end{aligned} \tag{3.1}$$

Here $x_{m+k} = x_k$ for $k = 1,2$ and $K(t,s) = \max\{1-t,\ 1-s\}$.

Theorem 3.1. *The singular system* (3.1) *has at least two positive solutions* $\big(x_1(t),\ x_2(t),\ \cdots,\ x_m(t)\big)$ *and* $\big(y_1(t),\ y_2(t),\ \cdots,\ y_m(t)\big)$ *with*

$$0 < \sum_{n=1}^{m}|x_n(t)| < 1 \text{ and } \sum_{n=1}^{m}|y_n(t)| \geq 1-t \text{ for } t \in (0,1).$$

Proof. Let $E = R^m$ with norm $\|x\| = \sum\limits_{n=1}^{m}|x_n|$ for $x \in E$ and $P = \{x = (x_1,x_2,\cdots,x_m) : \ x_n \geq 0,\ n = 1,2,\cdots,m\}$. Then P is a cone in E and system (3.1) can be regarded as an equation of the form (1.1), where $x = (x_1,x_2,\cdots,x_m)$, $f(t,x) = \big(f_1(t,x), f_2(t,x),\cdots,f_m(t,x)\big)$ and f_n is defined by

$$f_n(t,x) = \frac{\sqrt{1-t}}{5}\Big(\frac{1}{m\sum\limits_{k=1}^{m}|x_k|} + x_{n+1} + x_{n+2}^2\Big).$$

First notice that

$$K(t,s) \geq (1-t)K(\tau,s), \qquad \text{for all } t,\tau,s \in [0,1].$$

This means that condition (H1) is satisfied for $\mu(t) = 1 - t$.

Secondly, for each pair of positive numbers R and r with $R \geq r$, we have

$$f_{r,R}(s) \leq \frac{\sqrt{1-s}}{5}\Big(\frac{1}{(1-s)r} + R + R^2\Big).$$

Choosing $l = 1$, notice that

$$\begin{aligned}\int_0^1 K(t,s) f_{1,1}(s)ds &\leq \frac{1}{5}\int_0^t (1-s)\sqrt{1-s}\Big(\frac{1}{1-s} + 2\Big)ds \\ &\quad + \frac{1}{5}\int_t^1 (1-t)\sqrt{1-s}\Big(\frac{1}{1-s} + 2\Big)ds \\ &< 1.\end{aligned}$$

Thus (H2) is satisfied. Obviously, (H3) holds in this situation since R^m is finite dimensional.

Finally, it is easy to see that (H4) and (H5) are satisfied if we choose $\psi^* = \varphi^* = (1, 1, \cdots, 1)$. Now the conclusion follows from Theorem 2.1. $\square$

Example 2. Consider the following singular boundary value problem of infinite system of scalar differential equations:

$$\begin{cases} -x_n''(t) = \dfrac{\cos t}{\sqrt{t(1-t)}}\Big(1 + \dfrac{1}{n}(tx_{2n} + \ln(1+x_n)) + \dfrac{[t\,(1-t)]^\beta \arctan t}{\|x\|^\beta \ln(2+n)}\Big) \\ t \in (0,1) \\ x_n(0) = x_n(1) = 0, \quad n = 1, 2, \cdots, \end{cases} \tag{3.2}$$

where $\|x\| = \sup\limits_{n\geq 1} |x_n|$, $\beta \in \big(0, \frac{1}{2}\big)$.

Theorem 3.2. *The singular system* (3.2) *has at least one positive solution* $\big(x_1(t),\ x_2(t),\ \cdots,\ x_n(t), \cdots\big)$.

Proof. Let $E = l^\infty$ with norm $\|x\| = \sup\limits_{n\geq 1} |x_n|$ for $x \in E$ and $P = \{x = (x_1, x_2, \cdots, x_n, \cdots) :\ x_n \geq 0,\ n = 1, 2, \cdots\}$. Then P is a cone in E and system (3.2) can be translated into a singular integral equation of the form (1.1), where $x = (x_1, x_2, \cdots, x_n, \cdots)$,

$$K(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1 \\ t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$$

$f(t,x) = \big(f_1(t,x), f_2(t,x), \cdots, f_n(t,x), \cdots\big)$, and f_n is defined by

$$f_n(t,x) = \frac{\cos t}{\sqrt{t(1-t)}}\Big(1 + \frac{1}{n}(tx_{2n} + \ln(1+x_n)) + \frac{[t\,(1-t)]^{\beta}\arctan t}{\|x\|^{\beta}\ln(2+n)}\Big).$$

First it is easy to see that (H1) is satisfied if we choose $\mu(t) = t(1-t)$. Next for each pair of positive numbers R and r with $R \geq r$, we have

$$f_{r,R}(t) \leq \frac{\cos t}{\sqrt{t(1-t)}}\Big(1 + R + \ln(1+R) + \frac{\pi}{2r^{\beta}}\Big).$$

Therefore, from

$$\lim_{l\to+\infty}\frac{1 + l + \ln(1+l) + \frac{\pi}{2l^{\beta}}}{l} = 1$$

and

$$\begin{aligned}\int_0^1 K\,(t,s)\,\frac{\cos s}{\sqrt{s(1-s)}}ds &\leq \int_0^1 \sqrt{s\,(1-s)}ds\\ &\leq \frac{1}{2}\end{aligned}$$

for all $t \in [0,1]$, it follows that there exists a positive number l such that condition (H2) holds. As regards (H3), as in [9], Example 2.1.2, by using diagonal method, one can show that $L = 0$ in this situation.

Finally we see that condition (H4) is satisfied if we choose $\varphi^*(x) = x_1$ and $\psi_R(t) = \min\Big\{1, \dfrac{\cos t}{\sqrt{(1-t)}}\Big\}$. Now the conclusion follows from Corollary 2.1. □

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