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A MEAN THEOREM CONCERNING THE BEHAVIOUR OF SOME NONLINEAR FUNCTIONALS

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1. In the paper [3] Tiberiu Popoviciu has proved the following mean theorem concerning the behaviour of some linear functionals, in relation to the convexity of order n $(n \in Z,\ n \geq 0)$:

THEOREM A([3]). If the linear functional F, defined on $C[a,b]$, satisfies the conditions: 1) $F(1)=F(x)= \ldots = F(x^n) = 0$; 2) $F(f)>0$, for every function $f \in C[a,b]$, convex of order n ; then for each $f \in C[a,b]$, there exist $n+2$ distinct points $\xi_1, \xi_2, \ldots, \xi_{n+2}$ in $[a,b]$, such that $F(f)= K[\xi_1, \xi_2, \ldots, \xi_{n+2}; f]$, where K is a positive number not depending on f.

In [5] we have said that a real function f, defined on an interval, is strong $(\mathcal{P}_0, \mathcal{P}_{n+1}, \mathcal{P}_{n+2})$ - quasi-convex $(n \geq 0)$, if it satisfies the inequality:

$$(1)\quad 0<\max\{-[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+3}; f],\ [x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{n+3}; f]\} ;$$

for any system of distinct points $x_1 < x_2 < \ldots < x_{n+3}$ in its domain of definition and for every integers i and k such that $1 \leq k < i \leq n+3$, $i-k \geq 2$. For $n = 0$, the strong $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)$ - quasi-convexity coincides with the usual strict quasi-convexity.

In this paper, an analogous of theorem A, concerning the behaviour of some nonlinear functionals, in relation to the strong

$(\mathcal{P}_0, \mathcal{P}_{n+1}, \mathcal{P}_{n+2})$ - quasi-convexity, will be given.

Let F be a functional, defined on $C[a,b]$, which is assumed to satisfy the following conditions:

(i) $F(\lambda f) = \lambda F(f)$, for every $f \in C[a,b]$ and $\lambda > 0$;

(ii) $F(\lambda f) = |\lambda| F(f)$, for every $f \in \mathcal{P}_{n+1}$ and $\lambda \in \mathbb{R}$;

(iii) $F(f+g) \leq F(f) + F(g)$; if $f \in \mathcal{P}_{n+1}$ or $g \in \mathcal{P}_{n+1}$;

(iv) $F(f) > 0$, for every strong $(\mathcal{P}_0, \mathcal{P}_{n+1}, \mathcal{P}_{n+2})$-quasi-convex functions $f \in C[a,b]$.

THEOREM 1. <u>If the functional F, defined on $C[a,b]$, satisfies the conditions (i) - (iv), then for each function $f \in C[a,b]$ for which $F(f) < 0$, there exist n+2 distinct points $\xi_1, \xi_2, \dots, \xi_{n+2}$ in $[a,b]$, such that</u>

$$F(f) = K[\xi_1, \xi_2, \dots, \xi_{n+2} ; f], \tag{2}$$

<u>where K is a positive number not depending on f.</u>

Theorem A was generalized in abstract linear spaces by Elena Popoviciu [2] and M.Ivan [1] . Theorem 1 is also a particular case of a mean theorem that, in the following section, will be formulated and proved as part of the theory of the interpolation in abstract linear spaces.

2. Let X be a real linear space, $S_0 \subsetneqq S_1 \subsetneqq S_2$ three linear subspaces of X, S_1 a maximal proper subspace of S_2 and $\widetilde{S}_0$ a maximal proper subspace of S_1, with $S_0 \subset \widetilde{S}_0$. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two sets of linear interpolating operators relative to S_1, respectively to S_2, that is: if $U_1 \in \mathcal{U}_1$, then $U_1 : X \to S_1$ and $U_1 x = x$ for any $x \in S_1$ and if $U_2 \in \mathcal{U}_2$, then $U_2 : X \to S_2$ and $U_2 x = x$ for any $x \in S_2$. If we fix $y_1 \in S_1 \setminus \widetilde{S}_0$ and $y_2 \in S_2 \setminus S_1$, then we can associate [1] to each operator $U_1 \in \mathcal{U}_1$, the divided

difference functional $[U_1 ; \cdot] : X \to \mathbb{R}$ which satisfies : $[U_1 ; U_1 x] = [U_1 ; x]$ for all $x \in X$, $[U_1 ; x] = 0$ for any $x \in \widetilde{S}_0$ and $[U_1 ; y_1] = 1$ and to each operator $U_2 \in \mathcal{U}_2$, the divided difference $[U_2 ; \cdot] : X \to \mathbb{R}$ satisfying $[U_2 ; U_2 x] = [U_2 ; x]$ for all $x \in X$, $[U_2 ; x] = 0$ for any $x \in S_1$ and $[U_2 ; y_2] = 1$. Obviously, the divided differences of all operators of $\mathcal{U}_1(\mathcal{U}_2)$ take the same value on each element of S_1 (respectively, S_2) . In the sequel, S_1^+ denotes the set $\{x \in S_1 ; [U_1 ; x] > 0$ for any $U_1 \in \mathcal{U}_1\}$; S_2^+ denotes the set $\{x \in S_2 ; [U_2 ; x] > 0$ for any $U_2 \in \mathcal{U}_2\}$ and, in the set of all operators from X into X, the notation $U' \lhd U$ stands for $UU' = U'U = U'$.

We say that the triplet (U_1', U_1, U_1'') is a decomposition of the operator $U_2 \in \mathcal{U}_2$, if it satisfies the following conditions :

1° $U_1', U_1, U_1'' \in \mathcal{U}_1$, $U_1', U_1, U_1'' \lhd U_2$;

2° there exists $y \in S_2^+$ such that

$$[U_1' ; y] < [U_1 ; y] < [U_1'' ; y]. \tag{3}$$

In [5] we have proved that if (U_1', U_1, U_1'') is a decomposition of an operator U_2, then (3) holds for every $y \in S_2^+$.

We say that the element $x \in X$ is strong (S_0, S_1, S_2) - quasi-convex, respectively strong (S_0, S_1, S_2) - quasi-concave, if for each decomposition (U_1', U_1, U_1'') of some operator from $\mathcal{U}_2$, the inequality

$$A(x) = \max\{-[U_1' ; x], [U_1'' ; x]\} > 0 , \tag{4}$$

respectively

$$B(x) = \min\{-[U_1' ; x], [U_1'' ; x]\} < 0 , \tag{5}$$

holds.

Denote by $\mathcal{A}$ and $\mathcal{B}$ the sets of the functionals admitting the representation from (4), respectively from (5). We can easily see that if $F \in \mathcal{A}$, then F satisfies the following conditions:

(6) $F(\lambda x) = \lambda F(x)$, for every $x \in X$ and $\lambda > 0$;

(7) $F(\lambda x) = |\lambda| F(x)$, for every $x \in S_1$ and $\lambda \in R$;

(8) $F(x+y) \leqslant F(x) + F(y)$, if $x \in S_1$ or $y \in S_1$;

(9) $F(x) > 0$, for each strong (S_o, S_1, S_2)quasi-convex element $x \in X$;

while if $F \in \mathcal{B}$, then (6), (7) and also the following relations:

(10) $F(x+y) \geqslant F(x) + F(y)$, if $x \in S_1$ or $y \in S_1$;

(11) $F(x) < 0$, for each strong (S_o, S_1, S_2)-quasi-concave element $x \in X$;

hold.

The following assertion is a trite remark: if $A \in \mathcal{A}$ and for certain element $x \in X$ one has $A(x) < 0$, then there exists a functional $B \in \mathcal{B}$ such that $B(x) < 0$ too; but, it is a remarkable fact that the same conclusion is true even if in the place of A stands an arbitrary functional F satisfying the conditions (6)-(9):

THEOREM 2. <u>If the functional F, defined on X, satisfies the relations (6)-(9), then for each element $x \in X$ for which $F(x) < 0$, there exists a functional $B \in \mathcal{B}$ such that $B(x) < 0$</u> .

Proof. Let $s \in S_1^+$ be a fixed element and $k > 0$, the common value on s of the divided differences associated to the operators of $\mathcal{U}_1$. Since s is strong (S_o, S_1, S_2)-quasi-convex, by (9) we have $F(s) > 0$.

If we put $y = F(s)x - F(x)s$ and we take into account the properties (6)-(8), we obtain :

$$F(y) = F(F(s)x - F(x)s) \leqslant F(F(s)x) + F(-F(x)s) =$$
$$= F(s)F(x) + |F(x)|\, F(s) = 0 \; ;$$

which shows that y is not strong (S_o, S_1, S_2)-quasi-convex.

Consequently, there exists an operator $U_2 \in \mathcal{U}_2$ and a decomposition (U_1', U_1, U_1'') of U_2, such that $\max\{-[U_1' ; y], [U_1'' ; y]\} \leqslant 0$, that is $[U_1' ; y] \geqslant 0$ and $[U_1'' ; y] \leqslant 0$. Therefore ,
$F(s)[U_1';x] - F(x)[U_1';s] \geqslant 0$ and $F(s)[U_1'';x] - F(x)[U_1''; s] \leqslant 0$.

Hence, we deduce that $\frac{F(s)}{k}[U_1''; x] \leqslant F(x) \leqslant \frac{F(s)}{k}[U_1' ; x]$ and lastly that $\frac{F(s)}{k}\min\{-[U_1';x], [U_1'';x]\} \leqslant F(x) < 0$.

Since $F(s)/k > 0$, we see that the functional we have looking for is $B(z) = \min\{-[U_1'; z], [U_1''; z]\}$ $(z \in X)$.

In what follows we will give the abstract version of theorem 1. To formulate it we need a continuity type property in linear spaces.

DEFINITION 1. <u>We say that the element $x \in X$ has the property (C), if for every decomposition (U_1', U_1, U_1'') of some operator of $\mathcal{U}_2$ and for each number λ lying between the numbers $[U_1';x]$ and $[U_1'';x]$, there is at least one operator $\bar{U}_1 \in \mathcal{U}_1$ such that $[\bar{U}_1 ; x] = \lambda$</u> .

It is evident that if an element $x \in X$ has the property (C), then for every $s \in S_2$, the element $x + s$ also has this property.

THEOREM 3. <u>If the functional F, defined on X, satisfies the conditions (6) - (9), then for each element $x \in X$ having the property (C) and satisfying the inequality $F(x) < 0$, there exists an operator $V_1 \in \mathcal{U}_1$ such that</u>

(12) $$F(x) = K[V_1 ; x] ,$$

<u>where K is a positive number not depending on x.</u>

Proof. Let us come back at the proof of theorem 2, more precisely at the step where we have deduced the inequalities $[U_1';y] \geqslant 0$ and $[U_1'';y] \leqslant 0$. If $[U_1; y] = 0$ or $[U_1', y] = 0$ or $[U_1''; y] = 0$, then the required operator V_1 is U_1, U_1', respectively U_1''. Othewise, one has $[U_1 ; y] \neq 0$, $[U_1'' ; y] < 0 < [U_1' ; y]$ and since y, as x, has the property (C), there is at least one operator $V_1 \in \mathcal{U}_1$, such that $[V_1 ; y] = 0$ and therefore, that (12) be fulfilled.

REMARKS. 1° In order to obtain theorem 1 from theorem 3, let us consider $X = C[a,b]$, $S_o = \mathcal{P}_o$, $S_1 = \mathcal{P}_{n+1}$, $S_2 = \mathcal{P}_{n+2}$, $\tilde{S}_o = \mathcal{P}_n$, where $\mathcal{P}_k$ $(k \geq 0)$ denotes the set of the polynomials by degree $\leq k$; let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the sets of all Lagrange interpolating operators on $n + 2$, respectively on $n + 3$, distinct points in $[a,b]$. The divided differences associated with the operators of $\mathcal{U}_1$ and $\mathcal{U}_2$, are the ordinary ones on $n + 2$, respectively $n + 3$, distinct points in $[a,b]$.

As follows from [5, theorem 3], the strong $(\mathcal{P}_o, \mathcal{P}_{n+1}, \mathcal{P}_{n+2})$ - quasi-convex elements of $C[a,b]$, are precisely those that satisfy the inequalities (1) and by [5, lemma 3] we can state that each function $f \in C[a,b]$ has the property (C). To justify this assertion, let us consider the Lagrange interpolating operator $U_2 = L(\mathcal{P}_{n+2} ; x_1, x_2, \ldots, x_{n+3}; \cdot) \in \mathcal{U}_2$ and its decomposition [5] :

$$U_1' = L(\mathcal{P}_{n+1}; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+3} ; \cdot) ,$$

$$U_1 = L(\mathcal{P}_{n+1}; x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+3}; \cdot) ,$$

$$U_1'' = L(\mathcal{P}_{n+1}; x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+3}; \cdot) , \quad \text{where}$$

$a \leq x_1 < x_2 < \ldots < x_{n+3} \leq b$, and $1 \leq i < j < k \leq n + 3$. Define the function $D : [0,1] \longrightarrow \mathbb{R}$, which assigns to each $t \in [0,1]$, the divided difference of the function f, on the points contained by the vector $(1 - t)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+3})$ + + $t(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+3})$. We can easily see that the function D is continuous on $[0,1]$ and since $D(0) = [U_1' ; f]$ and

$D(1) = [U_1'' ; f]$, it follows that for each λ lying between $[U_1' ; f]$ and $[U_1'' ; f]$, there is at least one $t \in [0,1]$ such that $D(t) = \lambda$; this shows that f has the property (C).

2° The assumption that $F(f) < 0$, is essential in order that the conclusion of theorem 1 be true. Indeed, it the function f is concave of order n, then $F(f) > 0$, while all divided differences of f on $n + 2$ distinct points, are negative. So, (2) is not possible.

3° Other functionals than those of $\mathcal{A}$, which satisfy (6) - (9), can be given by using linear positive operators $L : X \longrightarrow X$, preserving the strong (S_o, S_1, S_2) - quasi-convexity property and satisfying $L(S_1) \subset S_1$. Indeed, if L is such an operator, then the functional $F : X \longrightarrow \mathbb{R}$, $F(x) = \max\{-[U_1' ; Lx] , [U_1'' ; Lx]\}$ $(x \in X)$, where U_1' and U_1'' arise from a decomposition (U_1', U_1, U_1'') of some operator of $\mathcal{U}_2$, satisfies (6) - (9) .

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