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# A FIXED POINT THEOREM OF MAIA TYPE IN SYNTOPOGENOUS SPACES

by

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## 0. Introduction

Banach's contraction principle in metric spaces was generalized in [9] to sets endowed with two metrics as follows :

Let $X$ be a nonempty set, $d$ and $d'$ two metrics on $X$ and $T:X \to X$ a mapping. Suppose that

(0.1) $d'(x,y) \leq d(x,y)$ for $x,y \in X$ ;

(0.2) $[X,d']$ is a complete metric space ;

(0.3) $T:[X,d'] \to [X,d']$ is continuous ;

(0.4) $d(Tx,Ty) \leq a\, d(x,y) \quad (x,y \in X)$ ,

for a certain $a \in [0,1[$ .

Then $T$ has a unique fixed point $x^*$ and $T^n x_0 \xrightarrow{d'} x^*$ as $n \to \infty$ , for any $x_0 \in X$ (here $T^n$ stands for the n-th iterate of $T$ ).

This theorem remains true (see [14], Remark 2.3.1) if condition (0.1) is replaced by

(0.5) $d'(T^k x, T^k y) \leq C\, d(x,y) \quad (x,y \in X)$ ,

for a certain $k \in \mathbb{N}$ and $C > 0$.

Let us remark that we may set instead of (0.1) :

(0.6) $T^k :[X,d] \to [X,d']$ is uniformly continuous (for a certain $k \in N$) such that Maia's theorem remains true.

The purpose of this paper is to extend this result to syntopogenous spaces. In particular we shall give a variant of Gheorghiu's theorem of Maia type (see[8]) in quasi-uniform spaces and we shall deduce from it Perov's theorem referring to contractions in generalized metric spaces. Also, we shall observe that, in a number of metrical fixed point theorems, generalized contractions as :

(0.7) $d(Tx,Ty) \leq a[d(x,Tx) + d(y,Ty)]$ $(x,y \in X)$ ,

where $a \in [0,1/2[$ ;

(0.8) $d(Tx,Ty) \leq a\, d(x,Tx) + b\, d(y,Ty) + c\, d(x,y)$ $(x,y \in X)$,

where a,b,c are nonnegative and $a + b + c < 1$ ;

(0.9) $d(Tx,Ty) \leq a \max(d(x,Tx),d(y,Ty))$ $(x,y \in X)$ ,

where $a \in [0,1[$ ;

(0.10) $d(Tx,Ty) \leq a \max(d(x,y),d(x,Tx),d(y,Ty),$
$d(x,Ty),d(y,Tx))$ $(x,y \in X)$ ,

where $a \in [0,1[$ (see[12] for a collection of various definitions of contractive mappings), are usual contractions with respect to certain uniform or quasi-uniform structures on X, associated to T. Consequently, the theorems of Maia type on such generalized contractions in metric spaces, can be deduced from the theorem of Maia type concerning usual contractions in uniform or quasi-uniform spaces.

Referring to syntopogenous spaces we shall follow, both in terminology and notation, the monograph [5], [6]. In addition we shall use the terms : Cauchy sequence and sequentially complete syntopogenous space. Let $[X,\mathcal{S}]$ be a syntopogenous space. We shall say that the sequence $(x_n)$ of elements of X is a Cauchy sequence if



the corresponding sequential filter base $\mathcal{R} = \{R_n : n=0,1,\dots\}$, where $R_n = \{x_i : i \geq n\}$, is Cauchy filter base. The syntopogenous space X is sequentially complete if any Cauchy sequence is convergent.

We shall especially use the ordering structures. Let us recall (see [5], (12.35),(12.10) and (12.26)) that for every syntopogenous structure $\mathcal{S}$ on X, there exists an ordering structure $\Phi$ compatible with $\mathcal{S}$, i.e. $\mathcal{S} \sim \mathcal{S}_\Phi$ and also

(0.11) $\mathcal{S}_\Phi \sim \bigcup_{\varphi \in \Phi^v} \mathcal{S}_\varphi = \{<_{\varphi,\varepsilon} : \varphi \in \Phi^v, \varepsilon > 0\}$.

This is the reason for which in the sequel we make no distinction between the syntopogenous structures : $\mathcal{S}$, $\mathcal{S}_\Phi$ and $\bigcup_{\varphi\in\Phi^v}\mathcal{S}_\varphi$ whenever $\Phi$ is an ordering structure compatible with $\mathcal{S}$.

Now, let $\Phi$ be an ordering structure on X compatible with $\mathcal{S}$ and $\alpha : \Phi \longrightarrow [0,\infty[$, $\beta : \Phi \longrightarrow \Phi$ two mappings. The mapping $T : X \longrightarrow X$ is called $\alpha-\beta-\Phi$ - continuous (see[11]) if for every $\varphi \in \Phi$ and $\varepsilon > 0$ we have

(0.12) $A <_{\varphi,\varepsilon} B$ implies $T^{-1}(A) <_{\beta(\varphi),\varepsilon/\alpha(\varphi)} T^{-1}(B)$ .

Next let us extend the mappings $\alpha$ and $\beta$ to the saturated ordering structure $\Phi^v$ in the following way : for each $\varphi \in \Phi^v \setminus \Phi$ we fix a system $\varphi_1,\dots,\varphi_n \in \Phi$ such that $\varphi = [\varphi_1,\dots,\varphi_n]$ and we put :

(0.13) $\alpha(\varphi) = \max(\alpha(\varphi_1),\dots,\alpha(\varphi_n))$ ,

(0.14) $\beta(\varphi) = [\beta(\varphi_1),\dots,\beta(\varphi_n)]$.

REMARK 0.1. <u>If T is $\alpha-\beta-\Phi$ - continuous, then it is also $\alpha-\beta-\Phi^v$ - continuous.</u>

<u>Proof</u>. Let $\varphi \in \Phi^v \setminus \Phi$ and let $\varphi_1,\dots,\varphi_n \in \Phi$ be those which were chosen in (0.13) and (0.14). Assume $A <_{\varphi,\varepsilon} B$. Then, as in the proof of proposition (12.20) from [5], we find the sets $A_i$

and $C_{ik}$ $(i = 1,\dots,m ;\ k = 1,\dots,n)$ such that $A = \bigcup_{i=1}^{m} A_i$, $X \setminus B = \bigcup_{k=1}^{n} C_{ik}$ $(i = 1,\dots,m)$ and

$A_i <_{\varphi_k,\varepsilon} X\setminus C_{ik}$ $(i = 1,\dots,m ;\ k = 1,\dots,n)$ .

Hence, by (0.12)

$$T^{-1}(A_i) <_{\beta(\varphi_k),\,\varepsilon/\alpha(\varphi_k)} T^{-1}(X\setminus C_{ik})$$

and since $\beta(\varphi_k)\subset\beta(\varphi)$ $(k = 1,\dots,n)$, we may infer that

$$T^{-1}(A_i) <_{\beta(\varphi),\,\varepsilon/\alpha(\varphi_k)} T^{-1}(X\setminus C_{ik}) ,$$

whence $T^{-1}(A_i) <_{\beta(\varphi),\,\varepsilon/\alpha(\varphi)} T^{-1}(X\setminus C_{ik})$ .

Thus, $T^{-1}(A_i) <_{\beta(\varphi),\,\varepsilon/\alpha(\varphi)} \bigcap_{k=1}^{n} T^{-1}(X\setminus C_{ik}) = T^{-1}(B)$

and in consequence

$$T^{-1}(A) = \bigcup_{i=1}^{n} T^{-1}(A_i) <_{\beta(\varphi),\,\varepsilon/\alpha(\varphi)} T^{-1}(B) ,$$

which completes the proof.

REMARK 0.2. <u>Any $\alpha-\beta-\Phi$ - continuous mapping is $(\mathcal{S},\mathcal{S})$ - continuous.</u>

<u>Proof</u>. If $T$ is $\alpha-\beta-\Phi$ - continuous, then by remark 0.1, it is $\alpha-\beta-\Phi^{v}$ - continuous too. Now the conclusion follows by (0.11).

Let $x_o,x,y\in X$ and $\varphi\in\Phi$. In order to formulate the contraction principle in syntopogenous spaces, we have assigned (see [11]) to each $\alpha-\beta-\Phi$ - continuous mapping $T$ the following families of series :

$$(0.15)\quad \Big\{\sum_{n=0}^{\infty}\alpha(\varphi)\alpha(\beta(\varphi))\dots\alpha(\beta^{n}(\varphi))\,|f_n(x_o)-f_n(Tx_o)| :\ f_n\in\beta^{n+1}(\varphi)\ ;\ n = 0,1,\dots\Big\},$$



$$(0.16)\quad \Big\{\sum_{n=0}^{\infty}\alpha(\varphi)\alpha(\beta(\varphi))\dots\alpha(\beta^{n}(\varphi))\,|f_n(x)-f_n(y)| :\ f_n\in\beta^{n+1}(\varphi);\ n=0,1,\dots\Big\}$$

and the following family of sequences :

$$(0.17)\quad \Big\{\big(\alpha(\varphi)\alpha(\beta(\varphi))\dots\alpha(\beta^{n}(\varphi))\,|f_n(x)-f_n(y)|\big)_{n\in\mathbb{N}} :\ f_n\in\beta^{n+1}(\varphi)\quad (n=0,1,\dots)\Big\}.$$

We say that a family of sequences of real numbers $\{(a^i_n)_{n\in\mathbb{N}} : i\in I\}$ converges uniformly to the family of real numbers $\{a^i : i\in I\}$ if for each $\varepsilon>0$ there exists a $N(\varepsilon)$ (independent of $i$) such that $|a^i_n - a^i|<\varepsilon$ for all $n\geq N(\varepsilon)$ and $i\in I$. In this case we say that the family of sequences is uniformly convergent. If in addition $a^i = 0$ $(i\in I)$, we say that the family of sequences converges uniformly to zero. A family of series is said to be uniformly convergent if the family of the sequences of partial sums is uniformly convergent.

## 1. <u>A theorem of Maia type in syntopogenous spaces</u>

In the paper [11] we have proved a theorem of Banach type for contractive mappings in a sequentially complete syntopogenous space. In the following theorem the contraction condition will be formulate with respect to a certain syntopogenous structure $\mathcal{S}$, while the sequentially completitude will be imposed to an other syntopogenous structure $\mathcal{S}'$, coarser in a certain sense then $\mathcal{S}$.

LEMMA 1.1. <u>Let $[X,\mathcal{S}]$ be a syntopogenous space, $\Phi$ an ordering structure compatible with $\mathcal{S}$ and $T : X \to X$ a $\alpha-\beta-\Phi$ - continuous mapping. If for some $x_o\in X$ the family of series (0.15) is uniformly convergent for each $\varphi\in\Phi$, then $(T^n x_o)_{n\in\mathbb{N}}$ is a $\mathcal{S}$ - Cauchy sequence.</u>

<u>Proof</u>. Consider the filter base $\mathcal{R} = \{R_n : n=0,1,\dots\}$, where

$R_n = \{T^i x_o : i \geq n\}$ $(n=0,1,\ldots)$. We must show that for any $\varphi \in \Phi^{\nu}$ and $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $A <_{\varphi,\varepsilon} B$ implies $R_k \subset X \setminus A$ or $R_k \subset B$.

Assume, a contrario, that there exist $\varphi \in \Phi^{\nu}$ and $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there are $A_k$ and $B_k$ satisfying

(1.1) $$A_k <_{\varphi,\varepsilon} B_k$$

and

(1.2) $$A_k \cap R_k \neq \emptyset \neq (X \setminus B_k) \cap R_k .$$

By (1.1), there exists $g_k \in \varphi$ such that

(1.3) $$g_k(A_k) <_{\varepsilon} \mathbb{R} \setminus g_k(X \setminus B_k) .$$

On the other hand, (1.2) implies that for each fixed $k$ there exist $p,q \in \mathbb{N}$ ; $p,q \geq k$ such that $T^p x_o \in A_k$ and $T^q x_o \in X \setminus B_k$. Then, by (1.3) we have

(1.4) $$g_k(T^q x_o) - g_k(T^p x_o) \geq \varepsilon .$$

Now, since $\varphi \in \Phi^{\nu}$, we have $\varphi = [\varphi_1, \varphi_2, \ldots, \varphi_n]$, where $\varphi_i \in \Phi$ $(i=1,\ldots,n)$ and in consequence (see [5], (12.1)) :

$$g_k = \min (h_1, \ldots, h_m) ,$$

where

$$h_i = \max (h_{i1}, \ldots, h_{in_i}) \qquad (1 \leq i \leq m)$$

and $$h_{ij} \in \varphi_{ij} \in \{\varphi_1, \ldots, \varphi_n\} \qquad (1 \leq i \leq m ;\ 1 \leq j \leq n_i) .$$

Suppose $g_k(T^p x_o) = h_r(T^p x_o)$, where $r = r(k) \in \{1, \ldots, m\}$.

Then, (1.4) yields to

$$h_r(T^q x_o) - h_r(T^p x_o) \geq \varepsilon ,$$

whence, if $h_r(T^q x_o) = h_{rs}(T^q x_o)$, where $1 \leq s \leq n_r$, we obtain

$$h_{rs}(T^q x_o) - h_{rs}(T^p x_o) \geq \varepsilon ,$$

where $h_{rs}$ belongs to $\bigcup_{i=1}^{n} \varphi_i$ .



Thus, for every $k \in \mathbb{N}$ there exist $p,q \in \mathbb{N}$, $p,q \geq k$ and a function $h \in \bigcup_{i=1}^{n} \varphi_i$ , such that :

(1.5) $$h(T^q x_o) - h(T^p x_o) \geq \varepsilon .$$

Let us fix for a moment $\psi \in \{\varphi_1, \ldots, \varphi_n\}$, $f \in \psi$ and $k \in \mathbb{N}$.

Denote $\theta = |f(T^{k+2}x_o) - f(T^{k+1}x_o)|$ and assume $\theta \neq 0$. Then, in the case when $f(T^{k+2}x_o) - f(T^{k+1}x_o) > 0$ , we have

(1.6) $$T^{k+1}x_o <_{\psi,\theta} X \setminus T^{k+2}x_o$$

and in the opposite case :

(1.7) $$T^{k+2}x_o <_{\psi,\theta} X \setminus T^{k+1}x_o .$$

Suppose that (1.6) holds. Then, since $T$ is $\alpha$-$\beta$-$\Phi$-continuous, we deduce

$$T^{-1}(T^{k+1}x_o) <_{\beta(\psi),\theta/\alpha(\psi)} T^{-1}(X \setminus T^{k+2}x_o)$$

whence, since $T^k x_o \in T^{-1}(T^{k+1}x_o)$ and $T^{-1}(X \setminus T^{k+2}x_o) \subset X \setminus T^{k+1}x_o$, we get $T^k x_o <_{\beta(\psi),\theta/\alpha(\psi)} X \setminus T^{k+1}x_o$ .

Hence, there exists $f_k \in \beta(\psi)$ such that

$$f_k(T^{k+1}x_o) - f_k(T^k x_o) \geq \theta / \alpha(\psi)$$

or, equivalently

$$0 < f(T^{k+2}x_o) - f(T^{k+1}x_o) \leq \alpha(\psi)(f_k(T^{k+1}x_o) - f_k(T^k x_o)) .$$

If relation (1.7) holds, then similarly to the above, we obtain

$$0 < f(T^{k+1}x_o) - f(T^{k+2}x_o) \leq \alpha(\psi)(f_k(T^k x_o) - f_k(T^{k+1}x_o)) .$$

Therefore, for every $f \in \psi$ and $k \in \mathbb{N}$ there exists (even if $\theta = 0$) $f_k \in \beta(\psi)$ such that

$$|f(T^{k+2}x_o) - f(T^{k+1}x_o)| \leq \alpha(\psi) |f_k(T^{k+1}x_o) - f_k(T^k x_o)| .$$

Hence, for every $f \in \psi$ and $q, r \in \mathbb{N}^*$ :

$$\sum_{i=q-1}^{q+r-2} \alpha(\psi)\alpha(\beta(\psi))\ldots\alpha(\beta^{i}(\psi))\left|f_i(Tx_0)-f_i(x_0)\right| \geq$$
$$\geq \left|f(T^{q+r}x_0) - f(T^{q}x_0)\right| ,$$

for certain $f_i \in \beta^{i+1}(\psi)$ $\quad (i = q-1,\ldots, q+r-2)$.

Whence, by the uniformly convergence of the family of series (0.15), we infer that for each $\varepsilon > 0$ there exists $q_\varepsilon(\psi) \in \mathbb{N}$ such that

(1.8) $$\left|f(T^{q+r}x_0) - f(T^{q}x_0)\right| < \varepsilon$$

for every $f \in \psi$ , $r \in \mathbb{N}$ and $q \geq q_\varepsilon(\psi)$.

If we put $q_\varepsilon = \max(q_\varepsilon(\varphi_1),\ldots, q_\varepsilon(\varphi_n))$ , then, clearly, (1.8) holds for every $f \in \bigcup_{i=1}^{n} \varphi_i$ , $r \in \mathbb{N}$ and $q \geq q_\varepsilon$ .

Finally, (1.5) where $k$ is taken equal with $q_\varepsilon$ and (1.8) yield a contradiction. This completes the proof.

LEMMA 1.2. <u>Let $\mathcal{S}$ and $\mathcal{S}'$ be two syntopogenous structures on $X$ and $T : X \to X$ be a mapping such that $T^k$ be $(\mathcal{S},\mathcal{S}')$ - continuous for a certain $k \in \mathbb{N}$. If the iterate sequence $(T^n x_0)_{n\in\mathbb{N}}$ is $\mathcal{S}$ - Cauchy, then it is $\mathcal{S}'$ - Cauchy as well.</u>

<u>Proof</u>. Assume that $(T^n x_0)_{n\in\mathbb{N}}$ is a $\mathcal{S}$ - Cauchy sequence and denote $\mathcal{R} = \{R_n : n \in \mathbb{N}\}$, where $R_n = \{T^i x_0 : i \geq n\}$ $(n \in \mathbb{N})$. Let $<' \in \mathcal{S}'$. Then, since $T^k$ is $(\mathcal{S},\mathcal{S}')$ - continuous, there exists $< \in \mathcal{S}$ such that

(1.9) $\quad A <' B$ implies $(T^k)^{-1}(A) < (T^k)^{-1}(B)$ .

On the other hand, the sequence $(T^n x_0)_{n\in\mathbb{N}}$ being $\mathcal{S}$ - Cauchy, for the order $< \in \mathcal{S}$ there exists $n \in \mathbb{N}$ such that if $C < D$ then $R_n \subset X \setminus C$ or $R_n \subset D$. Hence, by (1.9)

$$A <' B \text{ implies } R_n \subset X \setminus (T^k)^{-1}(A) \text{ or } R_n \subset (T^k)^{-1}(B) ,$$

whence

(1.10) $A <' B$ implies $R_{n+k} \subset X \setminus A$ or $R_{n+k} \subset B$ .



Therefore, for each $<' \in \mathcal{S}'$ there exists $n' = n + k \in \mathbb{N}$ satisfying (1.10). This shows that $\mathcal{R}$ is $\mathcal{S}'$ - Cauchy, which completes the proof.

LEMMA 1.3. <u>Let $\mathcal{S}$ and $\mathcal{S}'$ be two syntopogenous structures on $X$, $\Phi$ an ordering structure compatible with $\mathcal{S}$ and $T : X \to X$ a mapping. Suppose that</u>

(i) <u>$\mathcal{S}'$ satisfies the separation axiom $(T_0)$</u> ;

(ii) <u>there exists $k \in \mathbb{N}$ such that $T^k$ be $(\mathcal{S},\mathcal{S}')$-continuous</u>;

(iii) <u>$T$ is $\alpha$-$\beta$-$\Phi$ - continuous</u> ;

(iv) <u>for every $\varphi \in \Phi$ and $x,y \in X$ the family of sequences (0.17) converges uniformly to zero.</u>

<u>Then $T$ has at most one fixed point.</u>

<u>Proof</u>. Let $x^*$ and $y^*$ be fixed points of $T$. Assume $x^* \neq y^*$. Then, by (i), there exists $<' \in \mathcal{S}'$ such that $x^* <' X \setminus y^*$ or $y^* <' X \setminus x^*$. Suppose that $x^* <' X \setminus y^*$. Then, according to (ii), there exists $\psi \in \Phi^{\sigma}$ and $\varepsilon > 0$ such that

$$(T^k)^{-1}(x^*) <_{\psi,\varepsilon} (T^k)^{-1}(X \setminus y^*) ,$$

whence, since $x^* \in (T^k)^{-1}(x^*)$ and $(T^k)^{-1}(X \setminus y^*) \subset X \setminus y^*$ , we obtain

(1.11) $$x^* <_{\psi,\varepsilon} X \setminus y^* .$$

From (1.11), by a reasoning like that which has permeted to us to pass from (1.4) to (1.5), we deduce that there exists an $\varphi \in \Phi$ such that

$$x^* <_{\varphi,\varepsilon} X \setminus y^* ,$$

whence, by (iii) and since $x^*, y^*$ are fixed points of $T$, we have

$$x^* \in T^{-1}(x^*) <_{\beta(\varphi),\varepsilon/\alpha(\varphi)} T^{-1}(X \setminus y^*) = X \setminus T^{-1}(y^*) \subset X \setminus y^*$$

Hence

$$x^* <_{\beta(\varphi),\varepsilon/\alpha(\varphi)} X \setminus y^* .$$

Therefore, there exists $f_0 \in \beta(\varphi)$ such that

$$f_0(y^*) - f_0(x^*) \geqslant \frac{\varepsilon}{\alpha(\varphi)} .$$

Repeating this reasoning, we obtain

$$f_1(y^*) - f_1(x^*) \geqslant \frac{\varepsilon}{\alpha(\varphi)\alpha(\beta(\varphi))} ,$$

for a certain $f_1 \in \beta^2(\varphi)$ and generally

$$f_n(y^*) - f_n(x^*) \geqslant \frac{\varepsilon}{\alpha(\varphi)\alpha(\beta(\varphi))\dots\alpha(\beta^n(\varphi))} ,$$

for a certain $f_n \in \beta^{n+1}(\varphi)$ .

Thus, for every $n \in \mathbb{N}$ there exists $f_n \in \beta^{n+1}(\varphi)$ such that

$$\alpha(\varphi)\alpha(\beta(\varphi))\dots\alpha(\beta^n(\varphi))\left| f_n(y^*) - f_n(x^*)\right| \geqslant \varepsilon ,$$

which contradicts the uniformly convergence to zero of the family of sequences (0.17). Therefore, $x^* = y^*$ as desired.

REMARK 1.4. Lemma 1.3 remains true even if in (ii) we only require that $T^k$ be $(\mathcal{S}^p, \mathcal{S}')$ - continuous (see [5] , (4.7)).

Now we can state and prove the main result of this paper.

THEOREM 1.5. Let $\mathcal{S}$ and $\mathcal{S}'$ be two syntopogenous structures on $X$ , $\Phi$ an ordering structure compatible with $\mathcal{S}$ and $T : X \longrightarrow X$ a mapping. Suppose that the following conditions hold :

(i) $\mathcal{S}'$ satisfies the separation axiom $(T_2)$ ;

(ii) $\mathcal{S}'$ is sequentially complete ;

(iii) there exists $k \in \mathbb{N}$ such that $T^k$ be $(\mathcal{S}, \mathcal{S}')$-continuous;

(iv) $T$ is $(\mathcal{S}', \mathcal{S}')$ - continuous ;

(v) $T$ is $\alpha$ - $\beta$ - $\Phi$ - continuous.

Then a) If for a certain $x_0 \in X$ the family of series (0.15) is uniformly convergent for each $\varphi \in \Phi$ , then the sequence $(T^n x_0)_{n \in \mathbb{N}}$ is $\mathcal{S}'$ - convergent to a fixed point of $T$.



If in addition, for every $\varphi \in \Phi$ and $x, y \in X$, the family of sequences (0.17) converges uniformly to zero, then $T$ has a unique fixed point.

b) If for every $\varphi \in \Phi$ and $x, y \in X$ the family of series (0.16) is uniformly convergent, then for each $x_0 \in X$ the sequence $(T^n x_0)_{n \in \mathbb{N}}$ $\mathcal{S}'$ - converges to the unique fixed point of $T$.

Proof. a) By lemma 1.1 the sequence $(T^n x_0)_{n \in \mathbb{N}}$ is $\mathcal{S}$-Cauchy and so, according to lemma 1.2, it is $\mathcal{S}'$- Cauchy as well. Hence , since $\mathcal{S}'$ is sequentially complete, $T^n x_0 \longrightarrow x^*(\mathcal{S}')$ as $n \longrightarrow \infty$ . Whence, by (iv), we obtain (see [5] , (15.16)) $T^n x_0 \longrightarrow T x^*(\mathcal{S}')$ as $n \longrightarrow \infty$ . Now using (i) we may infer ( [5] , (15.15)) that $Tx^* = x^*$.

Next, under the additional assumption that the family of sequences (0.17) converges uniformly to zero for every $\varphi \in \Phi$ and $x, y \in X$, the unicity of the fixed point of $T$ follows by lemma 1.3.

Finally, b) is an immediate consequence of a). This completes the proof of theorem 1.5.

Let us assign to some mapping $T : X \longrightarrow X$ a relation on the set of all syntopogenous structures on $X$, denoted $\prec_T$ and defined by

$\mathcal{S}' \prec_T \mathcal{S}$ iff there exists $k \in \mathbb{N}$ such that $T^k$ be $(\mathcal{S}, \mathcal{S}')$-continuous. ($\mathcal{S}$ and $\mathcal{S}'$ being two syntopogenous structures on $X$).

REMARK 1.6. The relation $\prec_T$ is reflexive and transitive.

Also, if $\mathcal{S}$ is finer than $\mathcal{S}'$ in the usual sense, i.e. $\mathcal{S}' \prec \mathcal{S}$ , then $\mathcal{S}' \prec_T \mathcal{S}$ with respect to each mapping T. Indeed, if $\mathcal{S}' \prec \mathcal{S}$ , then obviously, the identity mapping $T^0$ on $X$ is $(\mathcal{S}, \mathcal{S}')$-continuous and consequently $\mathcal{S}' \prec_T \mathcal{S}$ .

Therefore, theorem 1.5 is all the more true if condition (iii) is replaced by $\mathcal{S}' \prec \mathcal{S}$ .

REMARK 1.7. <u>The contraction theorem in syntopogenous spaces, given in [11], is a consequence of theorem 1.5.</u>

Indeed, if we take $\mathcal{S}' = \mathcal{S}$, then condition (iii) is trivially satisfied, condition (iv) is a consequence of (v) and so, theorem 1.5 reduces to the contraction theorem in the space $[X,\mathcal{S}]$.

## 2. <u>A theorem of Maia type in quasi-uniform spaces</u>

Since the category of quasi-uniform spaces is isomorphic to a subcategory of that of syntopogenous spaces, in this section a theorem of Maia type in quasi-uniform spaces will be deduced from theorem 1.5. In uniform spaces it is a variant of Gheorgiu's theorem of Maia type (see[8]).

Next, since each quasi-uniformity can be derived from a family of quasi-metrics (see[5], (13.45)), by a quasi-uniform space we shall understand a pair $[X,\Sigma]$ of a nonempty set $X$ and a nonempty family $\Sigma$ of quasi-metrics on $X$ (here a quasi-metric on $X$ is a mapping $d: X \times X \longrightarrow [0,\infty[$ satisfying: $d(x,x) = 0$ and $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x,y,z \in X$). Recall that to each family $\Sigma$ of quasi-metrics on $X$ one attashes a syntopogenous structure on $X$, namely $\bigvee_{d\in\Sigma} \mathcal{S}_d$, where $\mathcal{S}_d = \{<_{d,\varepsilon} : \varepsilon > 0\}$ and $A <_{d,\varepsilon} B$ stands for $d(x,y) \geqslant \varepsilon$ for all $x \in A$ and $y \in X \setminus B$. Also we may consider the biperfect syntopogenous structure

$$\mathcal{S}_\Sigma = (\bigvee_{d\in\Sigma} \mathcal{S}_d)^b = (\bigvee_{d\in\Sigma} \mathcal{S}_d)^p \qquad \text{(see [5], (13.7)).}$$

Now let $[X,\Sigma]$ be a quasi-uniform space, where $\Sigma = \{d_i : i \in I\}$ and let $T$ be a mapping from $X$ into $X$. Consider two mappings $\alpha : I \longrightarrow [0,\infty[$ and $\beta : I \longrightarrow I$. In order to formulate a contraction principle in quasi-uniform spaces, it is imposed to $T$(see[7]) to satisfy conditions as :



1)

$$(2.1) \qquad d_i(Tx,Ty) \leqslant \alpha(i) d_{\beta(i)}(x,y) \qquad (x,y \in X,\ i \in I);$$

2) The series

$$(2.2) \qquad \sum_{n=0}^{\infty} \alpha(i)\alpha(\beta(i))\ldots\alpha(\beta^n(i))\max(d_{\beta^{n+1}(i)}(x_0,Tx_0),\ d_{\beta^{n+1}(i)}(Tx_0,x_0)),$$

is convergent for each $i \in I$ (for a fixed $x_0 \in X$) ;

3) The sequence

$$(2.3) \qquad (\alpha(i)\alpha(\beta(i))\ldots\alpha(\beta^n(i)) d_{\beta^{n+1}(i)}(x,y))_{n\in\mathbb{N}}$$

converges to zero for every $x,y \in X$ and $i \in I$ ;

4) The series

$$(2.4) \qquad \sum_{n=0}^{\infty} \alpha(i)\alpha(\beta(i))\ldots\alpha(\beta^n(i)) d_{\beta^{n+1}(i)}(x,y)$$

is convergent for every $x,y \in X$ and $i \in I$.

We shall express these conditions in terms of an ordering structure $\Phi_\Sigma$ on $X$. In order to do this, let us first define for each $d \in \Sigma$, the set $\varphi_d$ of all real functions $f$ defined on $X$ which can be represented as :

$$(2.5) \qquad f = \inf (f_i : i \in J),$$

where

$$(2.6) \qquad f_i = \max (f_{ij} : j = 1,\ldots,n_i)$$

and $f_{ij}$ has the form:

$$(2.7) \qquad f_{ij} = a_{ij} d(x_{ij},\cdot) + b_{ij},$$

with $a_{ij} \in \{0,1\}$, $b_{ij} \in \mathbb{R}$ and $x_{ij} \in X$ ($i \in J$ ; $j = 1,\ldots,n_i$). It is easy to see that $\varphi_d$ is an ordering family on $X$ (apply a reasoning similar with that from the proof of (12.1) in [5] ) .

Now let us set

$$(2.8) \qquad \Phi_{\Sigma} = \{ \varphi_d : d \in \Sigma \} .$$

LEMMA 2.1. <u>For all $d \in \Sigma$ and $\varepsilon > 0$ the following equality</u>

$$(2.9) \qquad <_{d,\varepsilon} = <_{\varphi_d, \varepsilon}$$

<u>holds</u>.

<u>Proof</u>. First suppose that $A <_{d,\varepsilon} B$. Then $d(x,y) \geq \varepsilon$ for all $x \in A$ and $y \in X \setminus B$. Define

$$f(x) = \inf \, (d(a,x) : a \in A) \qquad (x \in X).$$

Since $f \in \varphi_d$, $f(x) = 0$ for $x \in A$ and $f(x) \geq \varepsilon$ for $x \in X \setminus B$, we may infer that $A <_{\varphi_d, \varepsilon} B$. Therefore, $<_{d,\varepsilon} \subset <_{\varphi_d,\varepsilon}$.

Now assume $A <_{\varphi_d,\varepsilon} B$. Then there exists $f \in \varphi_d$ such that

$$f(y) - f(x) \geq \varepsilon \quad (x \in A, \ y \in X \setminus B) .$$

Let us fix for a moment $x \in A$ and $y \in X \setminus B$ and take $\eta > 0$ such that $\varepsilon > \eta$. By (2.5) there exists $i \in J$ such that $f_i(x) \leq f(x) + \eta$. Then

$$f_i(y) - f_i(x) \geq f(y) - f(x) - \eta \geq \varepsilon - \eta .$$

By (2.6) there exists $j \in \{1,\dots,n_i\}$ with $f_{ij}(y) = f_i(y)$. In consequence

$$f_{ij}(y) - f_{ij}(x) \geq f_i(y) - f_i(x) \geq \varepsilon - \eta > 0 ,$$

whence, taking into account (2.7), we find that

$$d(x,y) \geq d(x_{ij},y) - d(x_{ij},x) \geq \varepsilon - \eta$$

for a certain $x_{ij} \in X$. Now if we take $\eta \to 0$ we obtain $d(x,y) \geq \varepsilon$. Thus, $A <_{d,\varepsilon} B$ and consequently $<_{\varphi_d,\varepsilon} \subset <_{d,\varepsilon}$, as desired.

LEMMA 2.2. <u>If $[X, \Sigma]$ is a quasi-uniform space, then</u>

$$(2.10) \qquad \mathcal{S}_{\Phi_\Sigma} = \bigvee_{d \in \Sigma} \mathcal{S}_d .$$



<u>Proof</u>. By lemma 2.1 we have $\mathcal{S}_{\varphi_d} = \mathcal{S}_d$ and since $\mathcal{S}_{\Phi_\Sigma} = \bigvee_{d\in\Sigma} \mathcal{S}_{\varphi_d}$ ([5],(12.23)), equality (2.10) holds evidently.

Before passing to a transcription of conditions (2.1)-(2.4) in terms of syntopogenous structure $\mathcal{S}_{\Phi_\Sigma}$, let us make the convention that $\alpha$ and $\beta$ also denote the mappings:

$$\Phi_\Sigma \longrightarrow [0,\infty[ \ , \quad \varphi_{d_i} \longmapsto \alpha(i) \quad (i \in I) ,$$

respectively

$$\Phi_\Sigma \longrightarrow \Phi_\Sigma \ , \quad \varphi_{d_i} \longmapsto \varphi_{d_{\beta(i)}} \quad (i \in I) .$$

LEMMA 2.3. 1° <u>The mapping $T$ satisfies condition (2.1) if and only if it is $\alpha$ - $\beta$ - $\Phi_\Sigma$ - continuous.</u>

2° <u>The series (2.2) is convergent (for some $i \in I$) if and only if the family of series (0.15) is uniformly convergent for $\varphi = \varphi_{d_i}$.</u>

3° <u>The sequence (2.3) converges to zero (for some $i \in I$) if and only if the family of sequences (0.17) converges uniformly to zero for $\varphi = \varphi_{d_i}$.</u>

4° <u>The series (2.4) is convergent (for some $i \in I$) if and only if the family of series (0.16) is uniformly convergent for $\varphi = \varphi_{d_i}$.</u>

<u>Proof</u>. 1° Let $T$ be $\alpha$ - $\beta$ - $\Phi_\Sigma$ - continuous. Let $x$ and $y$ be any elements of $X$ and $i \in I$. Assume $\varepsilon = d_i(Tx,Ty) > 0$. Then $Tx <_{d_i,\varepsilon} X \setminus Ty$, whence, since $T$ is $\alpha$ - $\beta$ - $\Phi_\Sigma$ - continuous and by lemma 2.1, we get $x <_{d_{\beta(i)}, \varepsilon/\alpha(i)} X \setminus y$. Hence $d_{\beta(i)}(x,y) \geq \varepsilon / \alpha(i)$, as desired.

Conversely, assume now that $T$ satisfies condition (2.1). Let $A <_{\varphi_{d_i},\varepsilon} B$ and suppose that $T^{-1}(A) \not<_{\beta(\varphi_{d_i}),\, \varepsilon/\alpha(\varphi_{d_i})} T^{-1}(B)$. Then there exists $x \in T^{-1}(A)$ and $y \notin T^{-1}(B)$ such that

$d_{\beta(i)}(x,y) < \varepsilon/\alpha(i)$ , whence by (2.1) we get $d_i(Tx,Ty) < \varepsilon$ . On the other hand $Tx \in A$ and $Ty \notin B$ and since $A <_{d_i,\varepsilon} B$, we must have $d_i(Tx,Ty) \geqslant \varepsilon$ , a contradiction.

2° Assume that for $\varphi = \varphi_{d_i}$ the family of series (0.15) is uniformly convergent. If in (0.15) we set $f_n = d_{\beta^{n+1}(i)}(x_0,\cdot)$ if $d_{\beta^{n+1}(i)}(x_0,Tx_0) \geqslant d_{\beta^{n+1}(i)}(Tx_0,x_0)$ and $f_n = d_{\beta^{n+1}(i)}(Tx_0,\cdot)$ otherwise, then (0.15) reduces to (2.2).

Conversely, if the series (2.2) is convergent and we observe that $|f_n(x) - f_n(y)| \leqslant \max(d_{\beta^{n+1}(i)}(x,y), d_{\beta^{n+1}(i)}(y,x))$ ($f_n \in \beta^{n+1}(\varphi_{d_i})$), then the uniformly convergence of the family of series (0.15) is obvious.

The proof of parts 3° and 4° is similar.

THEOREM 2.4. Let $\Sigma$ and $\Sigma'$ be two families of quasi-metrics on $X$ and $T : X \longrightarrow X$ a mapping. Assume

(i) $\mathcal{S}_{\Sigma'}$ is separable ;

(ii) $\mathcal{S}_{\Sigma'}$ is sequentially complete ;

(iii) there exists $k \in \mathbb{N}$ such that $T^k$ be $(\mathcal{S}_{\Sigma}, \mathcal{S}_{\Sigma'})$ - continuous;

(iv) $T$ is $(\mathcal{S}_{\Sigma'}, \mathcal{S}_{\Sigma'})$ - continuous ;

(v) $T$ satisfies (2.1).

Then a) If for a certain $x_0 \in X$ the series (2.2) is convergent for each $i \in I_{\Sigma}$, then the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is $\mathcal{S}_{\Sigma'}$ - convergent to a fixed point of $T$.

If in addition, for every $i \in I_{\Sigma}$ and $x,y \in X$, the sequence (2.3) converges to zero, then $T$ has a unique fixed point.

b) If for every $i \in I_{\Sigma}$ and $x,y \in X$ the series (2.4) is convergent, then for each $x_0 \in X$ the sequence $(T^n x_0)_{n\in\mathbb{N}}$ $\mathcal{S}_{\Sigma'}$ - converges to the unique fixed point of $T$.



Proof. Apply lemma 2.3 and theorem 1.5 taking into account theorem (15.23) from [5] and remark 1.4.

REMARK 2.5. All the considerations about quasi-uniform structures remain valable for uniform structures, with the mention that in the case of a uniform space $[X,\Sigma]$ ( $\Sigma$ being a family of pseudo-metrics) we take $\varphi_d$ the set of all real functions defined on $X$ which can be represented as $f = \pm \inf(f_i : i \in J)$ , $f_i = \sup(f_{ij} : j \in K_i)$ where $f_{ij} = a_{ij}\, d(x_{ij},\cdot) + b_{ij}$ with $a_{ij} \in \{-1,0,1\}$ , $b_{ij} \in \mathbb{R}$ and $x_{ij} \in X$ .

## 3. Perov's fixed point theorem as a consequence of the theorem of Maia type in uniform spaces

In this section we show that Perov's fixed point theorem may be deduced from theorem 2.4.

COROLLARY 3.1 (A.I.Perov's theorem). Let $[X,d]$ be a complete generalized metric space with $d : X \times X \longrightarrow \mathbb{R}^r$ and let $T:X \longrightarrow X$ be a mapping such that

(3.1) $$d(Tx,Ty) \leqslant A\, d(x,y) \qquad (x,y \in X) ,$$

where $A \in M_{rr}(\mathbb{R}_+)$ and

(3.2) $$A^n \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Then $T$ has a unique fixed point $x^*$ and $T^n x_0 \longrightarrow x^*$ as $n \longrightarrow \infty$ for each $x_0 \in X$.

Proof. Denote by $d_i$ $(i=1,\dots,r)$ the pseudo-metrics for which $d(x,y) = (d_1(x,y),\dots,d_r(x,y))$ and consider $\Sigma' = \{d_i : i=1,\dots,r\}$ . Also, if $A^n = [a^n_{ij}]$ $(n \in \mathbb{N})$, we define the pseudo-metrics $d_{in} = \sum_{j=1}^{r} a^n_{ij} d_j$ $(i=1,\dots,r;\ n=0,1,\dots)$ and we consider $\Sigma = \{d_{in} : (i,n) \in I_{\Sigma} = \{1,\dots,r\} \times \mathbb{N}\}$. It is easy to see that conditions (i),(ii),(iii) (with $k = 0$) and (iv) of theorem 2.4 are fulfilled. Also $T$ satisfies condition (2.1), where

$\alpha : I_{\Sigma} \longrightarrow [0,\infty[$ , $\alpha(i,n) = 1$ $((i,n) \in I_{\Sigma})$ and

$\beta : I_{\Sigma} \longrightarrow I_{\Sigma}$ , $\beta(i,n) = (i,n+1)$ $((i,n) \in I_{\Sigma})$ .

On the other hand, the assumption of part b) of theorem 2.4 is also fulfilled. Indeed, by (3.2) we have $\sum_{n=0}^{\infty} A^{m+n+1} = A^{m+1}(I-A)^{-1}$ , whence $\sum_{n=0}^{\infty} A^{m+n+1} d(x,y) = A^{m+1}(I-A)^{-1} d(x,y)$. Hence, for all $x,y \in X$ and $(i,m) \in I_{\Sigma}$ , the series $\sum_{n=0}^{\infty}\sum_{j=1}^{r} a_{ij}^{m+n+1} d_j(x,y) = \sum_{n=0}^{\infty} d_{i,m+n+1}(x,y)$ which coincides with series (2.4), is convergent. Therefore, all the assumptions of theorem 2.4 being fulfilled, the conclusion of corollary 3.1 follows by theorem 2.4.

## 4. Theorems of Maia type for generalized contractions in metric spaces

In this section we show in what way theorems of Maia type concerning generalized contractions in metric spaces, can be deduced from theorem 2.4.

Let us first refer to generalized contractions (0.8). Let $[X,d]$ be a complete metric space and $T : X \longrightarrow X$ a mapping satisfying (0.8). In addition let us assume that $a + bc > 0$.

We have

$$d(Tx,T^2x) \leqslant a\, d(x,Tx) + b\, d(Tx,T^2x) + c\, d(x,Tx) ,$$

whence

$$d(Tx,T^2x) \leqslant r\, d(x,Tx) \qquad (x \in X), \tag{4.1}$$

where $r = (a+c)/(1-b)$ and $c < r < 1$ .

For each $n \in \mathbb{N}$ we define a quasi-metric on $X$, namely

$$d_n(x,y) = \frac{r^n - c^n}{r - c}[a\, d(x,Tx) + b\, d(y,Ty)] + c^n d(x,y) , \quad \text{for } x \neq y$$
$$= 0 , \quad \text{for } x = y. \tag{4.2}$$



By (0.8) and (4.1) we see that

$$d_n(Tx,Ty) \leqslant d_{n+1}(x,y) \qquad (x,y \in X,\ n \in \mathbb{N}). \tag{4.3}$$

Now let us set $\Sigma' = \{d\}$ ; $\Sigma = \{d_n : n \in \mathbb{N}\}$ ; $\alpha : \mathbb{N} \longrightarrow [0,\infty[$ , $\alpha(n) = 1$ $(n \in \mathbb{N})$ and $\beta : \mathbb{N} \longrightarrow \mathbb{N}$, $\beta(n) = n+1$ $(n \in \mathbb{N})$.

Then it is easy to see that assumptions (i)-(iii) (with $k=0$) and (v) of theorem 2.4 are fulfilled. Moreover, the hypothesis of part b) of theorem 2.4 is also fulfilled. Indeed, for an arbitrary $d_m \in \Sigma$ , series (2.4) is $\sum_{n=0}^{\infty} d_{m+n+1}(x,y)$, which, by (4.2) and by $0 \leqslant c < 1$ and $c < r < 1$, is obviously convergent.

Only assumption (iv) of theorem 2.4 is not fulfilled. But if we impose to $T$ to satisfy a certain condition of type (iv) , then the following result of Maia type (see [14] , Remark 2.3.2 and [2] ) can be deduced from theorem 2.4:

COROLLARY 4.1. Let $d$ and $d'$ be two metrics on $X$ and $T : X \longrightarrow X$ a mapping. Suppose that

(i) $[X,d']$ is a complete metric space ;

(ii) there exists $k \in \mathbb{N}$ such that $T^k : [X,d] \longrightarrow [X,d']$ be uniformly continuous;

(iii) $T : [X,d'] \longrightarrow [X,d']$ is continuous ;

(iv) $d(Tx,Ty) \leqslant a\, d(x,Tx) + b\, d(y,Ty) + c\, d(x,y)$ for all $x,y \in X$, where $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$ and $a + b + c < 1$.

Then $T$ has a unique fixed point $x^*$ and $T^n x_0 \xrightarrow{d'} x^*$ as $n \longrightarrow \infty$ , for each $x_0 \in X$.

Proof. If $a + bc > 0$ then apply theorem 2.4 for $\Sigma' = \{d'\}$ and $\Sigma = \{d_n : n \in \mathbb{N}\}$. For $a + bc = 0$ the proof is directly.

Let us remark that if $c = 0$ and $a = b \neq 0$, i.e. $T$ satisfies condition (0.7), we can take instead of $\Sigma$ a set of only two quasi-metrics (even pseudo-metrics), namely $d_0 = d$ and $d_1' = (1/a)d_1$.

In this case we must set $\alpha(d) = a$, $\alpha(d_1') = a/(1-a)$, $\beta(d) = d_1'$ and $\beta(d_1') = d_1'$.

Next les us assume that $T$ satisfies condition (0.10) (see [3] ). We define the following pseudo-metrics on $X$ :

(4.4) $$d_n(x,y) = \max (d(T^i x,T^j x), d(T^i y,T^j y), d(T^i x,T^j y) : i,j = 0,\dots,n) \quad \text{for } x \neq y$$
$$= 0, \quad \text{for } x = y$$

$(x,y \in X,\ n \in \mathbb{N})$.

In particular, for $n = 0$ one has $d_o = d$. Since, by (0.10)

$$d(T^i x,T^j x) \leq a\, d_n(x,y),\ d(T^i y,T^j y) \leq a\, d_n(x,y),$$
$$d(T^i x,T^j y) \leq a\, d_n(x,y)$$

for all $x,y \in X$ and $i,j \in \{1,\dots,n\}$, it follows that

(4.5) $$d_n(Tx,Ty) \leq a\, d_{n+1}(x,y) \qquad (x,y \in X;\ n \in \mathbb{N})$$

and also

(4.6) $$d_n(x,y) = \max (d(x,T^i x), d(y,T^i y), d(x,T^i y), d(y,T^i x) : i = 0,\dots,n)$$

$(x,y \in X\ ;\ n \in \mathbb{N})$.

If, for instance, $d_n(x,y) = d(x,T^i x)$ (with $1 \leq i \leq n$), we have

$$d_n(x,y) \leq d(x,Tx) + d(Tx,T^i x) \leq d(x,Tx) + a\, d_n(x,y).$$

Hence $$d_n(x,y) \leq \frac{1}{1-a} d(x,Tx) \leq \frac{1}{1-a} d_1(x,y).$$

Generally, we can see that

(4.7) $$d_n(x,y) \leq \frac{1}{1-a} d_1(x,y) \qquad (x,y \in X,\ n \in \mathbb{N}).$$

Now we are ready to deduce from theorem 2.4 the following.

COROLLARY 4.2. Let $d$ and $d'$ be two metrics on $X$ and $T : X \to X$ a mapping. Assume that

(i) $[X,d']$ is a complete metric space ;

(ii) $T^k : [X,d] \to [X,d']$ is uniformly continuous for a certain $k \in \mathbb{N}$;



(iii) $T : [X,d'] \to [X,d']$ is continuous ;

(iv) $d(Tx,Ty) \leq a \max (d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) \quad (x,y \in X)$,

where $a \in [0,1[$.

Then $T$ has a unique fixed point $x^*$ and $T^n x_o \xrightarrow{d'} x^*$ as $n \to \infty$, for each $x_o \in X$.

Proof. Let $\sum' = \{d'\}$ ; $\sum = \{d_n : n \in \mathbb{N}\}$ ; $\alpha : \mathbb{N} \to [0,\infty[$, $\alpha(n) = a\ (n \in \mathbb{N})$ and $\beta : \mathbb{N} \to \mathbb{N}$, $\beta(n) = n+1\ (n \in \mathbb{N})$. Then, conditions (i) - (v) of theorem 2.4 are fulfilled. Also, by (4.6), the hypothesis of part b) of this theorem is fulfilled too. Therefore, we may apply theorem 2.4.

Let us remark that if $T$ satisfies condition (0.9), then in the proof of the above corollary, we can take instead of $\sum$ a set consisting of only two pseudo-metrics, namely $d_o = d$ and $d_1'$ defined by

$$d_1'(x,y) = \max (d(x,Tx), d(y,Ty)), \quad \text{for } x \neq y$$
$$= 0, \quad \text{for } x = y.$$

In this case we must set $\alpha(d) = \alpha(d_1') = a$, $\beta(d) = d_1'$ and $\beta(d_1') = d_1'$.

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