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TOPOLOGICAL TRANSVERSALITY AND APPLICATIONS

by

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ABSTRACT. A generalized version of the topological transversality theorem due to A. Granas is proved by using Urysohn's lemma in Császár's variant. Some applications are suggested.

§ 1. PRELIMINARIES

Let X be a set and $\{<_n : n \in \mathbb{N}\}$ a sequence of relations $<_n \subset 2^X \times 2^X$ such that the following conditions hold for every $n \in \mathbb{N}$:

(1) $\emptyset <_n \emptyset, \quad X <_n X$;

(2) $M <_n N$ implies $M \subset N$;

(3) $M' \subset M <_n N \subset N'$ implies $M' <_n N'$;

(4) $M_i <_n N_i$, $i = 1, 2$ implies $M_1 \cup M_2 <_n N_1 \cup N_2$ and $M_1 \cap M_2 <_n N_1 \cap N_2$;

(5) if $M <_n N$, there is P with $M <_{n+1} P <_{n+1} N$.

DEFINITION 1 ([2]). A function $f : X \to I$, $I = [0, 1]$, is said to be <u>associated</u> with the sequence $\{<_n : n \in \mathbb{N}\}$ if

(6) $P, Q \subset I$, $d(P,Q) > 1/2^n$ implies $f^{-1}(P) <_{n+2} f^{-1}(I \setminus Q)$,

for every $n \in \mathbb{N}$, where $d(x, y) = |x-y|$.

LEMMA 1 ([2]). *Let $\{<_n : n \in \mathbb{N}\}$ be a sequence of relations on 2^X satisfying conditions (1)-(5). If $M <_0 N$, then there*

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exists a function f associated with $\{<_n : n \in N\}$ such that $f(x) = 0$ for $x \in M$ and $f(x) = 1$ for $x \in X \setminus N$.

§ 2. THE TOPOLOGICAL TRANSVERSALITY THEOREM

Let X be a set endowed with a sequence $\{<_n : n \in \mathbb{N}\}$ of relations on 2^X satisfying conditions (1)-(5), let $Y \subset X$ and $\emptyset \neq A \subset Y$. We consider a class of mappings

$$\mathcal{A}_A(Y;X) \subset \{T : Y \to X : Fix(T) \cap A = \emptyset\},$$

where $Fix(T) = \{x \in Y : T(x) = x\}$.

For a relation $<$ on 2^X we shall denote by $<|_Y$ the restriction of $<$ to Y([1], (6.19)), i.e.

(7) $M <|_Y N$ if $M, N \subset Y$ and $M < N \cup (X \setminus Y)$.

It is easily seen that the sequence $\{<_n|_Y : n \in N\}$ also satisfies conditions (1)-(5).

DEFINITION 2. A mapping $T \in \mathcal{A}_A(Y; X)$ is said to be essential if for each $T' \in \mathcal{A}_A(Y; X)$ having the same restriction to A as T, i.e. $T'|_A = T|_A$, one has $Fix(T') \neq \emptyset$. Otherwise, T is said to be inessential.

Let us consider an equivalence relation $\sim$ on $\mathcal{A}_A(Y; X)$ such that:

(i) if $T'|_A = T|_A$ then $T' \sim T$;

(ii) if $T' \sim T$ there exists $H : I \times Y \to X$ such that $H(0, .) = T'$, $H(1, .) = T$, $\bigcup\{Fix(H(t, .)) : t \in I\} = Z <_0 X \setminus A$ and $H(\theta(.), .) \in \mathcal{A}_A(Y; X)$ for any $\theta : Y \to I$ associated to $\{<_n|_Y : n \in \mathbb{N}\}$, with $\theta(x) = 1$ for $x \in A$.

LEMMA 2. Let $T \in \mathcal{A}_A(Y; X)$. T is inessential if and only if there exists $T' \in \mathcal{A}_A(Y; X)$ such that $T' \sim T$ and $Fix(T') = \emptyset$.

Proof. The necessity part follows from condition (i). Conversely, assume now that $T' \sim T$ and $Fix(T') = \emptyset$ and let H be a mapping as in (ii). If $Z = \emptyset$, then $Fix(H(1, .)) = \emptyset$ and thus

$T = H(1, .)$ is inessential. Suppose that $Z = \emptyset$. Since $Z <_o X \setminus A$, we have $Z <_o|_Y Y \setminus A$ and so, by Lemma 1, there exists a function $\theta : Y \to I$ associated to $\{<_n|_Y : n \in \mathbb{N}\}$, such that $\theta(x) = 0$ for all $x \in Z$ and $\theta(x) = 1$ for all $x \in A$. Define

$$H^* : Y \to X, \; H^*(x) = H(\theta(x), x) \text{ for } x \in Y.$$

According to (ii), $H^* \in \mathcal{A}_A(Y; X)$. In addition, $H^*|_A = H(1,.)|_A = T|_A$ and $Fix(H^*) = \emptyset$. Hence T is inessential.

THEOREM 1. *Let* T *and* T' *be in* $\mathcal{A}_A(Y; X)$ *such that* $T \sim T'$. *Then* T *and* T' *are both essential or both inessential.*

Proof. Assume that T is inessential. Then, by Lemma 2, there exists $T'' \in \mathcal{A}_A(Y; X)$ such that $T'' \sim T$ and $Fix(T'') = \emptyset$, whence, since $T \sim T'$ and relation $\sim$ is symmetric and transitive, it follows that $T'' \sim T'$, where $Fix(T'') = \emptyset$. This, again by Lemma 2, shows that T' is inessential too, which completes the proof.

REMARK. The assumption $Z <_o X \setminus A$ in (ii) is satisfied if we require that $Y \setminus A <_o X \setminus A$ and $Fix(H(t, .)) \cap A = \emptyset$ for all $t \in I$. Indeed, this last condition implies that $Z \subset Y \setminus A$ and then, by (3), $Z <_o X \setminus A$.

§ 3. APPLICATIONS

COROLLARY 1. *Let* X *be a normal topological space*, $\emptyset = A \subset Y \subset X$, A *and* Y *closed in* X. *Let*

$$\mathcal{A}_A(Y; X) \subset \{T : Y \to X : Fix(T) \cap A = \emptyset\}$$

and $\sim$ *be an equivalence relation on* $\mathcal{A}_A(Y; X)$ *satisfying condition* (i) *and*

(ii') *if* $T' \sim T$ *there exists* $H : I \times Y \to X$ *such that* $H(0, .) = T'$, $H(1, .) = T$, $cl(\bigcup\{Fix(H(t, .)) : t \in I\}) \cap A = \emptyset$ *and* $H(\theta(.), .) \in \mathcal{A}_A(Y; X)$ *for any continuous* $\theta : Y \to I$ *with* $\theta(x) = 1$ *for* $x \in A$.

If $T \sim T'$, *then* T *and* T' *are both essential or both inessential.*

Proof. According to classical Urysohn's lemma, condition (12.57) in [1] is satisfied. Consequently, by (12.56) in [1], the topology $\mathcal{T}_o = \{<\}$ on X can be derived from a symmetrical topogenenous structure $\mathcal{T}$, i.e. $\mathcal{T}_o = \mathcal{T}^p$. Moreover, by (12.58) in [1], we may assume that $\mathcal{T}$ is finer than any other symmetrical topogeneous structure having this property. Let $\mathcal{T} = \{<_o\}$. Obviously, the constant sequence of relations $\{<_n : n \in \mathbb{N}\}$, where $<_n = <_o$ for every $n \in \mathbb{N}$, satisfies conditions (1)-(5). Now, if $\theta : Y \to I$ is associated with the constant sequence $\{<_o|_Y\}$, then by (6), it is $(\mathcal{T}_o|_Y, \mathcal{H})$ - continuous and according to (1o.12) in [1] it is $(\mathcal{T}_o|_Y, \mathcal{H}^{tp})$-continuous too. Thus, θ is continuous (see (8.59) in [1]). Now, suppose that $T \sim T'$ and let H be a mapping as in (ii'). We want to whow that $Z <_o X \setminus A$, where $Z = \bigcup \{Fix(H(t, .)) : t \in I\}$. To do this, observe that since A is closed, we have $X \setminus A < X \setminus A$ and taking into account that $cl(Z) \subset X \setminus A$, we obtain that $cl(Z) < X \setminus A$. Similarly, from $A \subset X \setminus cl(Z)$ and $X \setminus cl(Z) < X \setminus cl(Z)$, we deduce that $A < X \setminus cl(Z)$. Now from $cl(Z) < X \setminus A$, $A < X \setminus cl(Z)$ and the fact that $\{<_o\}$ is the finest symmetrical topogeneous structure on X satisfying $< = <_o^p$ we may infer that $cl(Z) <_o X \setminus A$, whence $Z <_o X \setminus A$, as wished. Thus, Theorem 1 is applicable, which completes the proof.

In the paper [4] we use Corollary 1 to obtain fixed point theorems for several classes of nonlinear mappings.

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